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Unconditioned strong d-sequences

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A sequence $a_1, a_2, \ldots, a_s$ of elements in a commutative ring A is called a <u>d-sequence</u> if the equality

$$[(a_1, \ldots, a_{i-1}) : a_i a_j] = [(a_1, \ldots, a_{i-1}) : a_j]$$

holds for all $1 \leqq i \leqq j \leqq s$ and is called a <u>strong</u> d-sequence if $a_1^{n_1}, \ldots, a_s^{n_s}$ is a d-sequence for all $n_1, \ldots, n_s > 0$. Moreover if $a_1, a_2, \ldots, a_s$ is a (strong) d-sequence in any order, we will say that $a_1, a_2, \ldots, a_s$ is an <u>unconditioned</u> (strong) d-sequence ([9]).

The behaviour of parameter ideals in Buchsbaum rings are studied by S. Goto and many others (cf., [5] and [6]). The aim of this lecture is to give a new approach to them in terms of unconditioned strong d-sequences. This motif is suggested by Prof. S. Goto, and the writer wishes to thank him and the members of his seminar for helpful suggestions.

1. Let $a_1, a_2, \ldots, a_s$ be a sequence of elements in a commutative ring A and put $I = (a_1, \ldots, a_s)$. Our first result is stated as follows

Theorem 1. Let $a_1, a_2, \ldots, a_s$ be an unconditioned strong d-sequence. Then the following equalities hold:

(1) $\left[(a_1^{n_1+m_1}, \ldots, a_s^{n_s+m_s}) : \prod_{i=1}^{s} a_i^{m_i}\right] = \sum_{i=1}^{s} \left[(a_1^{n_1}, \ldots, \widehat{a_i^{n_i}}, \ldots, a_s^{n_s}) : a_i\right] + (a_1^{n_1}, \ldots, a_s^{n_s})$ for all $n_i, m_i > 0$ $(1 \leqq i \leqq s)$;

(2) $(a_1^{n_1}, \ldots, a_k^{n_k}) \cap I^n = \sum_{i=1}^{k} a_i^{n_i} \cdot I^{n-n_i}$ for all

$1 \leqq k \leqq s$, $n_1, \dots , n_k > 0$ and $n \in \mathbb{Z}$;

(3) $\left[(a_1^{n_1}, \dots , a_{k-1}^{n_{k-1}}): a_k\right] \cap I^n =$

$$\sum_{i=1}^{k-1} a_i^{n_i}. I^{n-n_i} + \sum_{\substack{F \subset \{1, \dots, k-1\} \\ \sum_{f\in F}(n_f-1) \geqq n}} \prod_{f\in F} a_f^{n_f-1}. \left[(a_f \mid f\in F): a_k\right]$$

for all $1 \leqq k \leqq s$, $n_1, \dots , n_{k-1} > 0$ and $n \in \mathbb{Z}$.

Let $G_I(A) = \bigoplus_{n\geq 0} I^n/I^{n+1}$ and $R_I(A) = \bigoplus_{n\geq 0} I^n$ denote the associated graded ring of I and the Rees algebra of I , respectively. $H_I^i(.)$ (resp. $H_{\underline{M}}^i(.)$ and $H_{\underline{N}}^i(.)$) stands for the $i^{\underline{th}}$ local cohomology defined by the direct limit of Koszul cohomology relative to I (resp. $\underline{M} = G_I(A)_+$ and $\underline{N} = I.R_I(A) + R_I(A)_+$). With this terminology we also have the following

Theorem 2. Let $a_1, a_2, \dots , a_s$ be an unconditioned strong d-sequence. Then

(1) $I. H_I^i(A/(a_1^{n_1}, \dots , a_j^{n_j})) = (0)$ for all $0 \leqq i + j < s$ and $n_1, \dots , n_j > 0$.

(2) $$\left[H_{\underline{M}}^i(G_I(A))\right]_n = H_I^i(A) \quad (n = - i) ,$$
$$= (0) \quad (n \neq - i)$$

for every $0 \leqq i < s$ and

$$\left[H_{\underline{M}}^s(G_I(A))\right]_n = (0) \quad (n > - s) .$$

(3) $$\left[H_{\underline{N}}^0(R_I(A))\right]_n = H_I^0(A) \quad (n = 0) ,$$
$$= (0) \quad (n \neq 0) ,$$
$$\left[H_{\underline{N}}^i(R_I(A))\right]_n = H_I^{i-1}(A) \quad (2 - i \leqq n \leqq - 1) ,$$
$$= (0) \quad (n \leqq 1 - i \text{ or } n \geqq 0)$$

for every $1 \leqq i \leqq s$ and

$$\left[H_{\underline{N}}^{s+1}(R_I(A))\right]_n = (0) \quad (n \geqq 0) .$$

(4) $H_{\underline{N}}^i(R_I(A)) = (0)$ for all $i > s + 1$.

All conclusions of our theorems are led by Lemma 4 which is given by S. Goto and another powerful application of this lemma can be found in [18]. So we belive that this lemma plays very important roles in the whole of our research.

In order to discuss an application of our results, let us recall some definition. For a while let A be a Noetherian local ring of $\dim A = d > 0$ and m the maximal ideal of A. Then A is called a Buchsbaum ring if the difference

$$l_A(A/q) - e_A(q)$$

is an invariant $I(A)$ of A not depending on the particular choice of a parameter ideal q of A, where $l_A(A/q)$ and $e_A(q)$ denote the length of the A-module A/q and the multiplicity of A relative to q, respectively. This is equivalent to saying that every system $a_1, a_2, \ldots, a_d$ of parameters for A is a weak-sequence, i.e., the equality $[(a_1, \ldots, a_{i-1}) : a_i] = [(a_1, \ldots, a_{i-1}) : m]$ holds for all $1 \leqq i \leqq d$ ([14]). C. Huneke showed in [9, (1.7)] that A is Buchsbaum if and only if every system of parameters for A forms a d-sequence. The theory of Buchsbaum rings (and modules) has rapidly developed and nowadays much is known about them (cf., e.g., [1], [3], [4], [7], [8], [10], [11], [12], [13], [15], [16], [20]). Let q be a parameter ideal of A and put

$$G_q(A) = \bigoplus_{n \geqq 0} q^n/q^{n+1} \quad \text{and} \quad R_q(A) = \bigoplus_{n \geqq 0} q^n$$

, the associated graded ring of q and the Rees algebra of q, respectively. As an application of our results we give an affirmative answer to the question posed by S. Goto in [2] as follows: is $R_q(A)$ a Buchsbaum ring if so is A ? Our answer is stated as follows

Theorem 3. Suppose that depth $A > 0$. Then the following conditions are equivalent.

(1) A is a Buchsbaum ring;

(2) $G_q(A)_{\underline{M}}$ is a Buchsbaum ring for every parameter ideal q of A;

(3) $R_q(A)_{\underline{N}}$ is a Buchsbaum ring for every parameter ideal q of A,

here $\underline{M}$ and $\underline{N}$ denote the unique graded maximal ideal of $G_q(A)$ and $R_q(A)$ respectively.

2. Sketch of Proof of Theorem 1.

Let $a_1, a_2, \dots, a_s$ be a sequence of elements in a commutative ring A and put $I = (a_1, \dots, a_s)$. Then

Lemma 4 (S. Goto). If $a_1, a_2, \dots, a_s$ is an unconditioned strong d-sequence modulo bA, then the equality

$$\left[(a_1^{n_1}, \dots, a_s^{n_s}) : b\right] = \sum_{F \subset \{1, \dots, s\}} \prod_{f \in F} a_f^{n_f - 1} \cdot \left[(a_f \mid f \in F) : b\right]$$

holds for all $n_1, \dots, n_s > 0$.

Lemma 5 ([6, Theorem (2.4)]). If $a_1, a_2, \dots, a_s$ is a d-sequence, then

$$\left[(a_1, \dots, a_{i-1}) : a_i\right] \cap I^n = (a_1, \dots, a_{i-1}) \cdot I^{n-1}$$

for all $1 \leqq i \leqq s$ and $n > 0$.

Lemma 6. If $a_1, a_2, \dots, a_s$ is an unconditioned strong d-sequence, then the following conditions are equivalent.

(1) $(a_f^{n_f} \mid f \in F) \cap I^n = \sum_{f \in F} a_f^{n_f} \cdot I^{n - n_f}$

for all $F \subset \{1, \dots, s\}$, $n_f > 0$ $(f \in F)$ and $n \in \mathbb{Z}$.

(2) $(a_f^{n_f} \mid f \in F) \cap I^n = \sum_{f \in F} a_f^{n_f} . I^{n-n_f}$

for all $F \subset \{1, \dots, s\}$, $n_f = 1, 2$ $(f \in F)$ and $n \in \mathbb{Z}$.

(3) $(a_1^{n_1}, \dots, a_s^{n_s}) \cap I^n = \sum_{i=1}^{s} a_i^{n_i} . I^{n-n_i}$

for all $n_i = 1, 2$ $(1 \leqq i \leqq s)$ and $n \in \mathbb{Z}$.

Lemma 7. If $a_1, a_2, \dots, a_s$ is an unconditioned strong d-sequence such that

$(a_1^2, \dots, a_{s-1}^2, a_s) \cap I^n = (a_1^2, \dots, a_{s-1}^2) . I^{n-2} + a_s . I^{n-1}$

for all $n \in \mathbb{Z}$, then the equality

$$([(a_1^2, \dots, a_{s-1}^2) : a_s] + a_s A) \cap I^n = (a_1^2, \dots, a_{s-1}^2) . I^{n-2} + a_s . I^{n-1} + \sum_{\substack{F \subset \{1, \dots, s-1\} \\ \# F \geqq n}} \prod_{f \in F} a_f . [(a_f \mid f \in F) : a_s]$$

holds for all $n \in \mathbb{Z}$.

Proof of Theorem 1. (1) Use induction on s and apply Lemma 4. (2) By Lemma 6, we must show that

$$(a_1^{n_1}, \dots, a_s^{n_s}) \cap I^n = \sum_{i=1}^{s} a_i^{n_i} . I^{n-n_i}$$

for all $n_i = 1, 2$ $(1 \leqq i \leqq s)$ and $n \in \mathbb{Z}$. Case 1: $n_i = 1$ for some i . Use induction on s and apply Lemma 5. Case 2 : $n_i = 2$ for all i . Apply Lemma 7 and Case 1. (3) Use induction on n and apply Lemma 4 and the above assertion (2).

3. Sketch of Proof of Theorem 2.

Let J be an ideal of A . A sequence $a_1, a_2, \dots, a_s$ is called a <u>weak-sequence</u> with respect to J if the equality

$$[(a_1, \dots, a_{i-1}) : a_i] = [(a_1, \dots, a_{i-1}) : J]$$

holds for all $1 \leqq i \leqq s$. We will use the words "strong" and "unconditioned" with the same meaning as d-sequences.

Lemma 8. Let J be an ideal of A which contains I. Then $a_1, a_2, \ldots, a_s$ is an unconditioned strong weak-sequence with respect to J if and only if it is an unconditioned strong d-sequence and an unconditioned weak-sequence w.r.t. J.

Proof of (1) of Theorem 2. By Lemma 8, we see that

$$I.\ H_I^0(A/(a_1^{n_1}, \ldots, a_j^{n_j})) = (0)$$

for every $0 \leqq j < s$, $n_1, \ldots, n_j > 0$. The assertion comes at once by descending induction on j and the similar discussion as in [20, Lemma 2].

We put $h_i = a_i \bmod I^2$ $(1 \leqq i \leqq s)$. Then

Proposition 9. $h_1, h_2, \ldots, h_s$ is an unconditioned strong d-sequence if so is $a_1, a_2, \ldots, a_s$.

Lemma 10. If $a_1, a_2, \ldots, a_s$ is an unconditioned strong d-sequence, then

(1) $\left[H_{\underline{M}}^0(G_I(A))\right]_n = H_I^0(A)$ $(n = 0)$,
$\qquad = (0)$ $(n \neq 0)$;

(2) $\left[H_{\underline{M}}^i(G_I(A))\right]_n = (0)$ $(n \neq -i)$

for every $1 \leqq i < s$;

(3) $\left[H_{\underline{M}}^s(G_I(A))\right]_n = (0)$ for all $n > -s$.

We put $a = a_1$. As $aA \cap I^n = a.I^{n-1}$ $(n \in \mathbb{Z})$ by Theorem 1, we get the following diagram:

$$\begin{array}{ccccccccc}
 & & & & & & & & 0 \\
 & & & & & & & & \uparrow \\
 & & & & & & & & G_I(A) \\
 & & & & & & & & \uparrow \\
0 \longrightarrow & [0 : a] & \longrightarrow & R_I(A) & \xrightarrow{\ a\ } & R_I(A) & \longrightarrow & R_I(A)/a.R_I(A) & \longrightarrow 0 \\
 & & & \searrow & & \nearrow & & \uparrow & \\
 & & & & a.R_I(A) & & & R_{I/aA}(A/aA)_+ & \\
 & & & \nearrow & & \searrow & & \uparrow & \\
 & & & 0 & & 0 & & 0 &
\end{array}$$

Lemma 11. If $a_1, a_2, \ldots, a_s$ is an unconditioned strong d-sequence, then

(1) $\left[H^0_{\underline{N}}(R_I(A))\right]_n = H^0_I(A) \quad (n = 0)$,

$= (0) \quad (n \neq 0)$;

(2) the canonical map $H^0_{\underline{N}}(R_I(A)) \longrightarrow H^0_{\underline{N}}(R_I(A)/a.R_I(A))$ is an isomorphism;

(3) $\left[H^i_{\underline{N}}(R_I(A))\right]_n = (0) \quad (n \leqq -i \text{ or } n \geqq 0)$ for every $1 \leqq i \leqq s$ and $\left[H^{s+1}_{\underline{N}}(R_I(A))\right]_n = (0) \quad (n \geqq 0)$.

Proof of Theorem 2. Consider the following two exact sequences:

$$0 \longrightarrow R_I(A)_+ \longrightarrow R_I(A) \longrightarrow A \longrightarrow 0 \quad ,$$

$$0 \longrightarrow R_I(A)_+(1) \longrightarrow R_I(A) \longrightarrow G_I(A) \longrightarrow 0 \quad .$$

Use **these sequences,** and apply Lemma 10 and 11.

4. Sketch of Proof of Theorem 3.

Let A be a Noetherian local ring of $\dim A = d > 0$ and m the maximal ideal of A . We say that A has finite local cohomology if the local cohomology $H^i_m(A)$ of A are finitely generated (i.e., the length $l_A(H^i_m(A))$ are finite) for all $i \neq d$. Let $a_1, a_2, \ldots, a_d$ be a system of parameters for A and put $q = (a_1, \ldots, a_d)$. We define that

$$I(a_1, \ldots, a_d; A) = l_A(A/q) - e_A(q)$$

, where $l_A(A/q)$ and $e_A(q)$ denote the length of an A-module A/q and the multiplicity of A relative to q , respectively.

Lemma 12. The following conditions are equivalent.

(1) $a_1, a_2, \ldots, a_d$ is an unconditioned strong d-sequence.

(2) $I(a_1, \ldots, a_d; A) = I(a_1^2, \ldots, a_d^2; A)$.

(3) A has finite local cohomology and

$$I(a_1, \dots , a_d; A) = \sum_{i=0}^{d-1} \binom{d-1}{i} . l_A(H_m^i(A)) .$$

Let G denote the associated graded ring $G_q(A) = \bigoplus_{n \geq 0} q^n/q^{n+1}$ of q and h_i the initial form of a_i $(1 \leq i \leq d)$.

Lemma 13. $h_1, h_2, \dots , h_d$ is an unconditioned strong d-sequence if and only if so is $a_1, a_2, \dots , a_d$.

In the 4th Symposium on Commutative Algebra at Karuizawa in Japan (Nov. 3-6, 1982), N. V. Trung introduces a standard system of parameters for an A-module E which is a system $a_1, a_2, \dots , a_d$ of parameters for E satisfying the same condition as (3) (and (2)) of Lemma 12, i.e., $H_m^i(E)$ is finitely generated for all $i \neq \dim_A E$ and $l_A(E/(a_1, \dots , a_d).E) - e_E(a_1, \dots , a_d) = \sum_{i=0}^{d-1} \binom{d-1}{i} . l_A(H_m^i(E))$, where $d = \dim_A E$. Hence a system of paramaters for A is an unconditioned strong d-sequence if and only if it is standard ([19]).

Let us recall that A is called a quasi-Buchsbaum ring if $m. H_m^i(A) = (0)$ for all $i \neq d$. This is equivalent saying that at least one (**and hence every**) system of parameters for A in m^2 is a weak-sequence (By a weak-sequence we mean a weak-sequence with respect to the maximal ideal of the local ring A), see ([17]).

Proposition 14 ([5], also [19]). The following two conditions are equivalent.

(1) $G_{\underline{M}}$ is a Buchsbaum ring, where $\underline{M} = m.G + G_+$.

(2) $a_1, a_2, \dots , a_d$ is an unconditioned strong weak-sequence.

In this case, A is a quasi-Buchsbaum ring and $I(G_{\underline{M}}) = I(A)$.

Theorem 15. Let A be a quasi-Buchsbaum ring. Then the following two conditions are equivalent.

(1) $G_{\underline{M}}$ is a quasi-Buchsbaum ring with $I(G_{\underline{M}}) = I(A)$.

(2) The equality

$$(a_1{}^2, \dots , a_d{}^2) \cap q^n = \sum_{i=1}^{d} a_i{}^2 . q^{n-2}$$

holds for all $n \in \mathbb{Z}$.

Lemma 16. Suppose that depth $A > 0$ and Proj $R_q(A)$ is Cohen-Macaulay. If $a_1, a_2, \dots , a_{d-1}$ is an unconditioned strong d-sequence modulo $a_d{}^n A$ for almost all $n > 0$, then the equality

$$[(a_1, \dots , a_{d-1}): a_d{}^2] = [(a_1, \dots , a_{d-1}): a_d]$$

holds.

Proposition 17 ([6]). Suppose that depth $A > 0$. Then the following two conditions are equivalent.

(1) $a_1, a_2, \dots , a_d$ is an unconditioned strong d-sequence.

(2) Proj $R_{(a_1{}^{n_1}, \dots , a_d{}^{n_d})}(A)$ is Cohen-Macaulay for all $n_1, \dots , n_d > 0$.

Proof of Theorem 3. The equivalence of (1) and (2) comes at once from Proposition 14. (3) $\Longrightarrow$ (1) follows from Proposition 17 and the converse is proved by Surjective Criterion in [13] and [15].

Finally we also have the following

Proposition 18 ([5]). If $a_1, a_2, \dots , a_d$ is an unconditioned strong d-sequence, then

$$e_A(q) \geqq \sum_{i=1}^{d-1} \binom{d-1}{i-1} . l_A(H_m^i(A)) \quad .$$

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