数理解析研究所講究録
第 485 巻 1983 年 88-111

Pointwise Completeness and Controllability by Linear Delay Feedback

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1. Introduction

Let a control system be described by

$$\frac{d}{dt}x(t) = Ax(t) + u(t)$$

on a Banach space X. We study the controllability of the system when the control $u(t)$ is given by a sum of delay feedbacks

$$u(t) = \sum_{r=1}^{m} A_r x(t-h_r), \qquad 0 < h_1 < \cdots < h_m.$$

This controllability leads to the following problem:

For the delay system

$$\begin{aligned} &\frac{d}{dt}x(t) = Ax(t) + \sum_{r=1}^{m} A_r x(t-h_r), \\ &x(0) = x_0, \qquad x(s) = g(s) \qquad s \in [-h_m, 0), \end{aligned} \tag{1.1}$$

does the reachable set (with respect to initial value x_0 and initial function g) fill X or become a proper subset of X ?

This problem is called pointwise completeness or pointwise degeneracy. The problem was first proposed by Weiss [18] in his study of controllability for retarded systems in Euclidean spaces.

In case of $X = R^n$, the pointwise completeness was investigated by several authors Brooks and Schmidt [3], Zmood and MaClamroch [19] and Zverkin [20] for autonomous and nonautonomous single delayed systems.

It was Popov [13] who gave an elegant answer of pointwise degeneracy for autonomous single delayed systems in R^n. His result was extended by Asner and Halanay [1] for autonomous systems with multiple commensurable delays. For such systems Charrier and Haugazeau [5] gave another extensions which depend on linear operator theory. Kappel [11] also obtained similar results for general nonautonomous retarded systems and gave a further analysis of systems with commensurable delays. For systems of neutral type in R^n, the problem was solved by Choudhury [6] and Asner and Halanay [2].

The only one paper which studies the problem in infinite dimensional space is Charrier [4]. The paper gives an example of pointwise degenerate system in a Hilbert space but does not give any detailed study as in [1-3,5,6,11,13, 19].

The purpose of this paper is to develop a general theory for pointwise completeness and degeneracy of (1.1) in infinite dimensional (Banach) spaces. We employ the delay system (1.1) studied by Datko [8]. In Section 2, we give a definition of exact and approximate pointwise completeness by taking into account of X being infinite dimensional. A necessary and sufficient condition and a negative result for exact pointwise completeness are establishe in Section 3. Section 4 studies approximate pointwise completeness and pointwise degeneracy as a complementary concept. A main theorem is contained in Section 4. In Section 5 we specify the results of Section 4 to the systems with commensurable delays. In specifying such results the representation of fundamental solution of (1.1) given by Nakagiri [14] is effectively used.

2. System Description and Definition

Let X be a reflexive Banach space with norm $\|\cdot\|$. Consider the differential system with multiple delays

$$S: \quad \frac{d}{dt}x(t) = Ax(t) + \sum_{r=1}^{m} A_r x(t-h_r), \qquad t > 0, \tag{2.1}$$

$$x(0) = x_0, \qquad x(s) = g(s) \quad s \in [-h, 0). \tag{2.2}$$

Here we assume that $0 < h_1 < \cdots < h_m = h$ are positive constants, $x(t)$, $g(t) \in X$, operators A_r $(r = 1, \cdots, m)$ are bounded on X and A generates a strongly continuous semi-group $T(t)$, $t \geq 0$ on X.

In the system S, $x_0 \in X$ and g are called an initial value and an initial function , respectively.

Under the above assumptions, R. Datko [8] has constructed the fundamental solution $G(t)$ of the system S by the delay perturbation of $T(t)$. That is, $G(t)$ satisfies the following relations:

i) $G(t) = O$ (the null operator on X) if $t < 0$. (2.3)

ii) $G(t)$ is strongly continuous on R^+ and satisfies

$$G(t) = T(t) + \sum_{r=1}^{m} \int_0^t T(t-s)A_r G(s-h_r)ds \qquad \text{if} \quad t \geq 0. \tag{2.4}$$

Let $x_0 \in X$ and $g \in L_p(-h,0;\ X)$, $p \in [1,\infty]$. Then the function

$$x(t) = G(t)x_0 + \sum_{r=1}^{m} \int_{-h_r}^{0} G(t-h_r-s)A_r g(s)ds \tag{2.5}$$

makes sense, the integrals being Bochner integrals in X, and is strongly continuous on R^+. Since $x(t)$ depends on x_0 and g, we denote this by $x(t;\ x_0,g)$. It is proved in [8] that $x(t;\ x_0,g)$ satisfies the integrated form of (2.1) and (2.2), i.e.,

$$x(t;\ x_0,g) = T(t)x_0 + \int_0^t \{T(t-s) \sum_{r=1}^{m} A_r x(s-h_r;\ x_0,g)\}ds$$

$$\text{if} \quad t \geq 0. \qquad (2.6)$$

In this sense this function $x(t;\ x_0,g)$ is called the mild solution of S.

In what follows we study pointwise completeness and pointwise degeneracy by means of the mild solutions.

To give a definition of pointwise completeness, the following set of reachability is needed.

$$R_t(L_p) = \{\ x \in X : x = x(t;\ x_0,g) \quad \text{where} \quad x_0 \in X \quad \text{and} \quad g \in L_p(-h,0;\ X)\}.$$

DEFINITION 2.1. The system S is said to be

(i) L_p-exactly pointwise complete at time t if $R_t(L_p) = X$;

(ii) L_p pointwise complete at time t if $\overline{R_t(L_p)} = X$.

3. Exact Pointwise Completeness

In this section we study exact pointwise completeness.

For Banach spaces X, W and a densely defined linear operator $L : D(L) \subset W \to X$, we denote their dual Banach spaces by X^*, W^* and its adjoint operator by L^*, respectively. The following abstract result is used to derive an equivalent condition for exact pointwise completeness.

Lemma 3.1. Let X and W be reflexive Banach spaces and let L be a bounded linear operator from W into X. Then the image of L fills X if and only if there exists $K > 0$ such that

$$\|x^*\|_{X^*} \leq K \|L^* x^*\|_{W^*} \qquad \text{for all} \quad x^* \in X^*. \qquad (3.1)$$

This lemma follows from inverse mapping theorem [15,p.83] (see also [7]).

Lemma 3.2. $G^*(t) = G(t)^*$ is strongly continuous on R^+.

Proof. Since X is reflexive, the weak topology of X coincides with the weak* topology. Then $T^*(t) = T(t)^*$ is weakly continuous on R^+. From the property $T^*(t+s) = T^*(t)T^*(s)$ and a well known result [10,p.306] that $T^*(t)$ is strongly continuous on R^+. By (2.3)

$$G^*(t) = T^*(t) \quad \text{is strongly continuous on} \quad [0, h_1],$$

so that

$$G^*(t) = T^*(t) + \int_0^t T^*(s-h_1)A_r^*T^*(t-s)\,ds$$

is also strongly continuous on $[h_1, h_2]$. Hence $G^*(t)$ is strongly continuous on $[0, h_2]$. Continuing this procedure it is verified by (2.4) that $G^*(t)$ is strongly continuous on R^+.

Now we can give an equivalent condition for exact pointwise completeness.

THEOREM 3.1. Let $p \in (1,\infty)$. Then the system S is L_p-exactly pointwise complete at time t if and only if there exists $K_t > 0$ such that

$$\| x^* \|_{X^*} \le K_t \operatorname{Max} \{ \| G^*(t)x^* \|_{X^*}, \ (\int_{-h}^{0} \| F_t^*(s)x^* \|_{X^*}^q ds)^{1/q} \} \quad \text{for all} \quad x^* \in X^*, \tag{3.2}$$

where $1/p + 1/q = 1$. Here the operator $F_t^*(s)$ is given by

$$F_t^*(s) = \sum_{r=1}^{m} \chi_r A_r^* G^*(t-h_r-s) \qquad \text{for all} \quad s \in [-h, 0] \tag{3.3}$$

and χ_r is the characteristic function of $[-h_r, 0]$ $(r = 1,\cdots, m)$.

Proof. Let W be the direct sum of X and $L_p(-h,0; X)$ whose norm $\|\cdot\|_W$

is introduced by

$$\| (x,g) \|_W = \| x \| + \| g(\cdot) \|_{L_p(-h,0;\ X)}.$$

We denote this Banach space W by $X \oplus L_p(-h,0;\ X)$. Let the operator $L_t : X \oplus L_p(-h,0;\ X) \to X$ be given by

$$L_t(x,g) = G(t)x + G_t g(\cdot) \qquad \text{for} \quad (x,g) \in X \oplus L_p(-h,0;\ X). \tag{3.4}$$

Here $G_t : L_p(-h,0;\ X) \to X$ is given by

$$G_t g(\cdot) = \sum_{r=1}^{m} \int_{-h_r}^{0} G(t-h_r-s)A_r g(s)ds \qquad \text{for} \quad g \in L_p(-h,0;\ X). \tag{3.5}$$

It is evident that L_t is linear and bounded. By (3.4) we have

$$L_t(X \oplus L_p(-h,0;\ X)) = R_t(L_p).$$

Then L_p-exact pointwise completeness at time t is equivalent to

$$L_t(X \oplus L_p(-h,0;\ X)) = X.$$

Since $p \in (1,\infty)$, $W = X \oplus L_p(-h,0;\ X)$ is reflexive and W^* is represented by $W^* = X^* \oplus L_q(-h,0;\ X^*)$ $(1/p + 1/q = 1)$ whose norm $\| \cdot \|_{W^*}$ is given by

$$\| (x^*,g^*) \|_{W^*} = \mathrm{Max}\ \{ \| x^* \|_{X^*},\ \| g^*(\cdot) \|_{L_q(-h,0;\ X^*)} \}. \tag{3.6}$$

To apply Lemma 3.1, we shall calculate L_t^*.

For $x^* \in X^*$ we have

$$\begin{aligned} \langle L_t(x,g),x^* \rangle &= \langle G(t)x,x^* \rangle + \langle \sum_{r=1}^{m} \int_{-h_r}^{0} G(t-h_r-s)A_r g(s)ds,x^* \rangle \\ &= \langle x,G^*(t)x^* \rangle + \int_{-h}^{0} \langle g(s),F_t^*(s)x^* \rangle ds \end{aligned}$$

$$= \langle (x,g), L_t^* x^* \rangle_{W,W^*} ,$$

where $F_t^*(s)$ is given by (3.3). Hence $L_t^* x^*$ is expressed by $(G^*(t)x^*, G_t^* x^*)$ and $G_t^* x^*$ is given by

$$\langle g, G_t^* x^* \rangle_{L_p(-h,0;\ X), L_q(-h,0;\ X^*)} = \int_{-h}^{0} \langle g(s), F_t^*(s) x^* \rangle ds$$

for each $g \in L_p(-h,0;\ X)$.

Since $G^*(t)$ is strongly continuous (Lemma 3.2), $F_t^*(s)$ is strongly continuous on $[-h, 0]$ except for $s = h_r$, $r = 1, \cdots, m-1$. Then it follows by (3.6) that

$$\| L_t^* x^* \| = \operatorname{Max} \{ \| G^*(t) x^* \| ,\ (\int_{-h}^{0} \| F_t^*(s) x^* \|_{X^*}^q ds)^{1/q} \} < \infty. \tag{3.7}$$

Then applying Lemma 3.1 with $W = X \oplus L_p(-h,0;\ X)$ and $L = L_t$, we obtain condition (3.3) for L_p-exact pointwise completeness from (3.7).

Since $G(t) = T(t)$ for $t \in [0, h_1]$, we have the following corollary.

<u>COROLLARY 3.1.</u> If $T(t)$ is a strongly continuous group, then S is L_p-exactly pointwise complete at any time $t \in [0, h_1]$ for each $p \in (1,\infty)$.

Next we are concerned with a negative result for exact pointwise completeness. Such a fact for mild solutions in non-delay systems is first pointed out by Kuperman and Lepin [12] and more detailed researches are given by Triggiani [16,17]. In these works some types of compactness of operators are assumed to show such a fact called the lack of exact controllability. We shall show that a similar situation, which is called the lack of exact pointwise completeness, can be viewed for our delay system S.

Lemma 3.3. Let $p \in (1,\infty]$ and $T(t)$ be compact for all $t > 0$. Then $G(t)$ and G_t are compact for all $t > 0$.

Proof. It is easily verified by (2.4) that $T(t)$ is compact for all $t > 0$ if and only if $G(t)$ is compact for all $t > 0$. By (3.5) and changes of integral variables the compactness of G_t follows from those of $G_t^r : L_p(0, h_r; X) \to X$ given by

$$G_t^r g(\cdot) = \int_0^{h_r} G(t-s)A_r g(s)ds \qquad \text{for } g \in L_p(0, h_r; X) \tag{3.8}$$

for each $t > 0$ and all $r = 1,\cdots, m$. It is not difficult to prove the compactness of G_t^r. We remark that this lemma does not hold if $p = 1$.

THEOREM 3.2. Let X be infinite dimensional and let $p \in (1,\infty]$. If $T(t)$ is compact for all $t > 0$, then S is never L_p-exact pointwise complete at any time $t > 0$.

Proof. We shall prove this theorem by Baire category theorem as in Triggiani [16,17]. Let $R_{nm} = L_n(S_m)$, where S_m is the closed ball in $X \oplus L_p(-h,0; X)$ of radius m with center the origin $(0,0)$. Since $L_t(x,g) = G(t)x + G_t g(\cdot)$, L_t is compact for all $t > 0$ by Lemma 3.3. Hence $\overline{R_{nm}}$ is compact in X for each $n, m = 1,2,\cdots$. Since X is infinite dimensional, $\overline{R_{nm}}$ cannot contain any open ball and hence is nowhere dense in X. This implies by Baire category theorem [15,p.80] that

$$X - \cup\{ \overline{R_{nm}} : n, m = 1,2,\cdots \} \neq \phi.$$

Since

$$\bigcup_{t>0} R_t(L_p) = \bigcup_{n=1} \bigcup_{m=1} R_{nm} \subset \cup \{ \overline{R_{nm}} : n, m = 1,2,\cdots \},$$

S is never L_p-exactly pointwise complete at any time $t > 0$.

4. Pointwise Completeness and Pointwise Degeneracy

It follows by Definition 2.1 and Hahn-Banach theorem that S is not L_p pointwise complete at time t if and only if there exists $x^* \neq 0$ in X^* such that $x^* \in R_t(L_p)^{\perp}$, i.e.,

$$\langle x,x^* \rangle = 0 \qquad \text{for all } x \in R_t(L_p).$$

In this case S is said to be L_p pointwise degenerate at time t with respect to x^*. If S is L_p pointwise degenerate at time t with respect to every $x^* \in E^*$ for $E^* \subset X^*$, S is called L_p pointwise degenerate at time t with respect to E^*.

The following lemma is fundamental in the arguments below.

<u>Lemma 4.1.</u> Let X and W be Banach spaces and let L be a densely defined linear operator from W into X. Then

$$\text{Ker } L^* = (\text{Range } L)^{\perp}.$$

Especially, $\overline{(\text{Range } L)} = X$ if and only if $\text{Ker } L^* = \{0\}$ in X^*.

By lemma 4.1, we obtain the following result.

<u>THEOREM 4.1.</u> Let $p \in [1,\infty)$. The system S is L_p pointwise degenerate at time $t > 0$ with respect to E^* if and only if

$$E^* \subset \text{Ker } G^*(t) \cap \{ \cap \{ \text{Ker } F_t^*(s) : s \in (-h, 0)\backslash\{ -h_1,\cdots, -h_m\}\} \}. \qquad (4.1)$$

Moreover, the system S is L_p pointwise complete at time $t > 0$ if and only if

$$\{0\} = \text{Ker } G^*(t) \cap \{ \cap \{ \text{Ker } F_t^*(s) : s \in (-h, 0)\backslash\{ -h_1,\cdots, -h_m\}\} \}. \qquad (4.2)$$

Proof. By Lemma 4.1, the system S is L_p pointwise degenerate at time t with respect to E^* if and only if

$$E^* \subset R_t(L_p)^{\perp} = (\text{Range } L_t)^{\perp} = \text{Ker } L_t^*.$$

Then by (3.6) it follows that

$$G^*(t)x^* = 0 \quad \text{in} \quad X^* \qquad \text{for all} \quad x^* \in E^*$$

and $$G_t^* x^* = 0 \quad \text{in} \quad L_q(-h,0;\ X^*) \qquad \text{for all} \quad x^* \in E^*,$$

where $1/p + 1/q = 1$ and $q \neq 1$. Hence if $x^* \in E^*$,

$$\langle g, G_t^* x^* \rangle_{L_p(-h,0;\ X),\ L_q(-h,0;\ X^*)} = \int_{-h}^{0} \langle g(s), F_t^*(s)x^* \rangle ds = 0$$

for all $g \in L_p(-h,0;\ X)$.

This implies

$$F_t^*(s)x^* = 0 \quad \text{in} \quad X^* \qquad \text{for a.e.} \quad s \in [-h,\ 0]. \qquad (4.3$$

Since $F_t^*(s)$ is strongly continuous on $[-h,\ 0]$ except for $s = h_i$, $i = 1, \cdots m-1$, we have by (4.3) that

$$F_t^*(s)x^* = 0 \quad \text{in} \quad X^* \qquad \text{for all} \quad s \in (-h,\ 0) \setminus \{-h_1, \cdots, -h_m\}.$$

Thus x^* belongs to the left hand side of (4.1).

The latter part of this theorem will be clear.

The condition (4.1) is equivalent to that for all $x^* \in E^*$,

$$G^*(t)x^* = 0 \quad \text{in} \quad X^* \quad \text{and} \quad F_t^*(s)x^* = 0 \quad \text{in} \quad X^*$$
$$\text{for all} \quad s \in (-h,\ 0) \setminus \{-h_1, \cdots, -h_m\} \qquad (4.4)$$

Since (4.1) and (4.2) do not depend on the space of initial functions $L_p(-h,0;\ X)$, we omit L_p in the terminology of L_p pointwise completeness and L_p pointwise degeneracy.

We now write the second condition in (4.4) by

$$\langle x, F_t^*(s)x^* \rangle = 0 \quad \text{for all} \quad x \in X \quad \text{and all} \quad s \in (-h_{j+1}, -h_j)$$

$$j = 0,1,\cdots, m-1. \qquad (4.5)$$

It then follows by (3.3) and changes of integral variables that (4.5) is equivalent to

$$\langle \sum_{r=j+1}^{m} G(t+s-h_r)A_r x, x^* \rangle = 0 \quad \text{for all} \quad x \in X \quad \text{and all} \quad s \in [h_j, h_{j+1}],$$

$$j = 0,1,\cdots, m-1. \qquad (4.6)$$

Lemma 4.2. The fundamental solution $G(t)$ of S satisfies the following relation:

$$G(t+s) = G(t)G(s) + \sum_{r=1}^{m} \int_{-h_r}^{0} G(t-\sigma-h_r)A_r G(\sigma+s)\,d\sigma \quad \text{for all} \quad s,\ t \geq 0. \qquad (4.7)$$

Proof. Since this lemma follows easily by direct substitutions, we omit its proof.

The following lemma is due to R. Datko [9,Lemma 2.4].

Lemma 4.3. If $x \in D(A)$, then $G(t)x$ is strongly differentiable for almost everywhere on R^+ and satisfies

$$\frac{d}{dt} G(t)x = AG(t)x + \sum_{r=1}^{m} A_r G(t-h_r)x$$

$$= G(t)Ax + \sum_{r=1}^{m} G(t-h_r)A_r x \quad \text{for a.e.} \quad t \in R^+. \qquad (4.8)$$

The following theorem gives an infinite dimensional version of the results by Zmood and MaClamroch [19,Theorem 2] and Kappel [11,Theorem 2.1].

THEOREM 4.2. The system S is pointwise degenerate at time $t > 0$ with respect to E^* if and only if

$$E^* \subset \bigcap_{s \geq t} \operatorname{Ker} G^*(s). \tag{4.9}$$

Moreover, the system S is pointwise complete at time $t > 0$ if and only if

$$\{0\} = \bigcap_{s \geq t} \operatorname{Ker} G^*(s). \tag{4.10}$$

Proof. It is sufficient to prove this theorem that

$$x^* \in \bigcap_{s \geq t} \operatorname{Ker} G^*(s) \tag{4.11}$$

if and only if

$$x^* \in \operatorname{Ker} G^*(t) \cap \{ \cap \{ \operatorname{Ker} F_t^*(s) : s \in (-h, 0)\setminus\{-h_1, \cdots, -h_m\}\} \}. \tag{4.12}$$

The condition (4.11) is equivalent to that

$$\langle x, G^*(s)x^* \rangle = \langle G(s)x, x^* \rangle = 0 \quad \text{for all } x \in X \text{ and all } s \geq t. \tag{4.13}$$

First we shall show that (4.12) implies (4.11). Let (4.12) be satisfied. Then by Lemma 4.2, we have

$$\langle G(t+s)x, x^* \rangle = \langle G(t)G(s)x, x^* \rangle + \int_{-h}^{0} \langle F_t(\sigma)G(\sigma+s)x, x^* \rangle d\sigma$$

$$= \int_{-h}^{0} \langle G(\sigma+s)x, F_t^*(\sigma)x^* \rangle d\sigma = 0 \quad \text{for all } x \in X \text{ and } s \geq 0.$$

This shows (4.11).

Conversely, let (4.11) be satisfied. Then by Lemma 4.2 and (4.13), we have

$$f(s,x) = \sum_{r=1}^{m} \int_{-h_r}^{0} \langle G(t-\sigma-h_r)A_r G(\sigma+s)x, x^*\rangle d\sigma$$

$$= \langle G(s+t)x, x^*\rangle - \langle G(t)G(s)x, x^*\rangle$$

$$= 0 \qquad \text{for all } x \in X \text{ and all } s \geq 0. \tag{4.14}$$

If $x \in D(A)$, by Lemma 4.3 we can differentiate $f(s,x)$ for a.e. $s \in R^+$ and obtain that

$$f'(s,x) = \sum_{r=1}^{m} \langle G(t+s-h_r)A_r x, x^*\rangle$$

$$+ \sum_{r=1}^{m} \int_{-h_r}^{0} \langle G(t-\sigma-h_r)A_r G(s+\sigma)Ax, x^*\rangle d\sigma$$

$$+ \sum_{r=1}^{m} \int_{-h_r}^{0} \langle G(t-\sigma-h_r)A_r (\sum_{j=1}^{m} G(s+\sigma-h_j)A_j)x, x^*\rangle d\sigma$$

$$= I_1(s) + I_2(s) + I_3(s) = 0 \qquad \text{for a.e. } s \in R^+. \tag{4.15}$$

Since $I_2(s) = f(s,Ax)$, we see from (4.14) that

$$I_2(s) = 0 \qquad \text{for all } s \geq 0.$$

$I_3(s)$ can be decomposed as

$$I_3(s) = \sum_{j=1}^{m} I_{3,j}(s) = \sum_{j=1}^{m} (\sum_{r=1}^{m} \int_{-h_r}^{0} \langle G(t-\sigma-h_r)A_r G(s+\sigma-h_j)A_j x, x^*\rangle d\sigma).$$

If $0 \leq s < h_j$, $G(s+\sigma-h_j) = 0$ for each $\sigma \in [-h, 0]$ so that

$$I_{3,j}(s) = 0 \qquad \text{for } 0 \leq s < h_j.$$

If $s \geq h_j$, we use Lemma 4.2 again and obtain from (4.13) that

$$I_{3,j}(s) = \langle G(t+s-h_j)A_jx, x^*\rangle - \langle G(t)G(s-h_j)A_jx, x^*\rangle = 0.$$

Hence $I_{3,j}(s) = 0$ for all $s \geq 0$ and $j = 1, \cdots, m$. Then by (4.5), $I_1(s) = 0$ for a.e. $s \in R^+$. Since $I_1(s)$ is piecewise continuous and $t+s-h_r \geq t$ if $s \geq h_r$, it follows by (4.13) that

$$I_1(s) = \sum_{r=j+1}^{m} \langle G(t+s-h_r)A_rx, x^*\rangle = 0 \qquad \text{for} \quad s \in [h_j, h_{j+1}].$$

Since $D(A)$ is dense in X, (4.6) holds. This proves that (4.12) implies (4.11) and completes the proof.

It follows from Theorem 4.2 that S is pointwise degenerate at all $t \geq t_0$ with respect to E^* if S is pointwise degenerate at time t_0 with respect to E^*.

By Theorem 4.2, we shall call $\bigcap_{s\geq t} \operatorname{Ker} G^*(s)$ the degenerate space of S at time t. It is clear that the degenerate space is a closed subspace of X^*.

<u>Example 4.1.</u> Let $X = L_2(0, 1)$. We define the semi-group $T(t)$ by

$$T(t)f = g\ ; \qquad g(s) = \begin{cases} f(s+t) & \text{if} \quad 0 \leq s+t \leq 1 \\ 0 & \text{if} \quad s+t > 1. \end{cases}$$

It is easy to verify that $T(t)$ is a strongly continuous semi-group on $L_2(0, 1)$, $T(t) = 0$ for all $t > 1$ and its infinitesimal generator $A = d/dt$.

We now consider the delay system

$$\frac{d}{dt}x(t) = Ax(t) + A_1x(t-1). \tag{4.16}$$

If $A_1 = I$ (the identity operator on X), then the system (4.16) is pointwise complete at all $t \geq 0$. Next consider the case where $A_1 = T(1/2)$. In this case the system (4.16) is pointwise degenerate at all $t > 0$ and the degenerate space of the system at time $t > 0$ is given by

$$\begin{cases} \{ g^* \in L_2(0, 1): g^* = \chi_{[1-t,\ 1]}x \quad \text{for} \quad x \in L_2(0, 1)\} & \text{if} \quad 0 < t \le 1/2 \\ \{ g^* \in L_2(0, 1): g^* = \chi_{[1/2,\ 1]}x \quad \text{for} \quad x \in L_2(0, 1)\} & \text{if} \quad 1/2 \le t \le 1 \\ \{ g^* \in L_2(0, 1): g^* = \chi_{[\frac{3}{2}-t,\ 1]}x \quad \text{for} \quad x \in L_2(0, 1)\} & \text{if} \quad 1 \le t < 3/2 \\ X^* \quad \text{if} \quad t \ge 3/2 , & \end{cases}$$

where χ_I is the characteristic function of I. Note that the degenerate space is infinite dimensional for all $t > 0$.

Example 4.2. (Extended Charrier's example [4]) We consider the single delay system (4.16) on a general Hilbert space X. The inner product and the norm are denoted by $< , >_X$ and $\| \ \|_X$, respectively. A in (4.16) is assumed to generate a semi-group $T(t)$ on X. Let $E^* = \overline{sp} \{ x_1^*, \cdots, x_n^* \}$ and $\{ x_1^*, \cdots, x_n^* \} \subset D(A^*)$. We assume that there exists a set $\{ y_1, \cdots, y_n\} \subset D(A)$ such that

$$<y_k, x_j^*>_X = \delta_{k,j}, \quad <T(1)y_k, x_j^*>_X = 0 \quad \text{and} \quad <T(1)Ay_k, x_j^*>_X = 0$$
$$\text{for all} \quad k, j = 1, \cdots, n.$$

Then if A_1 is given by

$$A_1 x = \sum_{k=1}^{n} \{ <T(1)x, x_k^*>_X Ay_k - <T(1)x, A^*x_k^*>_X y_k \} ,$$

the system (4.16) is pointwise degenerate at time 2 with respect to E^*.

It is possible to extend this example to the case where E^* is spanned by infinitely many elements $x_1^*, \cdots, x_n^*, \cdots$ in X^*.

5. Systems with Commensurable Delays

In this section we consider the system S with commensurable delays $h_r = r\tau$, $r = 1,\cdots, m$, $\tau > 0$.

To give a useful formula of $G(t)$ in this special system, we need some preparation. First we define the index sets $\Lambda(j,k)$ for all $j, k = 1,2,\cdots$ by

$$\Lambda(j,k) = \{ (i_1,\cdots, i_j): 1 \le i_1,\cdots, i_j \le m \text{ and } i_1+ \cdots + i_j = k \}.$$

Next we define the operators $T_k(t)$, $k = 1,2,\cdots$, by

$$T_1(t) = T(t)$$

$$T_k(t) = \sum_{j=1}^{k-1} \sum_{\Lambda(j,k-1)} \int_0^t T(t-s_{j-1})A_{i_1} \cdots\cdots \int_0^{s_1} T(s_1-s)A_{i_j} T(s)ds ds_1 \cdots ds_{j-1}, \quad k \geq 2. \tag{5.1}$$

Then $T_2(t) = \int_0^t T(t-s)A_1T(s)ds$, for example.

For each natural number i, we define the matrix of operators $\mathcal{T}_i$ by

$$\mathcal{T}_i(t) = \begin{pmatrix} T(t) & 0 & \cdots & 0 \\ T_2(t) & T(t) & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ T_i(t) & \cdots & T_2(t) & T(t) \end{pmatrix} \tag{5.2}$$

where $T_i(t)$ are given by (5.1).

We denote the transpose of i direct sum of X, $(X \oplus X \oplus \cdots \oplus X)^t$, by X^i.

<u>Lemma 5.1.</u> $\mathcal{T}_i(t)$ is a strongly continuous semigroup on X^i such that its infinitesimal generator $\mathcal{A}_i$ is represented by

$$
\mathcal{A}_i = \begin{pmatrix}
A & O & \cdots & \cdots & \cdots & O \\
A_1 & A & O & & & \vdots \\
\vdots & \ddots & \ddots & \ddots & & \vdots \\
\vdots & & \ddots & \ddots & O & \vdots \\
A_m & A_{m-1} & \cdots & A & \ddots & \vdots \\
O & \ddots & & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & & \ddots & O \\
O \cdots & O & A_m & A_{m-1} \cdots & A_1 & A
\end{pmatrix} \tag{5.3}
$$

on $D(A)^i = (D(A) \oplus D(A) \oplus \cdots \oplus D(A))^t$.

Lemma 5.1 says that $D(\mathcal{A}_i) = D(A)^i$. But we can not give such representations of $D(\mathcal{A}_i^k)$ in terms of the domains of operators A^k, $A^{k-1}A_r$ etc. for $k \geq 2$.

We define $D_r^k \subset X$ by $D_r^k = \bigcap_{j=1}^{k-1} D(A^{k-j}A_rA^{j-1})$ for $r = 1,\cdots, m$ and $k = 1,2,\cdots$. Then we have the following lemma.

<u>Lemma 5.2.</u>

$$
D(\mathcal{A}_i^k) \supset \begin{pmatrix} D(A^k) \cap D_1^k \cap \cdots \cap D_{\min(i,m)}^k \\ D(\mathcal{A}_{i-1}^k) \end{pmatrix}
\supset \begin{pmatrix} D(A^k) \cap D_1^k \cap \cdots \cap D_{\min(i,m)}^k \\ \vdots \\ D(A^k) \cap D_1^k \\ D(A^k) \end{pmatrix} . \tag{5.4}
$$

Since the proofs of Lemma 5.1 and Lemma 5.2 are complicated and tiresome, we omit the proofs. The following lemma is an easy consequence of (5.1) and (5.2).

<u>Lemma 5.3.</u> If $T(t)$ is analytic, then $\mathcal{T}_i(t)$ is also analytic for each $i = 1,2,\cdots$.

Now we define Z_i inductively by

$$Z_1 = I \quad \text{and} \quad Z_i = \begin{pmatrix} I \\ T_{i-1}(\tau) Z_{i-1} \end{pmatrix} \quad \text{for } i \geq 2. \tag{5.5}$$

It is shown in Nakagiri [14] that $G(t)$ is represented by

$$G(t) = I_i T_i (t-(i-1)\tau) Z_i \quad \text{when} \quad t \in [(i-1)\tau, i\tau), \tag{5.6}$$

where $I_i = [0, \cdots, 0, I]$.

Let X_i be the largest subspace of X such that $Z_i X_i \subset \bigcap_{n=0}^{\infty} D(A_i^n)$. It then follows by Lemma 5.2 that if

$$\bigcap_{k=0}^{\infty} (D(A^k) \cap D_1^k \cap \cdots \cap D_m^k)$$

is dense in X, then X_i is also dense in X.

The following theorem is a consequence from Theorem 4.2 and (5.6).

<u>THEOREM 5.1.</u> The system S is pointwise degenerate at time $t_0 \in [(k-1)\tau, k\tau$ with respect to E^* if

$$E^* \subset \bigcap_{i=k}^{\infty} \bigcap_{n=0}^{\infty} \operatorname{Ker} Z_i^* (A_i^*)^n I_i^*. \tag{5.7}$$

Moreover, the system S is pointwise complete at time $t_0 \in [(k-1)\tau, k\tau)$ if

$$\bigcap_{i=k}^{\infty} \bigcap_{n=0}^{\infty} \operatorname{Ker} Z_i^* (A_i^*)^n I_i^* = \{0\}. \tag{5.8}$$

Especially for the pointwise completeness, we obtain the next theorem.

<u>THEOREM 5.2.</u> The system S is pointwise complete at time $t_0 \in [(k-1)\tau, k\tau)$ if

$$X = \overline{\text{sp}} \{ I_i A_i^n Z_i X_i : i = k, k+1, \cdots, n = 0,1,2,\cdots \} \tag{5.9}$$

or, more generally, if

$$X = \overline{sp}\ \{\ I_i A_i^n T_i(s_i) Z_i X_i\ :\ i = k,\ k+1, \cdots,\ n = 0,1,2,\cdots\ \},$$
$$s_i \text{ arbitrary in } J_i, \qquad (5.10)$$

where $J_k = [t_0-(k-1)\tau,\ \tau)$ and $J_i = [0,\ \tau)$ for $i \geq k+1$.

Conversely if $T(t)$ is analytic, X_i is dense in X for each $i = k, k+1, \cdots$ and the system S is pointwise complete at time $t_0 \in [(k-1)\tau, k\tau)$, then

$$X = \overline{sp}\ \{\ I_i A_i^n T_i(s_i) Z_i X_i\ :\ i = k,\ k+1, \cdots,\ n = 0,1,2,\cdots\ \},$$
$$s_i \text{ arbitrary in } J_i-\{0\}. \qquad (5.11)$$

Remark 5.1. The condition (5.7) ((5.9)) is not necessary for pointwise degeneracy (pointwise completeness) in general. But if $X_i = X$ for all $i = k, k+1,\cdots$, i.e., A is bounded, (5.7) ((5.9)) is necessary and sufficient.

<u>COROLLARY 5.1.</u> Let A be bounded. Then S is pointwise degenerate at time $t_0 \in [(k-1)\tau, k\tau)$ with respect to E^* if and only if (5.7) holds or

$$E^* \subset (\ \overline{sp}\ \{\ I_i A_i^n Z_i X\ :\ i = k,\ k+1,\cdots,\ n = 0,1,2,\cdots\ \})^{\perp}. \qquad (5.12)$$

Furthermore, S is pointwise complete at time $t_0 \in [(k-1)\tau, k\tau)$ if and only if (5.8) holds or

$$X = \overline{sp}\ \{\ I_i A_i^n Z_i X\ :\ i = k,\ k+1,\cdots,\ n = 0,1,2,\cdots\ \}. \qquad (5.13)$$

To give a generalization of rank condition for pointwise completeness, we consider the condition (4.2). First we give the representation of $F_t(s)$ without using characteristic functions. Let $t \in [(k-1)\tau, k\tau)$ be fixed. Let $X_{i,r}$ $(i = 1,2,\cdots,\ r = 1,\cdots,\ m)$ be the largest subspace of X such that

$$Z_i A_r X_{i,r} \subset \bigcap_{n=0}^{\infty} D(A_i^n).$$

For negative integers $i = -1, -2, \cdots$ we put $X_{i,r} = X$ for each $r = 1,\cdots,$ m. We now define X_i^0, X_i^1 by

$$X_i^0 = \bigcap_{r=i-1}^{m} X_{k+i-r,r}, \qquad X_i^1 = \bigcap_{r=i}^{m} X_{k+i-r,r} \qquad \text{for } i = 1,\cdots, m.$$

Then we obtain by Theorem 4.1 and differentiations of (5.6) the following result.

THEOREM 5.3. The system S is pointwise complete at time $t_0 \in [(k-1)\tau, k\tau)$ if

$$X = \overline{sp}\ \{ G(t_0)X,\ (\sum_{r=i}^{m} I_{k+i-r} A^n_{k+i-r} Z_{k+i-r} A_r) X_i^1 : i=1,\cdots,m,\ n=0,1,2,\cdots \} \tag{5.14}$$

or, more generally if

$$X = \overline{sp}\ \{ G(t_0)X,\ (\sum_{r=i-1}^{m} I_{k+i-r} A^n_{k+i-r} T_{k+i-r}(s_i^0) Z_{k+i-r} A_{r+1}) X_i^0,$$

$$(\sum_{r=i}^{m} I_{k+i-r} A^n_{k+i-r} T_{k+i-r}(s_i^1) Z_{k+i-r} A_r) X_i^1 : i=1,\cdots,m,\ n=0,1,2,\cdots \},$$

$$s_i^0 \text{ arbitrary in } [0,\ t_0-(k-1)\tau) \text{ and } s_i^1 \text{ arbitrary in } [t_0-(k-1)\tau,\ \tau) \tag{5.15}$$

Conversely if $T(t)$ is analytic, X_i^0, X_i^1 are dense for all $i = 1,\cdots, m$ and S is pointwise complete at time $t_0 \in [(k-1)\tau, k\tau)$, then

$$X = \overline{sp}\ \{ G(t_0)X,\ (\sum_{r=i-1}^{m} I_{k+i-r} A^n_{k+i-r} T_{k+i-r}(s_i^0) Z_{k+i-r} A_{r+1}) X_i^0,$$

$$(\sum_{r=i}^{m} I_{k+i-r} A^n_{k+i-r} T_{k+i-r}(s_i^1) Z_{k+i-r} A_r) X_i^1 : i=1,\cdots,m,\ n=0,1,2,\cdots \},$$

$$s_i^0 \text{ arbitrary in } (0,\ t_0-(k-1)\tau) \text{ and } s_i^1 \text{ arbitrary in } (t_0-(k-1)\tau,\ \tau). \tag{5.16}$$

If A is bounded, then $G^*(t)$ is analytic in $t \in ((k-1)\tau, k\tau)$ for each $k = 1,2,\cdots$. Let $t_0 \in [(k-1)\tau, k\tau)$. Then we see easily that

$$\bigcap_{t \geq t_0} \mathrm{Ker}\, G^*(t) = \bigcap_{t \geq (k-1)\tau} \mathrm{Ker}\, G^*(t),$$

so that S is pointwise complete at $t_0 \in [(k-1)\tau, k\tau)$ if and only if S is pointwise complete at $t_0 = (k-1)\tau$. Thus we obtain the following corollary.

<u>COROLLARY 5.2.</u> Let A be bounded. Then S is pointwise degenerate at $t_0 \in [(k-1)\tau, k\tau)$ with respect to E^* if and only if

$$E^* \subset \mathrm{Ker}\, Z_k^* \mathcal{I}_k^* \cap \{ \bigcap_{i=1}^{m} \bigcap_{n=0}^{\infty} \mathrm{Ker}\, (\sum_{r=i}^{m} A_r^* Z_{k+i-r}^* (\mathcal{A}_{k+i-r}^*)^n \mathcal{I}_{k+i-r}^*) \} \tag{5.17}$$

or

$$E^* \subset (\overline{\mathrm{sp}} \{ \mathcal{I}_k Z_k X,\ (\sum_{r=i}^{m} \mathcal{I}_{k+i-r} \mathcal{A}_{k+i-r}^n Z_{k+i-r} A_r) X : i=1,\cdots,m,\ n=0,1,2,\cdots \})^{\perp}. \tag{5.18}$$

Furthermore, S is pointwise complete at $t_0 \in [(k-1)\tau, k\tau)$ if and only if

$$X = \overline{\mathrm{sp}} \{ \mathcal{I}_k Z_k X,\ (\sum_{r=i}^{m} \mathcal{I}_{k+i-r} \mathcal{A}_{k+i-r}^n Z_{k+i-r} A_r) X : i=1,\cdots,m,\ n=0,1,2,\cdots \}, \tag{5.19}$$

or

$$\mathrm{Ker}\, Z_k^* \mathcal{I}_k^* \cap \{ \bigcap_{i=1}^{m} \bigcap_{n=0}^{\infty} \mathrm{Ker}\, (\sum_{r=i}^{m} A_r^* Z_{k+i-r}^* (\mathcal{A}_{k+i-r}^*)^n \mathcal{I}_{k+i-r}^*) \} = \{0\}. \tag{5.20}$$

Especially if $m = 1$, (5.19) is reduced to

$$X = \overline{\mathrm{sp}} \{ \mathcal{I}_k Z_k X,\ \mathcal{I}_k \mathcal{A}_k^n Z_k A_1 X \ ;\ n = 0,1,2,\cdots \}. \tag{5.21}$$

Consider the finite dimensional case where $X = R^N$, A and A_1 are $N \times N$ matrices, so that $\mathcal{I}_k$, $\mathcal{A}_k$, Z_k are $N \times Nk$, $Nk \times Nk$, $Nk \times N$ matrices, respectively. In this case the condition (5.21) is equivalent to that

$$\mathrm{rank}\, [\mathcal{I}_k Z_k,\ \mathcal{I}_k Z_k A_1,\ \cdots,\ \mathcal{I}_k \mathcal{A}_k^{Nk-1} Z_k A_1] = N.$$

This fact follows by the Cayley-Hamilton theorem and gives the main result by Zmood and MaClamroch for Euclidean N-space [19,Theorem 3].

We next specify Theorem 5.2 and Theorem 5.3 in the special case where $m = 1$ and A_1 commutes with A. Let $X_\infty = \bigcap_{n=0}^{\infty} D(A^n)$. Clearly X_∞ is dense in X.

<u>COROLLARY 5.3.</u> Let $m = 1$, A_1 commute with A and $T(t)$ be analytic. Then S is pointwise degenerate at $t_0 \in [(k-1)\tau, k\tau)$ with respect to E^* if

$$E^* \subset \bigcap_{i=k}^{\infty} \bigcap_{n=0}^{\infty} \mathrm{Ker}\ (A_1^*)^i (A^*)^n$$

or if

$$E^* \subset \mathrm{Ker}\ G^*((k-1)\tau) \cap \{ \bigcap_{i=1}^{k} \bigcap_{n=0}^{\infty} \mathrm{Ker}\ (A_1^*)^i (A^*)^n \}.$$

Furthermore, S is pointwise complete at $t_0 \in [(k-1)\tau, k\tau)$ if

$$X = \overline{\mathrm{sp}}\ \{ A_1^i A^n X_\infty : i = k, k+1, \cdots, n = 0,1,2,\cdots \}$$

or if

$$X = \overline{\mathrm{sp}}\ \{ A_1^i A^n X_\infty,\ G((k-1)\tau)X : i=1,\cdots, m,\ n=0,1,2,\cdots \}.$$

We now recall the definition of approximate controllability. Let A genera a semi-group $T(t)$ on X and let B be a bounded operator from a Banach space U into X. Then the system $\{A, B\}$ is said to be approximately controllable if $\overline{\bigcup_{t>0} T(t)BU} = X$. We define $\tilde{Z}_i$ by

$$\tilde{Z}_1 = I \qquad \text{and} \qquad \tilde{Z}_i = \begin{pmatrix} I & 0 \\ 0 & T_{i-1}(\tau)\tilde{Z}_{i-1} \end{pmatrix} \qquad \text{for } i \geq 2.$$

The following corollary is immediate from Theorem 4.2 , Lemma 5.3 and (5.6).

<u>COROLLARY 5.4.</u> Let $T(t)$ be analytic and the system $\{ A_k, \tilde{Z}_k \}$ be approximately controllable on X^i. Then S is pointwise complete at any time $t \in [0, k\tau)$.

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