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Characterization of ω-Regular Languages by First-Order Formulas (An Extended Abstract)

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1. Introduction

Finite automata have been used to define not only languages but also ω-languages, i.e., sets of ω-words (or ω-sequences) $a_0a_1a_2\ldots$ over some alphabet. In the case of usual languages (of finite words), there is the standard notion of acceptance; the one to specify the set of final states. On the other hand, in the case of ω-languages there is a variety of notions of acceptance.

Suppose that $M = (S, \Sigma, \delta, s_0)$ is a (deterministic) finite automaton (S is the set of states, Σ is the input alphabet, δ is the transition function from $S \times \Sigma$ into S, and s_0 ($\in S$) is the initial state). For an ω-word $\alpha = a_0a_1a_2 \ldots$ over Σ let $\underline{Run}^M(\alpha)$ denote the set of states which M visits in reading α and let $\underline{\underline{Run}}^M(\alpha)$ denote the set of states which M visits infinitely often in reading α. Then, in the literature, following ω-languages are used as "the ω-languages accepted by M":

$$O_1(M, F) = \{\alpha \mid \underline{\mathrm{Run}}^M(\alpha) \cap F \neq \emptyset\},$$
$$O_2(M, F) = \{\alpha \mid \underline{\mathrm{Run}}^M(\alpha) \subseteq F\},$$
$$O_3(M, F) = \{\alpha \mid \underline{\underline{\mathrm{Run}}}^M(\alpha) \cap F \neq \emptyset\},$$
$$O_4(M, F) = \{\alpha \mid \underline{\underline{\mathrm{Run}}}^M(\alpha) \subseteq F\},$$
$$O(M, \mathcal{F}) = \{\alpha \mid \underline{\underline{\mathrm{Run}}}^M(\alpha) \in \mathcal{F}\}.$$

Here F ($\subseteq S$) is a set of states and $\mathcal{F}$ is a class of sets of states. Let $\mathbf{O}_1, \ldots, \mathbf{O}_4, \mathbf{R}$ denote the classes of ω-languages of the forms $O_1(M, F), \ldots, O_4(M, F), O(M, \mathcal{F})$ respectively.

There are some other definitions of acceptance (see, for example, [2] - [5]) but the resulting classes of ω-languages coincide with some of $\mathbf{O}_1, \ldots, \mathbf{O}_4, \mathbf{R}$ or their Boolean combinations.

Now a question arises: Can we think of any other kind of acceptance to define a new interesting class of ω-languages? Our work towards this question is motivated by the observation that all the conditions mentioned above and some others can be expressed by certain first-order formulas. For example, the condition $\underline{\underline{\mathrm{Run}}}^M(\alpha) \cap F \neq \emptyset$ used to define $O_3(M, F)$ can be expressed as $\bigvee_{s \in F} \forall i \, \exists j \, (i < j \wedge P_s(j))$, where $P_s(j)$ means "M is in state s at time j."

The first problem we are concerned with in this paper is: When we define the acceptance of finite automata with these first-order formulas, what do we get as the ω-languages accepted thereby? After providing necessary definitions in the next section, in section 3 we show that the class of such ω-languages is precisely the class $\mathbf{R}$.

The next problem we are interested in is a hierarchy of the ω-languages based on the complexity of the formulas. We define classes of ω-languages $\mathbf{\Sigma}_n^{fa}$ and $\mathbf{\Pi}_n^{fa}$ for $n \geq 0$ as the arithmetic hierarchy in the theory of recursive functions, based on the alternation of the quantifiers $\exists$ and $\forall$ in the defining formulas in prenex normal form. We show in sections 3 and 4 that $\mathbf{\Sigma}_1^{fa} = \mathbf{O}_1$, $\mathbf{\Pi}_1^{fa} = \mathbf{O}_2$, $\mathbf{\Sigma}_2^{fa} = \mathbf{O}_4$, $\mathbf{\Pi}_2^{fa} = \mathbf{O}_3$, $\mathbf{\Sigma}_3^{fa} = \mathbf{\Sigma}_4^{fa} = \ldots = \mathbf{\Pi}_3^{fa} = \mathbf{\Pi}_4^{fa} = \ldots = \mathbf{R}$.

From these results it seems reasonable to say that those (and possibly their Boolean combinations) exhaust the natural ways of acceptance of ω-languages by (deterministic) finite automata. In section 5, we remark on a "machine-independent" version of the first-order description of ω-languages.

2. Preliminaries

In the set Σ^* of all ω-words over Σ we can define a topology in a natural way, that is the product topology of the discrete topology on Σ. Let $\mathbf{G}$ ($\mathbf{F}$, resp.) be the class of open sets (closed sets), $\mathbf{G}'$ ($\mathbf{F}'$, resp.) be the class of denumerable intersections (unions) of open sets (closed sets), and $\mathbf{G}''$ ($\mathbf{F}''$, resp.) be the class of denumerable unions (intersections) of sets in $\mathbf{G}'$ ($\mathbf{F}'$).

Then $\mathbf{O}_1, \ldots, \mathbf{O}_4$ are characterized as $\mathbf{O}_1 = \mathbf{G} \cap \mathbf{R}$, $\mathbf{O}_2 = \mathbf{F} \cap \mathbf{R}$, $\mathbf{O}_3 = \mathbf{G}' \cap \mathbf{R}$, $\mathbf{O}_4 = \mathbf{F}' \cap \mathbf{R}$, and $\mathbf{R}$ is included in $\mathbf{G}'' \cap \mathbf{F}''$. The members of $\mathbf{R}$ are called ω-regular languages. We refer the reader to [1], [5], [6] for other properties of classes $\mathbf{O}_1, \ldots, \mathbf{O}_4, \mathbf{R}$.

With each finite automaton $M = (S, \Sigma, \delta, s_0)$ we associate a class of first-order formulas, called M-formulas, in the following way. Terms are expressions of the forms $i + n$ or n, where i is a variable and n is a natural number. Atomic M-formulas are expressions of the forms $P_s(t)$, $t = t'$, $t < t'$, where s is a state of M and t, t' are terms. Finally, M-formulas are constructed from atomic M-formulas using propositional connectives $\vee$ (or), $\wedge$ (and), $\neg$ (not), $\rightarrow$ (implication), $\leftrightarrow$ (equivalence), and quantifiers $\exists$, $\forall$. We may simply say "formulas" instead of M-formulas provided there is no fear of confusion.

Suppose ϕ is an M-formula, and α is an ω-word over Σ. We define the α-interpretation of ϕ as follows: The domain of the interpretation is the set $N = \{0, 1, 2, \dots\}$ of natural numbers. The terms $i + n$, n and atomic M-formulas $t = t'$, $t < t'$ are interpreted as usual. An atomic M-formula $P_s(t)$ is interpreted as "reading α, M is in state s at time t."

For any closed M-formula (i.e., an M-formula without free variables) ϕ, we say ϕ defines the ω-language

$$A(M, \phi) = \{\alpha \in \Sigma^\omega \mid \phi \text{ is true under the } \alpha\text{-interpretation}\}.$$

Finally we define types Σ_n^{fa}, Π_n^{fa} $(n = 0, 1, 2, \dots)$ of M-formulas. Suppose ϕ is an M-formula of the form

$$\exists i_{1,1} \exists i_{1,2} \dots \exists i_{1,e_1} \forall i_{2,1} \forall i_{2,2} \dots \forall i_{2,e_2}$$
$$\exists i_{3,1} \exists i_{3,2} \dots \exists i_{3,e_3} \dots Q i_{n,1} Q i_{n,2} \dots Q i_{n,e_n} \ \psi$$

where $e_p \geq 1$ $(1 \leq p \leq n)$, $i_{1,1}, \dots, i_{n,e_n}$ are different variables, Q is $\forall$ or $\exists$ according as n is even or odd, and ψ is an open M-formula (that is, an M-formula which contains no quantifiers). Then we say ϕ is an M-formula of type Σ_n^{fa}. An M-formula of type Π_n^{fa} is similarly defined with $\exists$ and $\forall$ replaced each other. In particular, we say open M-formulas are of type Σ_0^{fa} and Π_0^{fa}. For each $n \geq 0$, we denote by $\mathbf{\Sigma}_n^{fa}$ (or $\mathbf{\Pi}_n^{fa}$, resp.) the class of ω-languages of the form $A(M, \phi)$ with M-formulas ϕ of type Σ_n^{fa} (or Π_n^{fa}), and denote the class $\mathbf{\Sigma}_n^{fa} \cap \mathbf{\Pi}_n^{fa}$ by $\mathbf{\Delta}_n^{fa}$.

3. Characterization of $\mathbf{R}$

In this section we show that ω-regular languages are precisely the ω-languages defined by M-formulas, and the class of them is equal to $\mathbf{\Sigma}_3^{fa}$ and $\mathbf{\Pi}_3^{fa}$.

Theorem 1. For any closed M-formula ϕ, $A(M, \phi)$ is an ω-regular language.

Proof. We first extend the definition of $A(M, \phi)$ to all M-formulas ϕ (rather than just for closed M-formulas).

Suppose that $\phi(i_1, \ldots, i_n)$ is an M-formula that has n free variables $i_1, \ldots, i_n$ $(n \geq 0)$. Let $\Gamma = \{0, 1\}$ and let Γ_n be the alphabet $\Sigma \times \Gamma^n = \{(a, b_1, \ldots, b_n) \mid a \in \Sigma, b_1, \ldots, b_n \in \Gamma\}$. For an ω-word α on Σ and ω-words $\beta_1, \ldots, \beta_n$ on Γ let $\langle\alpha, \beta_1, \ldots, \beta_n\rangle$ denote the ω-word γ on Γ_n such that $\gamma(m) = (\alpha(m), \beta_1(m), \ldots, \beta_n(m))$ $(0 \leq m)$. Here, if $\alpha = a_0a_1a_2 \ldots$ then $\alpha(m)$ denotes a_m (that is, $\alpha(m)$ denotes the $\underline{m}$th symbol of α) and similarly for $\beta_1(m), \ldots, \beta_n(m), \gamma(m)$.

$A(M, \phi)$ denotes the ω-language

$$\{\langle\alpha, 0^{m_1}1\beta_1, \ldots, 0^{m_n}1\beta_n\rangle \mid \alpha \in \Sigma^\omega, 0 \leq m_1, \ldots, 0 \leq m_n, \beta_1, \ldots, \beta_n \in \Gamma^\omega, \phi(m_1, \ldots, m_n) \text{ is true under the } \alpha\text{-interpretation}\}.$$

Then we prove the theorem for M-formulas ϕ that are not necessarily closed. The proof is by induction on the structure of ϕ. It uses the fact that (1) ω-languages of the form $L\Sigma^\omega$

with (usual) regular languages L ($\subseteq \Sigma^*$) are in $\mathbf{R}$, and that (2) $\mathbf{R}$ is closed under Boolean operations and projections (that is, length preserving homomorphisms). We omit the details of the proof. □

The converse of theorem 1 is also true.

Theorem 2. For any ω-regular language A, there exists a closed M-formula ϕ such that $A(M, \phi) = A$.

Proof. Suppose that $A = A(M, \phi)$. Then we have

$$A = \{\alpha \in \Sigma^\omega \mid \underline{\underline{\text{Run}}}^M(\alpha) = F \text{ for some } F \in \mathcal{F}\}$$
$$= A(M, \bigvee_{F \in \mathcal{F}} (\exists i \, \forall j \, (j \geq i \rightarrow \bigvee_{s \in F} P_s(j)) \wedge \bigwedge_{s \in F} \forall j \, \exists k \, (k \geq j \wedge P_s(k)))). \quad \square$$

We can refine the above result as follows.

Theorem 3. For any $A \in \mathbf{R}$ there are finite automata M, M' and open formulas ψ, ψ' having free variables i, j, k such that

$$A = A(M, \exists i \, \forall j \, \exists k \, \psi(i, j, k))$$
$$= A(M', \forall i \, \exists j \, \forall k \, \psi'(i, j, k)).$$

Proof. We will prove only for $\exists i \, \forall j \, \exists k \, \psi(i, j, k)$ because $\mathbf{R}$ is closed under the operation of complement.

Suppose that $A = O(M_1, \mathcal{F})$ with $M_1 = (S_1, \Sigma, \delta, s_{01})$. Let n be $\max\{|\mathcal{F}|, |S_1|\}$, where $|\ |$ denotes the cardinality of sets.

We construct another finite automaton $M_2 = (S_2, \Sigma, \delta_2, s_{02})$. Intuitively, M_2 simulates M_1, and at the same time counts the time with modulo n. Formally, S_2 is the set $\{(s, h) \mid s \in S_1, 1 \leq h \leq n\}$, s_{02} is $(s_{01}, 1)$, and δ_2 is defined by $\delta_2((s, h), a) = (\delta_1(s, a), h')$, where $h' = h + 1$ if $h \leq n - 1$ and $h' = 1$ if $h = n$.

For each $F \in \mathcal{F}$ and $s \in S_1$ we can construct oper

M_2-formulas $\chi_{1,F}(i)$, $\chi_{2,s}(j)$, $\chi_{3,s}(k)$ having the following intuitive meaning:

$\chi_{1,F}(i) \Longleftrightarrow$ at time i, the second component h of the state (s, h) of M_2 is such that $h \leq |\mathcal{F}|$ and F is the hth element of $\mathcal{F}$;

$\chi_{2,s}(j) \Longleftrightarrow$ at time j, the second component h of the state (s, h) of M_2 is such that $h \leq |S_1|$ and s is the hth element of S_1;

$\chi_{3,s}(k) \Longleftrightarrow$ at time k, s is the first component of the state (s, h) of M_2.

Then, for each α, we have the following equivalence.

$$\begin{aligned}
&A \\
&= A(M_1, \bigvee_{F\in\mathcal{F}} (\exists i\ \forall j\ (j \geq i \rightarrow \bigvee_{s\in F} P_s(j)) \\
&\qquad \wedge \bigwedge_{s'\in F} \forall j\ \exists k\ (k \geq j \wedge P_{s'}(k)))) \\
&= A(M_2, \bigvee_{F\in\mathcal{F}} (\exists i\ \forall j\ (j \geq i \rightarrow \bigvee_{s\in F} \chi_{3,s}(j)) \\
&\qquad \wedge \bigwedge_{s'\in F} \forall j\ \exists k\ (k \geq j \wedge \chi_{3,s'}(k)))) \\
&= A(M_2, \bigvee_{F\in\mathcal{F}} \exists i\ (\forall j\ (j \geq i \rightarrow \bigvee_{s\in F} \chi_{3,s}(j)) \\
&\qquad \wedge \bigwedge_{s'\in F} \forall j\ \exists k\ (k \geq j \wedge \chi_{3,s'}(k)))) \\
&= A(M_2, \exists i\ (\forall j\ (j \geq i \rightarrow \bigvee_{F\in\mathcal{F}} (\chi_{1,F}(i) \wedge \bigvee_{s\in F} \chi_{3,s}(j))) \\
&\qquad \wedge \forall j\ \exists k\ (k \geq j \wedge \\
&\qquad \bigvee_{F\in\mathcal{F}} (\chi_{1,F}(i) \wedge \bigwedge_{s'\in F} (\chi_{2,s'}(j) \rightarrow \chi_{3,s'}(k)))))) \\
&= A(M_2, \exists i\ \forall j\ \exists k\ ((j \geq i \rightarrow \bigvee_{F\in\mathcal{F}} (\chi_{1,F}(i) \wedge \bigvee_{s\in F} \chi_{3,s}(j))) \\
&\qquad \wedge (k \geq j \wedge \\
&\qquad \bigvee_{F\in\mathcal{F}} (\chi_{1,F}(i) \wedge \bigwedge_{s'\in F} (\chi_{2,s'}(j) \rightarrow \chi_{3,s'}(k)))))).
\end{aligned}$$

The last M_2-formula is of the desired form. □

Corollary 4. For any ω-language A the following six conditions are equivalent.

(1) $A \in \mathbf{R}$.

(2) $A \in \Delta_n^{fa}$ $(n \geq 3)$.

(3) $A \in \Sigma_n^{fa}$ $(n \geq 3)$.

(4) $A \in \Pi_n^{fa}$ $(n \geq 3)$.

(5) $A = A(M, \exists i\ \forall j\ \exists k\ \psi)$ for a finite automaton M and an open M-formula ψ.

(6) $A = A(M, \forall i\ \exists j\ \forall k\ \psi)$ for a finite automaton M and an open M-formula ψ.

In closing this section, we mention a characterization of $\mathbf{R}$ which is a direct consequence of the proof of Theorem 1.

Corollary 5. The class of ω-regular languages is the closure of $\mathbf{O}_1$ (or $\mathbf{O}_2$) under Boolean operations and projections.

Compare the corollary with the fact that $\mathbf{R}$ is the class of projective images of the sets in $\mathbf{O}_3$, and also is the Boolean closure of $\mathbf{O}_4$ (see e.g., [1]).

4. Characterization of $\mathbf{O}_1$, $\mathbf{O}_2$, $\mathbf{O}_3$, $\mathbf{O}_4$

In this section we characterize $\mathbf{O}_1$, $\mathbf{O}_2$, $\mathbf{O}_3$, $\mathbf{O}_4$ by means of forms of M-formulas.

Theorem 6. For any ω-language A the following three conditions are equivalent.

(1) $A \in \mathbf{O}_1$.

(2) $A \in \Sigma_1^{fa}$.

(3) $A = A(M, \exists i\ \psi)$ for a finite automaton M and an open M-formula ψ.

A similar result holds when $\mathbf{O}_1$, Σ_1^{fa}, $\exists$ are replaced by $\mathbf{O}_2$, Π_1^{fa} , $\forall$, respectively.

Proof. We only prove the part (2) $\Rightarrow$ (1) for $\mathbf{O}_1$. Let $\exists i_1$ ...

$\exists i_p\ \psi(i_1, \ldots, i_p)$ be a closed M-formula defining A, where ψ is open. We may assume that ψ does not contain $\neg$ because we can eliminate $\neg$ by $\neg P_s(t) \Leftrightarrow \bigvee_{s' \neq s} P_{s'}(t)$, $\neg t < t' \Leftrightarrow t' < t \vee t = t'$, $\neg t = t' \Leftrightarrow t' < t \vee t < t'$.

The proof of Theorem 1 shows that if ϕ is an atomic M-formula then $A(M, \phi)$ is in $\mathbf{G}$ (that is, open). Hence, for any numbers $m_1, \ldots, m_p$, $A(M, \psi(m_1, \ldots, m_p))$ is obtained from sets in $\mathbf{G}$ by finite union and finite intersection. Hence it is in $\mathbf{G}$. Therefore we have

$$\begin{aligned} A &= A(M, \exists i_1 \ldots \exists i_p\ \psi(i_1, \ldots, i_p)) \\ &= \cup\ A(M, \psi(m_1, \ldots, m_p)) \\ &\in \mathbf{G} \cap \mathbf{R} = \mathbf{O}_1 \end{aligned}$$

($\cup$ ranges over all $m_1, \ldots, m_p$). □

Corollary 7. For any ω-language A in Σ^ω the following four conditions are equivalent.

(1) $A \in \mathbf{O}_1 \cap \mathbf{O}_2$.

(2) $A \in \mathbf{\Delta}_1^{fa}$.

(3) $A \in \mathbf{\Delta}_0^{fa}$ $(= \mathbf{\Sigma}_0^{fa} = \mathbf{\Pi}_0^{fa})$.

(4) $A = L\Sigma^\omega$ for a finite subset L of Σ^*.

Proof. For the equivalence of (1) and (4), see, e.g., [6]. □

Theorem 8. For any ω-language A the following three conditions are equivalent.

(1) $A \in \mathbf{O}_3$.

(2) $A \in \mathbf{\Pi}_2^{fa}$.

(3) $A = A(M, \forall i\ \exists j\ \psi)$ for a finite automaton M and an open M-formula ψ.

A similar result holds when $\mathbf{O}_3$, $\mathbf{\Pi}_2^{fa}$, $\forall$, $\exists$ are replaced with $\mathbf{O}_4$, $\mathbf{\Sigma}_2^{fa}$, $\exists$, $\forall$ respectively.

Proof of (2) $\Rightarrow$ (1) for $\mathbf{O}_3$. The proof is the same as that of (2) $\Rightarrow$ (1) of Theorem 6 except that we use

$$\begin{aligned} A &= A(M, \forall i_1 \ldots \forall i_p \exists j_1 \ldots \exists j_q \\ &\qquad\qquad \psi(i_1, \ldots, i_p, j_1, \ldots, j_q)) \\ &= \cap \cup A(M, \psi(m_1, \ldots, m_p, n_1, \ldots, n_q)) \\ &\in \mathbf{G}' \cap \mathbf{R} = \mathbf{O}_3 \end{aligned}$$

($\cap$ is over all $m_1, \ldots, m_p$ and $\cup$ is over all $n_1, \ldots, n_q$).□

Corollary 9. For any ω-language A the following three conditions are equivalent.

(1) $A \in \mathbf{O}_3 \cap \mathbf{O}_4$.

(2) $A \in \mathbf{\Delta}_2^{fa}$.

(3) $A = \{\alpha \in \Sigma^\omega \mid \underline{\mathrm{Run}}^M(\alpha) \in \mathcal{F}\}$ for a finite automaton $M = (S, \Sigma, \delta, s_0)$ and a class $\mathcal{F}$ of subsets of S.

Proof. The equivalence of (1) and (3) was shown by Staiger and Wagner ([5]). □

5. Concluding Remarks

In this paper we introduced the M-formulas associated with a finite automaton M, and studied classes of ω-languages accepted in the ways specified by the M-formulas. The result can be restated in terms of "machine-independent" first-order description of ω-languages.

For a given sequence $\mathbf{L} = (L_1, L_2, \ldots, L_m)$ of regular languages ($\subseteq \Sigma^*$, $m \geq 1$), let us define $\mathbf{L}$-formulas as the first-order formulas which are constructed from the atomic formulas of the forms $\alpha[t] \in L_p$ $(1 \le p \le m)$, $t = t'$, and $t < t'$ where t and t' are either i + n or n for a variable i and a natural number n. In other words, **L**-formulas are same as the M-formulas except that atomic formulas $P_s(t)$ $(s \in S)$ are now replaced by $\alpha[t] \in L_p$ $(1 \le p \le m)$. For a given ω-word α over Σ, we define the α-interpretation in the same way as for M-formulas except that the meaning of $\alpha[t] \in L$ is "the prefix of α of length t is in L."

For a closed **L**-formula φ we define

$$A(\mathbf{L}, \phi) = \{\alpha \in \Sigma^{\omega} \mid \alpha \text{ is true under the } \alpha\text{-interpretation}\}.$$

Lastly we define the types Σ_n^{reg} and Π_n^{reg} $(n \ge 0)$ of **L**-formulas in prenex normal form as in the case of Σ_n^{fa} and Π_n^{fa} (except that "M-formulas" are now replaced by "**L**-formulas"). Under these definitions we can easily see the followings.

(1) For a given finite automaton M and a closed M-formula φ of type Σ_n^{fa} (or Π_n^{fa}), there exists a sequence $\mathbf{L} = (L_1, L_2, \dots, L_m)$ of regular languages and an **L**-formula φ' of type Σ_n^{reg} (or Π_n^{reg}) such that $A(M, \phi) = A(\mathbf{L}, \phi')$.

(2) Conversely for a given sequence $\mathbf{L} = (L_1, L_2, \dots, L_m)$ of regular languages and a closed **L**-formula φ of type Σ_n^{reg} (or Π_n^{reg}) one can construct a finite automaton M and an M-formula φ' of type Σ_n^{fa} (or Π_n^{fa}) such that $A(\mathbf{L}, \phi) = A(M, \phi')$.

Therefore we can get characterizations of **R** and its subclasses $\mathbf{O}_1 \cap \mathbf{O}_2$, $\mathbf{O}_1$, $\mathbf{O}_2$, $\mathbf{O}_3 \cap \mathbf{O}_4$, $\mathbf{O}_3$, $\mathbf{O}_4$ by means of **L**-formulas of the types Σ_n^{reg} and Π_n^{reg} exactly as before. This can be seen as a generalization of the characterization (or definition) of $\mathbf{O}_1$ - $\mathbf{O}_4$ stated in terms of regular languages;

$$A \in \mathbf{O}_1 \iff A = \{\alpha \in \Sigma^\omega \mid \exists i\ (\alpha[i] \in L)\},$$
$$A \in \mathbf{O}_2 \iff A = \{\alpha \in \Sigma^\omega \mid \forall i\ (\alpha[i] \in L)\},$$
$$A \in \mathbf{O}_3 \iff A = \{\alpha \in \Sigma^\omega \mid \forall i\ \exists j\ (i < j \wedge \alpha[j] \in L)\},$$
$$A \in \mathbf{O}_4 \iff A = \{\alpha \in \Sigma^\omega \mid \exists i\ \forall j\ (i < j \rightarrow \alpha[j] \in L)\}.$$

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