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Some Applications of Fourier Analysis to Uniform Distribution Mod 1.

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We say that a sequence $(\lambda_n)_1^\infty$ of real numbers is uniformly distributed mod 1 (u.d. mod 1) if we have for any $0 \leqq \alpha < \beta \leqq 1$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{\substack{k=1 \\ \alpha \le \{\lambda_k\} \le \beta}}^{n} 1 = \beta-\alpha,$$

where $\{\lambda_k\}$ denotes the fractional part of λ_k. Then Weyl proved that $(\lambda_n)_1^\infty$ is u.d. mod 1 iff

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} e^{2\pi i h\lambda_k} = 0,$$

for all fixed natural numbers h. Among other important results, he gave the following "metric result".

Theorem A. If $(a_n)_1^\infty$ is a sequence of distinct natural numbers, then $(xa_n)_1^\infty$ is u.d. mod 1 for almost all fixed x.

In fact, behind his proof of this theorem, a much more general principle was laid. For instance, the following generalization is possible [3].

Theorem B. If $(\lambda_n)_1^\infty$ is a sequence of real numbers such that

$$\inf_{n \neq m} | \lambda_n - \lambda_m | > 0,$$

then the sequence $(x\lambda_n)_1^\infty$ is u.d. mod 1 for almost all real numbers x.

The main object of this article is to indicate another approach so that we can improve these results to some extent.

Let

(1) $$\sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

be the Fourier series of $f \in L^1$.

In 1920 arround, Russian mathematician Lusin made the conjecture that for all $f \in L^2$, (1) converges for almost all x. About ten years later, Kolmogorov constructed an $f \in L^1$ such that (1) diverges for all x. However, in 1966 L. Carleson [1] has proved the Lusin conjecture affirmatively. A version of his theorem is

Theorem C. If $\sum_{n=1}^{\infty} (a_n^2 + b_n^2) < \infty$, then (1) converges for almost all x.

It will be worth stating here an open problem due to Orlicz [4]: Give an example of (1) divergent for all x and such taht

$$\sum_{n=1}^{\infty} (a_n^{2+\varepsilon} + b_n^{2+\varepsilon}) < \infty$$

for every $\varepsilon>0$.

We may now apply Theorem C to the Fourier integral

$$\int_0^{\to\infty} f(t)\, e^{ixt} dt$$

and obtain the following

Theorem 1. If $\lambda_{k+1} - \lambda_k \geq \kappa > 0$ $(k=1,2,\dots)$ and

$$\sum_{k=1}^{\infty} |a_k|^2 < \infty ,$$

then

(2) $$\sum_{k=1}^{\infty} a_k e^{i\lambda_k x}$$

converges for almost all x.

Corollary 1. Under the assumption of Theorem 1,

$$\prod_{k=1}^{\infty} (1 + a_k e^{i\lambda_k x})$$

converges for almost all x.

Corollary 2. If $\lambda_{k+1} - \lambda_k \geq \kappa > 0$ $(k=1,2,\dots)$, then

$$\sum_{k=1}^{N} e^{i\lambda_k xh} = 0\left(\sqrt{N \log N}\ (\log\log N)^{\delta}\right)$$

for almost all x, where $h \in \mathbb{N}$ is fixed and $\delta > 1/2$ an arbitrary constant.

More generally, we can in fact prove

Theorem 2. If

(3) $$\lambda_{k+1} - \lambda_k \geq \mu_k > 0, \quad (k=1,2,\dots)$$

$$\lambda_k \to \infty \quad (k \to \infty),$$

and for almost all x

(4) $$\sum_{k=1}^{\infty} \frac{\mu_k}{\sin^2(\mu_k x/2)} |a_k|^2 < \infty ,$$

then (2) converges for almost all x.

Corollary 3. The Dirichlet series

$$\sum_{n=1}^{\infty} \frac{c_n}{n^s} \qquad (s = \sigma+it)$$

converges on $\sigma = 1/2$ for almost all t if

$$\sum_{n=1}^{\infty} |c_n|^2 < \infty .$$

From Theorem 2 we obtain the following

Theorem 3. If (3) holds and for almost all x

$$\text{(5)} \qquad \sum_{k=1}^{\infty} \frac{\mu_k}{k^2 \sin^2(\mu_k x/2)} < \infty ,$$

then the sequence $(x\lambda_k)_1^{\infty}$ is u.d. mod 1 for almost all x.

If we denote by p_n the n th prime ($p_1=2$), then for sufficiently large n we have

$$(\log p_{n+1})^{\delta} - (\log p_n)^{\delta} >> \frac{(\log n)^{\delta-2}}{n}, \quad (\delta > 0)$$

by the prime number theorem and Hoheisel's theorem. Thus the following interesting result is immediately deduced from Theorem 3.

Corollary 4. The sequence $(x(\log p_n)^{\delta})_1^{\infty}$ is u.d. mod 1 for almost all x, provided $\delta > 3$.

Our method cannot afford anything if $0 < \delta \leq 3$, and we are tempted to conjecture that Corollary 4 is still true if $\delta > 1$. We can prove, however, that $(x(\log p_n)^{\delta})_1^{\infty}$ is $(M, (\log n)^{\delta-1}/n)$ - u.d. mod 1 for almost all x if $\delta > 1$ (cf. Corollary 5 below). A sequence $(\lambda_n)_1^{\infty}$ of real numbers is said to be (M, α_n) - u.d. mod 1 if there exists a sequence of positive numbers such that

$$\alpha_1 > \alpha_2 > \cdots > \alpha_n > \cdots$$

$$\sum_{n=1}^{\infty} \alpha_n = \infty$$

and

$$\sum_{n=1}^{N} \alpha_n e^{2\pi i h \lambda_n} = 0 \left(\sum_{n=1}^{N} \alpha_n\right)$$

for any fixed $h \in \mathbb{N}$.

It is easy to see that any u.d. sequence is (M, α_n) - u.d. for any α_n.

By a similar argument due to Cossar [2], we obtain

Theorem 4. If (3) holds and for almost all x

$$\sum_{k=1}^{\infty} \left|\frac{c_k}{\lambda_k}\right|^p \cdot \frac{\mu_k}{|\sin(\mu_k x/2)|^p} < \infty ,$$

for some $1 < p \leqq 2$, then

$$\sum_{k=1}^{n} c_k e^{i\lambda_k x} = 0 (\lambda_{n+1})$$

for almost all x.

Corollary 5. If $\lambda_n << \sum_{k=1}^{n} c_k$ and

$$\lambda_{k+1} - \lambda_k >> 1/k(\log k)^\alpha$$

for some $0 \leqq \alpha < 1$, then $(x\lambda_k)_1^\infty$ is (M, c_n) - u.d. mod 1 for almost all x.

We remark that our argument breaks down if $\alpha = 1$. So it does not follow that $(x \log p_n)_1^\infty$ is $(M, 1/n)$- u.d. mod 1 for almost all x.

References

[1] Carleson, L. : On convergence and growth of partial sums of Fourier series, Acta Math. 116 (1966), 135-157.

[2] Cossar, J : The Bohr spectrum of a function. Proc. AMS. 10 (1959), 294-295.

[3] Kuipers, L. & Niederreiter, H. : Uniform distribution of sequences, Wiley, 1974.

[4] Mauldin, R. D. (Ed.) : The Scottish Book, Birkhäuser, 1981.