数理解析研究所講究録
第 499 巻 1983 年 91-99

Mean Value and Difference Type Functional Equations

岡山理大 春木 茂 (Shigeru Haruki)

§1. Introduction.

$x, y, x_i, y_i \in \mathbb{R}^n$ or a linear space X

$t, r, t_i, r_i \in \mathbb{R}$

$f: \mathbb{R}^n \to \mathbb{R}$ or $f: X \longrightarrow Y$ (another linear space)

Type I

$$\boxed{f(x) = F\{f(x+t_1 y), \cdots, f(x+t_k y), x, y\}}$$

$\forall x, y \in \mathbb{R}^n$ (or some given subsets of $\mathbb{R}^n$),

fixed $t_i \in \mathbb{R}$

Type II

$$\boxed{f(x) = F\{f(x+t y_1), \cdots, f(x+t y_k), x, t\}}$$

$\forall x \in \mathbb{R}^n$, $t \in \mathbb{R}$ (or subsets of $\mathbb{R}^n$ and $\mathbb{R}$),

fixed $y_i \in \mathbb{R}^n$

Note : $n = 1$, Type I = Type II

Problem :

1. the general solution
2. a very weak regularity assumption
   $\Rightarrow$ continuous, or C^∞
   very often that f is a polynomial on $\mathbb{R}^n$

Examples of Type I

The finite difference equations in $\mathbb{R}^n$

(1) $$\Delta_y^{k+1} f(x) = g(x,y) \quad \forall\, x, y \in \mathbb{R}^n$$

Equations (1) include as particular cases:

(a) Cauchy's functional equation in $\mathbb{R}^n$

$$f(x+y) = f(x) + f(y) \iff \Delta_y^1 f(x) = f(y)$$

(b) Jensen's functional equation in $\mathbb{R}^n$

$$f\left(\frac{u+v}{2}\right) = \frac{1}{2}\{f(u) + f(v)\} \iff \Delta_y^2 f(x) = 0$$

(c) Quadratic functional equation in $\mathbb{R}^n$

$$f(u+v) + f(u-v) = 2f(u) + 2f(v)$$

Set $u = x+y$, $v = y$

$$\iff \Delta_y^2 f(x) = 2f(y)$$

(d) $$f(u+v) + f(u-v) = 2f(u) + 2f(v) + g(u,v)$$

⋮

<u>*Example of Type II*</u>

The mean value equations in $\mathbb{R}^n$, for example,

$$(2) \qquad \sum_{i=1}^{k} \lambda_i f(x+ty_i) = f(x), \quad \lambda_i \in \mathbb{R}, \quad \sum_{i=1}^{k} \lambda_i = 1$$

Here, the y_i are interpreted as the position vectors of the vertices of a k-gon in $\mathbb{R}^n$ (regular or not), and the λ_i are weights at each vertex. Equation of type (2) then demand that the value $f(x)$ of f at the center x of the k-gon should equal the (weighted) mean value of f at the vertices of the k-gon.

Equations (2) include as particular cases:

(a) Jensen's functional equation

$$f\left(\frac{u+v}{2}\right) = \frac{1}{2}\{f(u)+f(v)\} \iff f(x+y)+f(x-y) = 2f(x)$$

(b) Kakutani-Nagumo-Walsh's functional equation

$$f: \mathbb{C} \to \mathbb{R}, \quad \theta = \exp(2\pi i/n), \quad n \geqq 3,$$

$$\sum_{\sigma=0}^{n-1} f(x+\theta^{\sigma} y) = n f(x)$$

⋮

§2. Linear Type I: Regular Solutions

Consider equations of the form

$$f_0(x) + \sum_{i=1}^{k} f_i(x + t_i y) = \phi(x, y) \quad \text{for fixed } 0 \neq t_i \in \mathbb{R} \tag{3}$$

where the equation is assumed to hold for:

$$\left\{\begin{array}{l} \text{all } x \in D \subseteq \mathbb{R}^n, \text{ where } D \text{ is open and connected,} \\ \text{all } y \in E \subseteq \mathbb{R}^n, \text{ where } \mu(E) > 0, \\ \phi \text{ is a given mapping } \phi: D \times E \to \mathbb{R}. \\ f_0: D \to \mathbb{R} \text{ and } f_i: D + t_i E \to \mathbb{R} \text{ are unknown functions.} \end{array}\right. \tag{4}$$

Theorem 1 (J.H.B. Kemperman 1957). Let f_0 and $\bar{f}_0$ be any two solutions of (3), given (4), and assume that f_0 and $\bar{f}_0$ are bounded in absolute value on sets S_0, $\bar{S}_0$ where $S_0 \subseteq D$, $\bar{S}_0 \subseteq D$, $\mu(S_0) > 0$, $\mu(\bar{S}_0) > 0$. (Note: no assumptions concerning the remaining f_i and $\bar{f}_i$!). Then:

i) if, for each $y_0 \in E$, $\phi(x, y_0)$ is continuous, or a polynomial, or analytic in $x \in D$, then f_0 and $\bar{f}_0$ are also continuous, or polynomials, or analytic in D.

ii) if, for each $y_0 \in E$, $\phi(x, y_0)$ is continuous for $x \in D$, then $f_0 - \bar{f}_0$ is a polynomial of degree at most $k-1$ on D.

iii) if, for each $y_0 \in E$, $\phi(x, y_0)$ is a polynomial for $x \in D$ of degree $\leq P$, then f_0 (and also $\bar{f}_0$) are polynomials of degree $\leq k + P$ on D.

<u>Examples of Theorem 1.</u> In the following examples, it is only assumed that the solution is bounded in absolute value on a set S, $\mu(S) > 0$ for a set $S \subseteq \mathbb{R}^n$.

Example 1. Write Pexider's equation

$$f(x+y) = g(x) + h(y)$$

in the form $f_1(x+y) - f_0(x) = \phi(y)$, as an equation in $\mathbb{R}^n$. Then we have that $g(x)$ is a polynomial of degree ≤ 1.

Example 2. Write the equation

$$f(x+y) + k(x-y) = 2g(x) + 2h(y)$$

in the form $f_2(x+y) + f_1(x-y) - 2f_0(x) = 2\phi(y)$, whence $g(x)$ is a polynomial of degree ≤ 2.

Example 3. For the equation

$$\Delta_y^k f(x) = g(y),$$

f is a polynomial of degree $\leq k$.

⋮

§ 3. Linear Type I : General Solutions

Let X and Y denote arbitrary linear spaces over the reals $\mathbb{R}$. Consider the general finite difference functional equation

$$(5) \qquad \Delta_y^{k+1} f(x) = 0 \qquad \forall\, x, y \in X$$

$$\Updownarrow$$

$$\sum_{i=0}^{k+1} \binom{k+1}{i} (-1)^{k+1-i} f(x+iy) = 0 \qquad \forall\, x, y \in X$$

where $f : X \to Y$. The most extensive treatment of the general solution of (5) was given by S. Mazur and W. Orlicz 1934 (among others), and of

$$\Delta_y^{k+1} f(x) = \phi(y) \qquad \forall\, x, y \in X$$

by G. van der Lijn 1945 (among others).

In order to describe the most general solution of these equations, we introduce the following notation:

> $A_p : X^p \to Y$ denotes a symmetric multi-additive function, that is, $A_p(x_1, \dots, x_p) = A_p(x_{i_1}, \dots, x_{i_p})$ for all permutations $(i_1, \dots, i_p)$ of $(1, \dots, p)$ and A_p satisfies Cauchy's functional equation in each x_α.

> $A^p : X \to Y$ is defined by $A^p(x) = A_p(x, x, \dots, x)$ for all $x \in X$.

A function of the form A^p (that is, one for which such an A_p exists) is called rational homogeneous of order p. We take $A_0 = A^0$ to be the constant functions, rational homogeneous of order 0.

The results of Mazur, Orlicz, and van der Lijn include

Theorem 2. A necessary and sufficient condition that f be rational homogeneous of order p is that f satisfy

$$\Delta_y^p f(x) = p!\, f(y)$$

for all $x, y \in X$.

Example of Theorem 2. The well-known quadratic equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y)$$

may be written, as in (C),

$$\Delta_y^2 f(x) = 2f(y).$$

Hence the general solution is $f(x) = A_2(x,x)$ where $A_2(x,y)$ is an arbitrary symmetric and bi-additive function on $X \times X \to Y$.

Theorem 3. A necessary and sufficient condition that

$$\Delta_y^p f(x) = \phi(y)$$

have a solution is that $\phi(y) = A^p(y)$.

<u>Theorem 4.</u> A necessary and sufficient condition that

$$(5) \qquad \Delta_y^{k+1} f(x) = 0 \qquad \forall\, x, y \in X$$

is that $f(x) = \sum_{p=0}^{k} A^p(x)$.

Let G and H be additive Abelian groups. Let S be any field and G, H be a unital S-modules. Let $f : G \to H$ satisfy the equation

$$(6) \qquad \sum_{i=0}^{n} \gamma_i f(x+\alpha_i y) = 0 \qquad \forall\, x, y \in G,$$

where $n > 2$ is a given integer, $\gamma_i \neq 0$, $\alpha_i \neq 0\ (= \alpha_0)$ for $i = 0, 1, \cdots, n$ are fixed elements in S and $\alpha_j \neq \alpha_k$ for $j \neq k$. Equation (6) is a generalization of (5). More generally we have

<u>Theorem 5</u> (S. Haruki 1981). Let $f_i : G \to H$ for $i = 0, 1, \cdots, n$ satisfy the equation

$$(7) \qquad \sum_{i=0}^{n} f_i(x+\alpha_i y) = 0 \qquad \forall\, x, y \in G,$$

where $\alpha_i \neq 0$ for $i = 0, 1, \cdots, n$ are fixed elements in S and $\alpha_j \neq \alpha_k$ for $j \neq k$. Then equation (7) implies

$$(8) \qquad \Delta_u^n f_i(x) = 0$$

for each $i = 0, 1, \cdots, n$ and for all $x, u \in G$.

§4. Linear Type II : General and Regular Solutions

Consider equations of the form

$$(9) \qquad \sum_{i \in I} \lambda_i f(x + t y_i) = 0$$

where λ_i, $t \in \mathbb{R}$, the λ_i fixed and $\sum_{i \in I} \lambda_i = 0$, and where $x, y_i \in \mathbb{R}^n$ for fixed y_i. We use I to denote the set $\{1, 2, \ldots, k\}$. The equation (9) is to hold for all $x \in \mathbb{R}^n$, $t \in \mathbb{R}$ such that $x + t y_i \in D$ for all $i \in I$, and $D \subseteq \mathbb{R}^n$ is some (connected) domain.

<u>Theorem 6</u> (M. A. McKiernan 1970). If $\sum_{i \in J} \lambda_i \neq 0$ for all subsets $J \subset I$ (proper subsets) then (9) implies that on D,

$$\Delta^p_{t y_i} f(x) = 0$$

for all $i \in I$, and all $p > k(k-1)/2$.

Hence, if $\sum_{i \in J} \lambda_i \neq 0$ for all $J \subset I$, then the type II, in $\mathbb{R}^n$, is related to the type I in $\mathbb{R}^n$. In particular it easily follows that

<u>Theorem 7</u> (M. A. McKiernan 1970). If further the y_i span $\mathbb{R}^n$, and if f is bounded in absolute value on a set $S \subseteq D$, $\mu(S) > 0$, then f is C^∞ on D, and hence a polynomial on D.