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ON THE EXISTENCE OF CONDENSER-TYPE MEASURES WITH RESPECT TO FUNCTION-KERNELS

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§1. Introduction.

Let X be a locally compact Hausdorff space with a countable base, G be a continuous function-kernel on X and N be a lower semicontinuous function-kernel which is locally bounded outside the diagonal set. Further, assume that each non-empty open set is not G-negligible and not N-negligible. On this note we shall ask necessary and sufficient conditions in order that for any pair <K, F> of disjoint compact sets there exists a (G, N)-condenser-type measure α of each $\lambda \in M_K^+$ of which $N\lambda$ is locally bounded on K, i.e., there exists a measure $\alpha = \mu_o - \mu_1$ such that

$$\mathrm{supp}(\mu_o) \subset K, \quad \mathrm{supp}(\mu_1) \subset F,$$
$$G\alpha = N\lambda \text{ G-n.e. on } K, \quad G\alpha = 0 \text{ G-n.e. on } F,$$
$$0 \le G\alpha \le N\lambda \quad \text{on } X.$$

In case N = 1 (the constant kernel) this problem has been solved in [6], i.e., G satisfies the condenser principle if and only if, G is non-degenerate and it satisfies the domination and the classical maximum principles.

In §3 we shall show that for any pair <K, F> of a non-G-negligible compact set K and a compact set F with $A(K) \cap F = \phi$, there exists a (G, N)-condenser-type measure of each $\lambda \in M_K^+$ of which $N\lambda$ is bounded on K if and only if, G satisfies the domination principle and the relative domination principle with respect to N.

It is well-known that in Dirichlet spaces there exists a condenser measure for any pair $<\overline{U}, \overline{V}>$ of a relatively compact open set U and an open set V with $\overline{U} \cap \overline{V} = \phi$. Further, C. Berg has proved in [1] the existence of the condenser measure for any above pair $<\overline{U}, \overline{V}>$ with respect to the kernel $\kappa = \int \mu_t dt$, where $(\mu_t)_{t>0}$ is a transient convolution semigroup on a locally compact abelian group.

In §4 we shall ask similar problems for any pair $<K, F>$ of a non-G-negligible compact set K and a closed set F with $A(K) \cap F = \phi$.

§2. Various domination principles.

Let X be a locally compact Hausdorff space with a countable base. A lower semicontinuous function-kernel on X is a non-negative lower semicontinuous function defined on $X \times X$ which is strictly positive on the diagonal set Δ of $X \times X$. A continuous function-kernel is a lower semicontinuous kernel which is continuous in the extended sense in $X \times X$ and finite-valued outside Δ. We use G and N for denoting lower semicontinuous function-kernels and introduce various domination principles.

Definition 1. We say that G satisfies the relative domination principle with respect to N, if for $\mu, \nu \in M_K^+$ with $\int G\mu d\mu < +\infty$

$$G\mu \leq N\nu \quad \text{on } \operatorname{supp}(\mu) \quad \text{implies} \quad G\mu \leq N\nu \quad \text{in } X.$$

We use the symbol $G \prec N$ for this principle.

"$G \prec G$" is said to be simply the domination principle.

"$G \prec 1$" is said to be the classical maximum principle.

Further, we say that G satisfies the elementary relative domination principle with respect to N if for $\mu \in M_K^+$ with $\int G\mu d\mu < +\infty$ and for $x_o \in C\operatorname{supp}(\mu)$

$$G\mu \leq N\varepsilon_{x_o} \quad \text{on } \operatorname{supp}(\mu) \quad \text{implies} \quad G\mu \leq N\varepsilon_{x_o} \quad \text{on } X.$$

Definition 2. We say that G satisfies the transitive domination principle with respect to N, if for $\mu, \nu \in M_K^+$ with $\int G\mu d\mu < +\infty$

$$G\mu \le G\nu \text{ on } supp(\mu) \text{ implies } N\mu \le N\nu \text{ in } X.$$

We use the symbol G [N for this principle.

Definition 3. We say that G satisfies the relative balayage principle with respect to N, if for every compact set K and every $\mu \in M_K^+$ such that $N\mu \not\equiv +\infty$ on K there exists $\nu \in M_K^+$ such that

$$supp(\nu) \subset K, \quad G\nu = N\mu \quad G\text{-n.e. on } K, \quad G\nu \le N\mu \text{ in } X.$$

Here "$G\nu = N\mu$ G-n.e. on K" means that the set $\{x \in K;\ G\nu(x) \ne N\mu(x)\}$ is a G-negligible set and a Borel measurable set A of X is called G-negligible if any measure $\lambda \in M_K^+$ with $supp(\lambda) \subset A$ and $\int G\lambda d\lambda < +\infty$ is equal to 0. Such a measure ν is called a relatively balayaged measure of μ onto K with respect to (G, N). If G satisfies the relative balayage principle with respect to G, we say that G satisfies the balayage principle and a relatively balayaged measure with respect to (G, G) is called a G-balayaged measure, simply a balayaged measure. If G satisfies the relative balayage principle with respect to the constant kernel 1, we say that G satisfies the equilibrium principle.

Definition 4. We say that G satisfies the continuity principle if for $\mu \in M_K^+$ $G\mu$ is finite and continuous everywhere whenever it is finite and continuous on $supp(\mu)$.

We denote by $\mathcal{L}(G)$ (resp. $\mathcal{F}(G)$) the set of measures $\mu \in M_K^+$ such that $G\mu$ are locally bounded (resp. $G\mu$ are finite and continuous).

Hereafter we shall assume that G is a continuous function-kernel on X and N is a lower semicontinuous function-kernel on X which is locally bounded outside the diagonal set Δ of $X \times X$.

Proposition 1. Assume that each non-empty open set is not G-negligible and not N-negligible. If $G \sqsubset N$, then $\mu \in \mathcal{L}(G)$ implies $\mu \in \mathcal{L}(N)$.

Proof. Let μ be a measure in $\mathcal{L}(G)$. Obviously the potential $N\mu$ is locally bounded outside $\mathrm{supp}(\mu)$. Let x_0 be an arbitrary point in $\mathrm{supp}(\mu)$. Since $G(x_0, x_0) > 0$, we can find a compact neighborhood K of x_0 and a constant c such that $cG(x, y) \geq 1$ on $K \times K$. Denote by μ_0 the restriction of μ to K. Since $G\mu_0$ is also locally bounded, there are a neighborhood K_1 of x_0 and a constant b such that $K_1 \subset K$ and $G\mu_0 \leq b$ on K_1. First, assume that x_0 is an isolated point. Then $\{x_0\}$ is not G-negligible and not N-negligible by the assumption and we have $G(x_0, x_0) < +\infty$ and $N(x_0, x_0) < +\infty$. Denote by μ_1 the restriction of μ_0 to K_1. Then $G\mu_1 \leq bcG\varepsilon_{x_0}$ on K_1 and hence $N\mu_1 \leq bcN\varepsilon_{x_0}$ on X by $G \sqsubset N$. Since $N\varepsilon_{x_0}$ and $N(\mu - \mu_1)$ are bounded on K_1, $N\mu$ is also bounded on K_1. Secondly, we consider the case where x_0 is not an isolated point. Take $x_1 \in K_1 \setminus \{x_0\}$ and find a compact neighborhood K_2 of x_0 satisfying $K_2 \subset K_1$ and $x_1 \notin K_2$. If we denote by μ_2 the restriction of μ_0 to K_2, we have $G\mu_2 \leq bcG\varepsilon_{x_1}$ on K_2 and hence $N\mu_2 \leq bcN\varepsilon_{x_1}$ on X. Since $N\varepsilon_{x_1}$ and $N(\mu - \mu_2)$ are bounded on K_2, $N\mu$ is bounded on K_2. Thus we obtain that $N\mu$ is locally bounded.

Proposition 2. Assume that each non-empty open set is not G-negligible and not N-negligible. If G satisfies the continuity principle and $G \sqsubset N$, then $\check{G}$ satisfies the relative balayage principle with respect to $\check{N}$. Here $\check{G}$ (resp. $\check{N}$) is the adjoint kernel of G (resp. N).

Proof. Let K be a non-G-negligible compact set and μ be a measure in M_K^+ with $\check{N}\mu \not\equiv +\infty$ on K. Put

$$S := \{u = G\sigma - G\tau;\ \sigma \in M_K^+,\ \tau \in \mathfrak{F}(G),\ \mathrm{supp}(\tau) \subset K\}$$

and define for each $f \in C(K)$

$$p(f) := \inf\{\int N\sigma d\mu - \int N\tau d\mu;\ u = G\sigma - G\tau \in S,\ u \geq f \text{ on } K\}.$$

We remark that $N\tau$ is locally bounded by Proposition 1 and $\int N\tau d\mu < +\infty$. By the assumption we can find, for each $f \in C(K)$, $\sigma_o \in \mathfrak{F}(G)$ satisfying $|f| \leq G\sigma_o$ on K. Then $p(f) \leq \int N\sigma_o d\mu < +\infty$. Take $u = G\sigma - G\tau \in S$ with $u \geq f$ on K and $\mathrm{supp}(\tau) \subset K$. Then $G\tau \leq G\sigma + G\sigma_o$ on K and $G \sqsubset N$, we have $N\tau \leq N\sigma + N\sigma_o$. Consequently $p(f) \geq -\int N\sigma_o d\mu > -\infty$. Since the mapping $f \to p(f)$ is sublinear on $C(K)$, there exists a linear functional ν on $C(K)$ such that

$$\nu(f) \leq p(f) \quad \text{for all } f \in C(K).$$

If $f \leq 0$, it follows that $\nu(f) \leq p(f) \leq 0$. Hence ν is a positive measure on K. For each $\sigma \in M_K^+$, we have

$$\int G\sigma d\nu = \sup\{\int f d\nu;\ 0 \leq f \leq G\sigma \text{ on } K,\ f \in C(K)\}$$
$$\leq \sup\{p(f);\ 0 \leq f \leq G\sigma \text{ on } K,\ f \in C(K)\} \leq \int N\sigma d\mu.$$

Especially, $\int G\varepsilon_x d\nu \leq \int N\varepsilon_x d\mu$ and hence $\check{G}\nu(x) \leq \check{N}\mu(x)$ for each $x \in X$. For each $\tau \in \mathfrak{F}(G)$ with $\mathrm{supp}(\tau) \subset K$, we have

$$\int -G\tau d\nu \leq p(-G\tau) \leq \int -N\tau d\mu.$$

Thus $\int \check{G}\nu d\tau = \int \check{N}\mu d\tau$ for each $\tau \in \mathfrak{F}(G)$ with $\mathrm{supp}(\tau) \subset K$. Since G satisfies the continuity principle by the assumption, it follows that $G\nu = N\mu$ G-n.e. on K.

§3. Condenser-type theorems for disjoint compact sets.

Hereafter we assume that <u>G is a continuous function-kernel and N is a lower semicontinuous function-kernel which is locally bounded outside Δ. Further, assume that each non-empty open set is not G-negligible and not N-negligible.</u>

For each $x \in X$ and for a closed set K, we define

$$A(x) := \{y \in X; \quad b > 0,\ G(z, y) = bG(z, x) \text{ for all } z \in X\},$$

$$A(K) ;= \bigcup_{x \in K} A(x).$$

We shall consider the necessary and sufficient conditions in order that (G, N) has the following property:

(b) Let K be a non-G-negligible compact set, F be a compact set with $A(K) \cap F = \phi$ and λ be a measure in M_K^+ of which $N\lambda$ is bounded on K. Then there exists a (G, N)-condedser-type measure $\alpha = \mu_o - \mu_1$ of λ onto <K, F>, i.e., there exist $\mu_o \in \mathcal{L}(G)$ and $\mu_1 \in \mathcal{L}(G)$ such that

b_1) $\quad \text{supp}(\mu_o) \subset K, \quad \text{supp}(\mu_1) \subset F,$

b_2) $\quad G\alpha = N\lambda$ G-n.e. on K, $\quad G\alpha = 0$ G-n.e. on F,

b_3) $\quad 0 \leq G\alpha \leq N\lambda$ on X.

<u>Theorem 1</u>. Let N and G be continuous function-kernels. Then the following assertions (1), (2) are equivalent:

(1) (G, N) has the property (b),

(2) $G \prec G$ and $G \prec N$.

<u>Proof</u>. (1) → (2): Let K be a non-G-negligible compact set and λ be a measure in M_K^+ such that $N\lambda$ is bounded on K. Then there exists a (G, N)-condenser-type measure $\mu_o - \mu_1$ of λ onto <K, ϕ> by the assumption. Since $\mu_1 = 0$, μ_o is a relatively balayaged measure of λ onto K with respect to (G, N). Using this, we can show that G satisfies the elementary relative domination principle (cf. Proof of Lemma 2 in [3]). By Theorem 1 in [3], we have $G \prec N$. Next, we shall show that $G \prec G$. Assume that $\check{G}\mu \leq \check{G}\varepsilon_x$ on supp(μ), where $\mu \in M_K^+$ with $\int G\mu d\mu < +\infty$ and $x \in C(\text{supp}(\mu))$. If there is a point $x_1 \in X \setminus \text{supp}(\mu)$ such that $\check{G}\mu(x_1) > \check{G}\varepsilon_x(x_1)$, we can find a compact neighborhood K of x_1 such that $\check{G}\mu(y) > \check{G}\varepsilon_x(y)$ for all $y \in K$. Remark that $A(K) \cap \text{supp}(\mu)$

$= \phi$. By the assumption there exists a (G, N)-condedser-type measure $\mu_o - \mu_1$ of ε_z for $z \in \mathrm{supp}(\mu)$ onto <K, supp(μ)>. It is obvious that $\mu_o \neq 0$. We obtain

$$0 = \int (G\mu_o - G\mu_1) d\mu = \int \check{G}\mu d\mu_o - \int \check{G}\mu d\mu_1 > \int \check{G}\varepsilon_x d\mu_o - \int \check{G}\varepsilon_x d\mu_1$$
$$= G\mu_o(x) - G\mu_1(x) \geq 0.$$

This is a contradiction. Thus we have $\check{G}\mu \leq \check{G}\varepsilon_x$ on X. Since $G \prec N$ implies that G satisfies the continuity principle (cf. [3]). Consequently $\check{G}$ satisfies the domination principle (cf. [5, Theorem II.3]) and G does (cf. [2]).

(2) → (1): Since $G \prec G$, G satisfies the continuity principle. Let K be a non-G-negligible compact set, F be a compact set with $A(K) \cap F = \phi$ and λ be a measure in M_K^+ such that $N\lambda$ is bounded on K. The assumption $G \prec N$ implies $\check{G} [\check{N}$. By Proposition 2, G satisfies the relative balayage principle with respect to N. If ν_o is a relatively balayaged measure of λ onto K with respect to (G, N), $G\nu_o$ is bounded on K. Since there is $\alpha_o \in \mathcal{F}(G)$ satisfying $G\nu_o \leq G\alpha_o$ on X, $G\nu_o$ is locally bounded. Further we can choose successively a sequence $\{\nu_n\}$ of measures in $\mathcal{L}(G)$ with the following properties:

(i) $\mathrm{supp}(\nu_{2m}) \subset K$, $\mathrm{supp}(\nu_{2m+1}) \subset F$,

(ii) ν_{2m+1} is a G-balayaged measure of ν_{2m} onto F,

(iii) ν_{2m+2} is a G-balayaged measure of ν_{2m+1} onto K.

Then $\{G\nu_n\}$ is decreasing and $\lim_{m\to\infty} G\nu_{2m} = \lim_{m\to\infty} G\nu_{2m+1}$. Since $G \prec G$, we have $\check{G} \prec \check{G}$. Let β be an arbitrary measure in $\mathcal{F}(\check{G})$. Then we can find $\sigma \in \mathcal{L}(\check{G})$ and $\tau \in \mathcal{L}(\check{G})$ satisfying

$$\check{G}\sigma - \check{G}\tau = \check{G}\beta \text{ G-n.e. on K}, \quad \check{G}\sigma - \check{G}\tau = 0 \text{ G-n.e. on F},$$
$$0 \leq \check{G}\sigma - \check{G}\tau \leq \check{G}\beta \text{ G-n.e. on X (cf. [8])}.$$

Remarking that $\mathrm{supp}(\nu_{2m}) \subset K$ and $\mathrm{supp}(\nu_{2m+1}) \subset F$, we have

$$\int \lim_{m\to\infty} G\nu_{2m} d\beta = \lim_{m\to\infty} \int G\nu_{2m} d\beta = \lim_{m\to\infty} \int \check{G}\beta d\nu_{2m} = \lim_{m\to\infty} \int (\check{G}\sigma - \check{G}\tau) d\nu_{2m}$$

$$= \int (\lim_{m\to\infty} G\nu_{2m}) d\sigma - \int (\lim_{m\to\infty} G\nu_{2m}) d\tau$$

$$= \int (\lim_{m\to\infty} G\nu_{2m+1}) d\sigma - \int (\lim_{m\to\infty} G\nu_{2m+1}) d\tau$$

$$= \lim_{m\to\infty} \int (\check{G}\sigma - \check{G}\tau) d\nu_{2m+1} = 0.$$

Consequently $\lim_{m\to\infty} G\nu_{2m} = 0$ G-n.e. on X and hence $\lim_{n\to\infty} G\nu_n = 0$ G-n.e. on X. Thus we see that the alternative series $\sum_{n=0}^{\infty} (-1)^n G\nu_n$ converses G-n.e. on X. Put

$$g := \sum_{m=0}^{\infty} (G\nu_{2m} - G\nu_{2m+1}).$$

Since $G\nu_{2m} - G\nu_{2m+1} = 0$ G-n.e. on F and $G\nu_{2m+1} - G\nu_{2m+2} = 0$ G-n.e. on K, we have

$$(3.1) \quad \begin{aligned} & g = G\nu_o = N\lambda \text{ G-n.e. on K}, \quad g = 0 \text{ G-n.e. on F}, \\ & 0 \le g \le G\nu_o \le N\lambda \text{ on X} \end{aligned}$$

Further we can choose γ, $\delta \in \mathcal{L}(\check{G})$ satisfying

$$\check{G}\gamma - \check{G}\delta \ge 1 \text{ G-n.e. on K}, \quad \check{G}\gamma - \check{G}\delta = 0 \text{ G-n.e. on F},$$

$$0 \le \check{G}\gamma - \check{G}\delta \text{ G-n.e. on X}.$$

For any natural number p, it follows that

$$\sum_{m=0}^{p} \int d\nu_{2m} \le \sum_{m=0}^{p} \int (\check{G}\gamma - \check{G}\delta) d\nu_{2m} - \sum_{m=0}^{p} \int (\check{G}\gamma - \check{G}\delta) d\nu_{2m+1}$$

$$= \sum_{m=0}^{p} \int (G\nu_{2m} - G\nu_{2m+1}) d\gamma - \sum_{m=0}^{p} \int (G\nu_{2m} - G\nu_{2m+1}) d\delta$$

$$\le \int g d\gamma = \int G\nu_o d\gamma < +\infty$$

and hence $\sum_{m=0}^{\infty} \int d\nu_{2m} < +\infty$. Put $\mu_o = \sum_{m=0}^{\infty} \nu_{2m}$. Then μ_o is a positive measure supported by K. If we can choose $\beta_o \in \mathcal{L}(\check{G})$ satisfyimg $\check{G}\beta_o \ge 1$ on F, we have

$$\sum_{m=0}^{\infty} \int d\nu_{2m+1} \le \sum_{m=0}^{\infty} \int \check{G}\beta_o d\nu_{2m+1} = \sum_{m=0}^{\infty} \int_G G\nu_{2m+1} d\beta_o$$

$$\leq \sum_{m=0}^{\infty}\int G\nu_{2m}d\beta_o = \sum_{m=0}^{\infty}\int G\beta_o d\nu_{2m} < +\infty.$$

Consequently $\mu_1 := \sum_{m=0}^{\infty} \nu_{2m+1}$ is a positive measure supported by F. Obviously it follows that $\sum_{m=0}^{\infty} G\nu_{2m} = G\mu_o$, $\sum_{m=0}^{\infty} G\nu_{2m+1} = G\mu_1$ and $g = G\mu_o - G\mu_1$ on X. Since $G\mu_o \leq G\mu_1 + G\nu_o$ and $G\mu_1$ is a continuous on K, $G\mu_o$ is bounded on K and hence it is locally bounded on X. Using $G\mu_1 \leq G\mu_o$, $G\mu_1$ is also locally bounded on X. Therefore $\mu_o - \mu_1$ is a (G, N)-condenser-type measure of λ onto <K, F>. Thus Theorem 1 has been proved.

Putting N = G in Theorem 1, we obtain the following corollary.

Corollary 1. G satisfies the domination principle if and only if (G, G) has the property (b).

Definition 5. We say that G satisfies the condenser principle if for each non-G-negligible compact set K and each compact set F with $K \cap F = \phi$, there exists a measure $\alpha = \mu_o - \mu_1$ ($\mu_o \in \mathcal{L}(G)$, $\mu_1 \in \mathcal{L}(G)$) such that

$$\operatorname{supp}(\mu_o) \subset K \quad \text{and } \operatorname{supp}(\mu_1) \subset F,$$
$$G\alpha = 1 \ \text{ G-n.e. on K}, \quad G\alpha = 0 \ \text{ G-n.e. on F},$$
$$0 \leq G\alpha \leq 1 \quad \text{on X}.$$

If $A(x) = \{x\}$ for any $x \in X$, we say that G is non-degenerate. Putting N = 1 (the constant kernel) in Theorem 1, we have easily the following well-known result (cf. [6]).

Corollary 2. Let G be non-degenerate. Then G satisfies the condenser principle if and only if $G \prec G$ and $G \prec 1$.

Corollary 3. If $G \prec G$, $G \prec N$ and G is non-degenerate, then for any pair of a non-G-negligible compact set K and a

F with $K \cap F = 0$, there exists uniquely a measure $\alpha = \mu_o - \mu_1$ (μ_o, $\mu_1 \in \mathcal{L}(G)$) satisfying b_1), b_2), b_3) in Theorem 1.

<u>Proof</u>. Let K be a non-G-negligible compact set and F be a compact set with $K \cap F = \phi$. Since G is non-degenerate, we have $A(K) \cap F = \phi$. By Theorem 1 there exists a measure $\alpha = \mu_o - \mu_1$ (μ_o, $\mu_1 \in \mathcal{L}(G)$) satisfying b_1), b_2), b_3). Let $\beta = \nu_o - \nu_1$ (ν_o, $\nu_1 \in \mathcal{L}(G)$) be another measure satisfying b_1), b_2), b_3). Since $G \prec G$ and G is non-degenerate, $\check{G} \prec \check{G}$ and $\check{G}$ in also non-degenerate. By Corollary 1 there exists, for each $\gamma \in \mathcal{L}(\check{G})$, a measure $\delta = \sigma - \tau$ (σ, $\tau \in \mathcal{L}(\check{G})$) such that

$$\check{G}\delta = \check{G}\gamma \quad \text{G-n.e. on } K, \qquad \check{G}\delta = 0 \quad \text{G-n.e. on } F,$$

$$0 \le \check{G}\delta \le \check{G}\gamma \quad \text{on } X.$$

Then we have

$$\begin{aligned}\int \check{G}\gamma d\mu_o &= \int(\check{G}\sigma - \check{G}\tau)d\mu_o - \int(\check{G}\sigma - \check{G}\tau)d\mu_1 \\ &= \int(G\mu_o - G\mu_1)d\sigma - \int(G\mu_o - G\mu_1)d\tau \\ &= \int(G\nu_o - G\nu_1)d\sigma - \int(G\nu_o - G\nu_1)d\tau \\ &= \int(\check{G}\sigma - \check{G}\tau)d\nu_o = \int \check{G}\gamma d\nu_o.\end{aligned}$$

Let f be a non-negative continuous cunction on X with compact support. Since G is non-degenerate, we can find sequences $\{\sigma_n\}$, $\{\tau_n\}$ in $\mathcal{L}(\check{G})$ satisfying

$$0 \le \check{G}\sigma_n - \check{G}\tau_n \le \check{G}\lambda \quad \text{on X for some } \lambda \in \mathcal{F}(\check{G}),$$

$$\lim_{n\to\infty} (\check{G}\sigma_n - \check{G}\tau_n) = f \quad \text{G-n.e. on } X.$$

Consequently we have $\mu_o = \nu_o$. If F is non-G-negligible, we have $\mu_1 = \nu_1$ analogously. If F is G-negligible, it follows that $\mu_1 = \nu_1 = 0$. Thus we have $\mu_o = \nu_o$ and $\mu_1 = \nu_1$.

§4. (G, N)-condenser-type theorem for a pair <K, F> of a compact set K and a closed set F.

We say that (G, N) has the property (b') if the following property (b') is satisfied:

(b') Let K be a non-G-negligible compact set and F be a closed set with $A(K) \cap F = \phi$. Then for each $\lambda \in \mathcal{L}(N)$ there exists a measure $\alpha = \mu_o - \mu_1$ satisfying

b_1') $\mathrm{supp}(\mu_o) \subset K$, $\mathrm{supp}(\mu_1) \subset F$,

b_2') $\int G\mu_1 d\beta < +\infty$ for each $\beta \in \mathcal{L}(\check{G})$,

b_3') $\mu_o(B) = \mu_1(B) = 0$ for each G-negligible set B,

b_4') $G\alpha = N\lambda$ G-n.e. on K, $G\alpha = 0$ G-n.e. on F,

b_5') $0 \le G\alpha \le N\lambda$ on X if F is compact and $0 \le G\alpha \le N\lambda$ G-n.e. on X if F is not compact.

In this section we ask a necessary and sufficient condition in order that (G, N) has the property (b').

Theorem 2. Assume that $G \prec G$ and $\mathcal{L}(N) \neq \{0\}$. Then (G, N) has the property (b') if and only if (G, N) has the following three properties (c), (d), (e):

(c) Let K be a non-G-negligible compact set. Then for every $\mu \in \mathcal{L}(N)$ there exists a measure $\nu \in M_K^+$ such that

$$\mathrm{supp}(\nu) \subset K, \quad G\nu = N\mu \text{ G-n.e. on } K, \quad G\nu \le N\mu \text{ on } X.$$

(d) Any $\mu \in \mathcal{L}(G)$ is balayable onto any closed set F, i.e., there exists a positive measure ν such that

$$G\nu = G\mu \text{ G-n.e. on } F, \quad G\nu \le G\mu \text{ on } X.$$

(e) Let K be a non-G-negligible compact set and F be a closed set with $A(K) \cap F = \phi$. Then there exist sequences $\{\sigma_n\}$, $\{\tau_n\}$ (σ_n, τ_n

$\in \mathcal{L}(\check{G})$) satisfying

e_1) $\lim_{n\to\infty} (\check{G}\sigma_n - \check{G}\tau_n) = 0$ G-n.e. on F,

$\lim_{n\to\infty} (\check{G}\sigma_n - \check{G}\tau_n) \geq 1$ G-n.e. on K,

e_2) $0 \leq \lim_{n\to\infty} (\check{G}\sigma_n - \check{G}\tau_n) \leq \check{G}\eta$ G-n.e. on X for some $\eta \in \mathcal{L}(\check{G})$,

e_3) $\check{G}\sigma_n \leq \check{G}\sigma_{n+1}$ and $\check{G}\tau_n \leq \check{G}\tau_{n+1}$ for each natural number n,

e_4) $\liminf_{n\to\infty} \int \check{N}\sigma_n d\beta < +\infty$ and $\liminf_{n\to\infty} \int \check{N}\tau_n d\beta < +\infty$ for each $\beta \in \mathcal{L}(\check{N})$.

<u>Proof</u>. Suppose that (G, N) has the property (b'). First, we shall show that (G, N) has the property (c). Let K be a non-G-negligible set and μ be a measure in $\mathcal{L}(N)$. Then there exists a (G, N)-condenser-type measure $\alpha = \mu_o - \mu_1$ of μ onto $<K, \phi>$. Since $\mu_1 = 0$, we have

$$G\mu_o = N\mu \quad \text{G-n.e. on K}, \quad G\mu_o \leq N\mu \quad \text{on X}.$$

Secondly, we shall show that any measure in $\mathcal{L}(G)$ is balayable onto any closed set F. For this purpose it is sufficient to prove that for each closed set F

(4.1) $\lim_{n\to\infty} (\check{G}\sigma_n - \check{G}\tau_n) = 0$ G-n.e. on F implies $\lim_{n\to\infty} (\check{G}\sigma_n - \check{G}\tau_n)=0$ G-n.e. on X, where $\{\sigma_n\}, \{\tau_n\} \subset \mathcal{L}(\check{G})$ satisfying

$\mathrm{supp}(\sigma_n) \subset F$, $\mathrm{supp}(\tau_n) \subset F$,

$0 \leq \check{G}\sigma_n - \check{G}\tau_n \leq \check{G}\gamma$ G-n.e. F for some $\gamma \in \mathcal{L}(\check{G})$,

$\check{G}\sigma_{n+1} - \check{G}\tau_{n+1} \leq \check{G}\sigma_n - \check{G}\tau_n$ G-n.e. on F (cf. [8, Theorem 1]).

To show (4.1), we suppose that it does not hold that $\lim_{n\to\infty} (\check{G}\sigma_n - \check{G}\tau_n) = 0$ G-n.e. on X. Since $0 \leq \check{G}\sigma_n - \check{G}\tau_n$ on X, there exists a non-G-negligible set contained in $\{x \in CF;\ \lim_{n\to\infty} (\check{G}\sigma_n(x) - \check{G}\tau_n(x)) > 0\}$. We remark that $A(K) \cap F = \phi$. By the assumption there exists, for $\lambda \in \mathcal{L}(N)$ $(\lambda \neq 0)$, a measure $\mu_o - \mu_1$ satisfying $b_1') \sim b_5')$. Since $\lambda \neq 0$, we have $\mu_o \neq 0$. Consequently

$$0 = \lim_{n\to\infty}\{\int(G\mu_o - G\mu_1)d\sigma_n - \int(G\mu_o - G\mu_1)d\tau_n\}$$
$$= \int\lim_{n\to\infty}(\check{G}\sigma_n - \check{G}\tau_n)d\mu_o - \lim_{n\to\infty}(\check{G}\sigma_n - \check{G}\tau_n)d\mu_1$$
$$= \int\lim_{n\to\infty}(\check{G}\sigma_n - \check{G}\tau_n)d\mu_o > 0.$$

This is a contradiction. Thus $\lim_{n\to\infty}(\check{G}\sigma_n - \check{G}\tau_n) = 0$ G-n.e. on F implies $\lim_{n\to\infty}(\check{G}\sigma_n - \check{G}\tau_n) = 0$ G-n.e. on X. Finally, we shall show that (G, N) has the property (e). Let $\{K_n\}$ be an increasing sequence of compact sets with $\bigcup_{n=1}^{\infty} K_n = X$. Suppose that K is a non-G-negligible compact set and F be a closed set with $A(K) \cap F = \phi$. Put $F_n := F \cap K_n$. We can find $\lambda_o \in \mathcal{L}(\check{G})$ satisfying $\check{G}\lambda_o \geq 1$ on K. Since $\check{G} \prec \check{G}$, there exists, by Corollary 1, $\delta_n = \alpha_n - \beta_n$ such that

$$\mathrm{supp}(\alpha_n) \subset K, \quad \mathrm{supp}(\beta_n) \subset F_n,$$
$$\check{G}\delta_n = \check{G}\lambda_o \quad \text{G-n.e. on } K, \quad \check{G}\delta_n = 0 \quad \text{G-n.e. on } F_n,$$
$$0 \leq \check{G}\delta_n \leq \check{G}\lambda_o \quad \text{on } X.$$

Since $\check{G}\alpha_n - \check{G}\beta_n \leq \check{G}\alpha_{n-1} - \check{G}\beta_{n-1}$ on $K \cup F_n$, it follows that $\check{G}\alpha_n - \check{G}\beta_n \leq \check{G}\alpha_{n-1} - \check{G}\beta_{n-1}$ on X. We remark that $\check{G}\alpha_n \leq \check{G}\alpha_{n+1}$ on X. In fact, there exists, for each $\eta \in \mathcal{L}(G)$, a measure $\sigma_n - \tau_n$ $(\sigma_n, \tau_n \in \mathcal{L}(G))$ satisfying

$$\mathrm{supp}(\sigma_n) \subset K, \quad \mathrm{supp}(\tau_n) \subset F_n,$$
$$G\sigma_n - G\tau_n = G\eta \quad \text{G-n.e. on } K, \quad G\sigma_n - G\tau_n = 0 \quad \text{G.n.e. on } F_n$$
$$0 \leq G\sigma_n - G\tau_n \leq G\eta.$$

Then we have

$$\int\check{G}\alpha_n d\eta = \int G\eta d\alpha_n$$
$$= \int(G\sigma_n - G\tau_n)d\alpha_n - \int(G\sigma_n - G\tau_n)d\beta_n$$
$$= \int(\check{G}\alpha_n - \check{G}\beta_n)d\sigma_n - \int(\check{G}\alpha_n - \check{G}\beta_n)d\tau_n$$
$$\leq \int(\check{G}\alpha_{n+1} - \check{G}\beta_{n+1})d\sigma_n - \int(\check{G}\alpha_{n+1} - \check{G}\beta_{n+1})d\tau_n$$
$$= \int(G\sigma_n - G\tau_n)d\alpha_{n+1} - \int(G\sigma_n - G\tau_n)d\beta_{n+1}$$

$$\leq \int G\eta d\alpha_{n+1} = \int \check{G}\alpha_{n+1} d\eta .$$

Consequently $\check{G}\alpha_n \leq \check{G}\alpha_{n+1}$ G-n.e. on X and hence $\check{G}\alpha_n \leq \check{G}\alpha_{n+1}$. Similarily we have $\check{G}\beta_n \leq \check{G}\beta_{n+1}$ on X. Let λ be an arbitrary measure in $\mathcal{L}(N)$. By the assumption we can find $\alpha = \mu_o - \mu_1$ satisfying $b_1') \sim b_5')$. Then we have

$$+\infty > \int \check{G}\lambda_o d\mu_o \geq \int \lim_{n\to\infty} (\check{G}\alpha_n - \check{G}\beta_n) d\mu_o - \int \lim_{n\to\infty} (\check{G}\alpha_n - \check{G}\beta_n) d\mu_1$$

$$= \lim_{n\to\infty} \{ \int (G\mu_o - G\mu_1) d\alpha_n - \int (G\mu_o - G\mu_1) d\beta_n \}$$

$$= \lim_{n\to\infty} \int N\lambda d\alpha_n = \lim_{n\to\infty} \int \check{N}\alpha_n d\lambda .$$

Since $\check{G}\beta_n \leq \check{G}\alpha_n$ on X and $\alpha_n, \beta_n \in \mathcal{L}(\check{G})$, we have, using (c),

$$\int \check{N}\beta_n d\lambda \leq \int \check{N}\alpha_n d\lambda \quad \text{for all } \lambda \in \mathcal{L}(N).$$

Consequently $\liminf_{n\to\infty} \int \check{N}\beta_n d\lambda \leq \lim_{n\to\infty} \int \check{N}\alpha_n d\lambda < +\infty$.

Conversely, suppose that (G, N) has the properties (c), (d), (e). Let K be a non-G-negligible compact set and F be a closed set with $A(K) \cap F = \phi$ and λ be a measure in $\mathcal{L}(N)$. We denote by ν_o a relatively balayaged measure of λ onto K with respect to (G, N). Since G has the property (d), we can choose a sequence $\{\nu_n\}$ of positive measures satisfying (i) $\sim$ (iii) in Theorem 1. Obviously $\{G\nu_n\}$ is decreasing. We shall show that $\lim_{n\to\infty} G\nu_n = 0$ G-n.e. on X. Let β be an arbitrary measure in $\mathcal{L}(\check{G})$ and b be a positive real number satisfying $b\check{G}\beta \leq 1$ on K. Using the property (e), for $u = \lim_{n\to\infty} \check{G}\sigma_n$, $v = \lim_{n\to\infty} \check{G}\tau_n$ in (e) we have

$$\int u d\nu_n = \lim_{p\to\infty} \int \check{G}\sigma_p d\nu_n = \lim_{p\to\infty} \int G\nu_n d\sigma_p$$

$$\leq \liminf_{p\to\infty} \int N\lambda d\sigma_p = \liminf_{p\to\infty} \int \check{N}\sigma_p d\lambda < +\infty .$$

Since the sequence $\{\int u d\nu_n\}$ is decreasing, it follows that

$$\lim_{m\to\infty}\int u\,d\nu_{2m} = \lim_{m\to\infty}\int u\,d\nu_{2m+1} < +\infty.$$

Analogously

$$\lim_{m\to\infty}\int v\,d\nu_{2m} = \lim_{m\to\infty}\int v\,d\nu_{2m+1} < +\infty.$$

Using these relations, we obtain

$$\begin{aligned} b\int \lim_{m\to\infty} G\nu_{2m}d\beta &= \lim_{m\to\infty}\int bG\nu_{2m}d\beta = \lim_{m\to\infty}\int b\check{G}\beta d\nu_{2m} \\ &\le \lim_{m\to\infty}\int \lim_{p\to\infty}(\check{G}\sigma_p - \check{G}\tau_p)d\nu_{2m} \\ &= \lim_{m\to\infty}\int u\,d\nu_{2m} - \lim_{m\to\infty}\int v\,d\nu_{2m} \\ &= \lim_{m\to\infty}\int u\,d\nu_{2m+1} - \lim_{m\to\infty}\int v\,d\nu_{2m+1} \\ &= \lim_{m\to\infty}\int \lim_{p\to\infty}(\check{G}\sigma_p - \check{G}\tau_p)d\nu_{2m+1} = 0. \end{aligned}$$

Consequently $\int \lim_{m\to\infty} G\nu_{2m}d\beta = 0$ for all $\beta \in \mathcal{L}(\check{G})$. Thus we have $\lim_{m\to\infty} G\nu_{2m} = 0$ G-n.e. on X and hence $\lim_{n\to\infty} G\nu_n = 0$ G-n.e. on X. Thus we see that the alternative series $\sum_{n=0}^{\infty}(-1)^n G\nu_n$ converges G-n.e. on X. Put

$$g := \sum_{m=0}^{\infty}(G\nu_{2m} - G\nu_{2m+1}).$$

Then g has the property (3.1). Next we shall show that $\sum_{m=0}^{\infty}\nu_{2m}$ is a positive measure. For any natural number q we have

$$\begin{aligned} \sum_{m=0}^{q}\int d\nu_{2m} &\le \sum_{m=0}^{q}\int \lim_{p\to\infty}(\check{G}\sigma_p - \check{G}\tau_p)d\nu_{2m} - \sum_{m=0}^{q}\int \lim_{p\to\infty}(\check{G}\sigma_p - \check{G}\tau_p)d\nu_{2m+1} \\ &= \lim_{p\to\infty}\sum_{m=0}^{q}\int (G\nu_{2m} - G\nu_{2m+1})d\sigma_p \\ &\quad - \lim_{p\to\infty}\sum_{m=0}^{q}\int (G\nu_{2m} - G\nu_{2m+1})d\tau_p \\ &\le \liminf_{p\to\infty}\int g\,d\sigma_p \le \liminf_{p\to\infty}\int N\lambda\,d\sigma_p < +\infty. \end{aligned}$$

Consequently $\sum_{m=0}^{\infty} \int d\nu_{2m} \le \int \liminf_{p\to\infty} \int N\lambda d\sigma_p < +\infty$ and hence $\mu_o := \sum_{m=0}^{\infty} \nu_{2m}$ is a positive measure supported by K. Let f be a non-negative continuous function on X with compact support and η be a measure in $\mathcal{F}(\check{G})$ with $\check{G}\eta \ge f$ on X. Then

$$\int f d\nu_{2m+1} \le \int \check{G}\eta d\nu_{2m+1} = \int G\nu_{2m+1} d\eta \le \int G\nu_{2m} d\eta = \int \check{G}\eta d\nu_{2m}.$$

Hence

$$\sum_{m=0}^{\infty} \int f d\nu_{2m+1} \le \sum_{m=0}^{\infty} \int \check{G}\eta d\nu_{2m} < +\infty.$$

This implies that the positive linear functional: $f \mapsto \sum_{m=0}^{\infty} \int f d\nu_{2m+1}$ on the space of continuous functions on X with compact support is a positive measure μ_1 supported by F. Obviously $\sum_{m=0}^{\infty} G\nu_{2m} = G\mu_o$ and $\sum_{m=0}^{\infty} G\nu_{2m+1} = G\mu_1$. For any $\beta \in \mathcal{L}(\check{G})$ we obtain

$$\int G\mu_1 d\beta = \sum_{m=0}^{\infty} \int G\nu_{2m+1} d\beta \le \sum_{m=0}^{\infty} \int G\nu_{2m} d\beta \le \sum_{m=0}^{\infty} \int \check{G}\beta d\nu_{2m} < +\infty.$$

Thus both $G\mu_1$ and $G\mu_o$ are finite on G-n.e. on X and it follows that $g = G\mu_o - G\mu_1$ G-n.e. on X. We remark that $g = G\mu_o - G\mu_1$ on X if F is compact. Therefore $\mu_o - \mu_1$ is a (G, N)-condenser-type measure of λ onto <K, F>. Thus Theorem 2 has been proved.

We define

$$S(\check{G}) := \{u = \lim_{n\to\infty} \check{G}\sigma_n \ ; \ \sigma_n \in \mathcal{L}(\check{G}),\ \check{G}\sigma_n \le \check{G}\sigma_{n+1},\ \int u d\beta < +\infty \text{ for all } \beta \in \mathcal{F}(G)\}.$$

Putting G = N, we have easily the following corollary.

<u>Corollary 4</u>. Assume that $G \prec G$. Then (G, G) has the property (b') if and only if G has the following properties (d), (e'):

(d) Any $\mu \in \mathcal{L}(G)$ is balayable onto any closed set F,

(e') Let K be a non-G-negligible compact set and F be a closed set with $A(K) \cap F = \emptyset$. Then there exist u, v $\in S(\check{G})$ such that

$u - v \geq 1$ G-n.(

$0 \leq u - v \leq \check{G}\eta$ G-ı

<u>Proposition 3</u>. Assume that G is non-degenerate and $G \prec G$. Further assume that G has the following property (f):

(f) For any $\lambda \in \mathcal{F}(\check{G})$, for any $\varepsilon > 0$ and for any compact set K there exist $u, v \in S(\check{G})$ and a compact set K' such that

$$v = \lim_{n\to\infty} \check{G}\tau_n, \quad \mathrm{supp}(\tau_n) \subset K',$$

$$(4.2) \qquad u - v \leq \varepsilon \ \text{G-n.e. on } K, \quad u - v \geq \check{G}\lambda \ \text{G-n.e. on } CK',$$

$$0 \leq u - v \ \text{G-n.e. on } X.$$

Then (G, G) has the property (b').

<u>Proof</u>. First we remark that we can choose $u, v \in S(\check{G})$ in (f) satisfying (4.2) and

$$u - v = \check{G}\lambda \ \text{G-n.e. on } CK', \quad 0 \leq u - v \leq \check{G}\lambda \ \text{G-n.e. on } X.$$

In fact, since $(u - v) \wedge \check{G}\lambda = u \wedge (\check{G}\lambda + v) - v$ and $u \wedge (\check{G}\lambda + v)$ $S(\check{G})$, we can take $(u - v) \wedge \check{G}\lambda$ instead of $u - v$. By Proposition 4 in [8] any $\mu \in \mathcal{L}(G)$ is balayable to any closed set. Further, let K be a non-G-negligible compact set and F be a closed set with $K \cap F = \phi$. Choose $\lambda_o \in \mathcal{F}(\check{G})$ satisfying $\check{G}\lambda_o \geq 2$ on K. By the assumption (f) there exist $u, v \in S(\check{G})$ and a compact set K' with $K \subset K'$ such that

$$u - v = \check{G}\lambda_o \ \text{G-n.e. on } CK', \quad u - v \leq 1 \ \text{G-n.e. on } K,$$

$$0 \leq u - v \leq \check{G}\lambda_o \ \text{G-n.e. on } X.$$

Then we obtain, putting $w_1 := \check{G}\lambda_o + v - u$,

$$w_1 \geq 1 \ \text{G-n.e. on } K, \quad w_1 = 0 \ \text{G-n.e. on } CK',$$

$$0 \leq w_1 \leq \check{G}\lambda_o \ \text{G-n.e. on } X.$$

Since $(F \cap K') \cap K = \phi$ and $\check{G} \prec \check{G}$, it follows from Corollar there exists $w_2 = \check{G}\alpha - \check{G}\beta$ $(\alpha, \beta \in \mathcal{L}(\check{G}))$ such that

$$w_2 = \frac{1}{2}\check{G}\lambda_o \geq 1 \quad \text{G-n.e. on } K, \qquad w_2 = 0 \quad \text{G-n.e. on } F \cap K',$$
$$0 \leq w_2 \leq \frac{1}{2}\check{G}\lambda_o \quad \text{G-n.e. on } X.$$

We can write $w_1 \wedge w_2 = u_1 - u_2$ $(u_1, u_2 \in S(\check{G}))$ G-n.e. on X and obtain

$$u_1 - u_2 \geq 1 \quad \text{G-n.e. on } K, \qquad u_1 - u_2 = 0 \quad \text{G-n.e. on } F,$$
$$0 \leq u_1 - u_2 \leq \frac{1}{2}\check{G}\lambda_o \quad \text{G-n.e. on } X.$$

Thus (G, G) has the property (b') by Corollary 4.

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