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Basic Representations of Extended Affine Lie Algebras

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1. As is well known, an affine Lie algebra $\underline{g}(A)$ has two kinds of natural realizations in terms of an underlying finite-dimensional simple Lie algebra. I would like to start my talk with a short sketch of them following [7]. Let $A = (a_{ij})_{i,j=0,\ldots,\ell}$ be a generalized Cartan matrix of type $X_N^{(k)}$, and $\mathfrak{k}$ the complex finite-dimensional simple Lie algebra of type X_N, where X is A, B, C,..., or G. Let μ be the automorphism of $\mathfrak{k}$, which permutes the Chevalley generators of $\mathfrak{k}$ according to an automorphism of order k of its Dynkin diagram. Set $\varepsilon = \exp(2\pi i/k)$ and denote by $\mathfrak{k}_j(\mu)$ the eigenspace of μ with eigenvalue ε^j. Then $\sum_{j\in\mathbb{Z}} t^j \otimes \mathfrak{k}_j(\mu) + \mathbb{C}c + \mathbb{C}d$ is an affine Lie algebra associated to A, where c is a generator of the center and $d = t\frac{d}{dt}$.

The another realization is constructed in the following way. Let σ be a Coxeter transformation on $\mathfrak{k}$ which commutes with μ, and h be the order of σ. Then each eigenvalue of $\sigma\mu$ has the form ω^j, where $\omega = \exp(2\pi i/kh)$. Set

$$\mathfrak{h}_j(\mu,\sigma) = \{ X \in \mathfrak{h} \ ; \ \sigma\mu X = \omega^j X \}$$

for $j \in \mathbb{Z}$. Then one has a so-called "principal realization"

$$\underline{g}(A) = \sum_{j \in \mathbb{Z}} t^j \otimes \mathfrak{h}_j(\mu,\sigma) + \mathbb{C}c + \mathbb{C}d_0 .$$

Note that under these two realizations d does not correspond to d_0, because $\langle d, \alpha_i \rangle = \delta_{0,i}$ and $\langle d_0, \alpha_i \rangle = 1$ for every simple root α_i.

2. Associated to either of these two realizations, an affine Lie algebra $\underline{g}(A)$ admits an extension

$$\tilde{\underline{g}}_{J'}(A) = \sum_{j \in \mathbb{Z}} t^j \otimes \mathfrak{h}_j(\mu) + \mathbb{C}c + \sum_{n \in k\mathbb{Z}} \mathbb{C}d'_n$$

or

$$\tilde{\underline{g}}_{J}(A) = \sum_{j \in \mathbb{Z}} t^j \otimes \mathfrak{h}_j(\mu,\sigma) + \mathbb{C}c + \sum_{n \in \mathbb{Z}} \mathbb{C}d_n ,$$

where $d'_n = t^{n+1}\frac{d}{dt}$ and $d_n = t^{nkh+1}\frac{d}{dt}$. Functions J' and J define the Lie brackets

$$[d'_m, d'_n] = (n-m)\, d'_{m+n} + J'(m)\,\delta_{m,-n}\, c ,$$

$$[d_m, d_n] = kh(n-m)\, d_{m+n} + J(m)\,\delta_{m,-n}\, c .$$

It is known that if J (resp. J') is not trivial the subalgebra $\mathcal{V} = \sum_{n \in \mathbb{Z}} \mathbb{C}d_n + \mathbb{C}c$ (resp. $\mathcal{V}' = \sum_{n \in k\mathbb{Z}} \mathbb{C}d'_n + \mathbb{C}c$) is isomorphic to the Virasoro algebra.

In this talk, we shall call $\underline{\tilde{g}}'_{J'}(A)$ (resp. $\underline{\tilde{g}}_J(A)$) the extension of the 1 st (resp. 2 nd) type.

Note that there is a remarkable difference between these two extensions. Denote by $\underline{\mathring{g}}$ the finite-dimensional subalgebra of $\underline{g}(A)$ generated by $\{e_i, f_i; i=1,\ldots,\ell\}$. Then in the 1st extension $\underline{\mathring{g}}$ commutes with $\mathcal{V}'$, while in the 2nd extension the centralizer $Z_{\underline{\tilde{g}}_J(A)}(\mathcal{V})$ of $\mathcal{V}$ in $\underline{\tilde{g}}_J(A)$ coincides with $\mathbb{C}c + \underline{\mathring{h}}$, where $\underline{\mathring{h}} = \underline{\mathring{g}} \cap \underline{h}$ is a Cartan subalgebra of $\underline{\mathring{g}}$.

The 1st type extension may be said to be the picture of G. Segal [10] and I. B. Frenkel [2][3]. If $k = 1$ and $J'(m) = J'_0(m) = (m^3-m)/12$, then $\mathcal{V}'$ is just the standard form of the Virasoro algebra. We set $\underline{\tilde{g}}'(A) = \underline{\tilde{g}}'_{J'_0}(A)$. The basic representation of an extended affine Lie algebra $\underline{\tilde{g}}'(A)$ was first discovered by G. Segal [10] in case of $A = A_1^{(1)}$, and the theory has been developed by I. B. Frenkel in case of $k = 1$ and $J' = J'_0$.

It seems to me that there exists no isomorphism between $\underline{\tilde{g}}'_{J'}(A)$ and $\underline{\tilde{g}}_J(A)$.

Now I want to consider representations of the 2nd type extensions. My conjecture is that "for any dominant integral form $\Lambda \in P_+$, there will exist a unique central extension J, such that the action of $\underline{g}(A)$ on $L(\Lambda)$ can be extended to that of $\underline{\tilde{g}}_J(A)$."

At present, I can verify this conjecture only for a few special cases: $\Lambda = \Lambda_0$ and $A = A_1^{(1)}$, $A_2^{(1)}$ or $A_2^{(2)}$.

3. In case of the basic representation of $A_1^{(1)}$, the whole story is most simple and most beautiful. So, from now on, let me restrict my talk on $A = A_1^{(1)}$ and $\Lambda = \Lambda_0$. It is well known that the basic representation $L(\Lambda_0)$ of $A_1^{(1)}$ is constructed on the space $\mathbb{C}[x_1, x_3, x_5, \ldots]$ of polynomial functions in x_i's with odd indices. We can prove that $\underline{\tilde{g}}_J(A_1^{(1)})$ acts on $L(\Lambda_0)$ if and only if $J(m) = m(2m^2+1)/6$, and that the action is given by

$$\pi \; : -d_n \longmapsto L_{2n} = \frac{1}{2} \sum_{\substack{j \in \mathbf{Z} \\ j = \text{odd}}} : a_j a_{2n-j} : \quad ,$$

where $a_j = \dfrac{\partial}{\partial x_j}$ and $a_{-j} = j x_j$ for a positive odd integer j, and : : denotes the normal product. These operators $\{ L_{2n} \}_{n \in \mathbf{Z}}$ are called the Virasoro operators.

Now look at the weights diagram of $L(\Lambda_0)$. It is known that the set $P(\Lambda_0)$ of all weights of $L(\Lambda_0)$ is given by

$$P(\Lambda_0) = \left\{ \Lambda_0 + q\delta + p\alpha_1 \; ; \; \begin{array}{l} p, q \in \mathbf{Z} \\ q \leqq -p^2 \end{array} \right\} \quad ,$$

and the set of all maximal weights is given by

$$\mathrm{Max}(\Lambda_0) = W \cdot \Lambda_0 = \left\{ \Lambda_0 + q\delta + p\alpha_1 \; ; \; \begin{array}{l} p, q \in \mathbf{Z} \\ q = -p^2 \end{array} \right\} ,$$

where W is the Weyl group and $\delta = \alpha_0 + \alpha_1$ is the fundamental imaginary root.

The multiplicity of each weight can be calculated by the Weyl-Kac character formula, and one has

$$\mathrm{Mult}_{\Lambda_0}(\lambda - n\delta) = p(n) \quad ,$$

where λ is a maximal weight and $p(n)$ is the partition number, i.e.,

$$\varphi(x)^{-1} = \prod_{j=1}^{\infty}(1-x^j)^{-1} = \sum_{n=0} p(n)x^n \quad .$$

Now we are interested in the problem to write down functions in each weight space explicitly. It is easily seen that a function in a weight space $L(\Lambda_0)_{\Lambda_0+q\delta+p\alpha_1}$ has the degree $-(2q + p)$, where the degree of a function is counted with the rule that the degree of x_j is equal to j. Consider the action of the Virasoro algebra $\mathcal{V}$. From $[d_0, d_n] = 2nd_n$, one sees that the operator L_{2n} maps a weight space $L(\Lambda_0)_\lambda$ to $L(\Lambda_0)_{\lambda+n\delta}$;

$$L_{2n} \;:\; L(\Lambda_0)_\lambda \longrightarrow L(\Lambda_0)_{\lambda+n\delta} \quad .$$

So $\sum_{n\in\mathbb{Z}} L(\Lambda_0)_{\lambda+n\delta}$ is stable under the action of $\mathcal{V}$, and by counting the multiplicity, we see that $\sum_{n\in\mathbb{Z}} L(\Lambda_0)_{\lambda+n\delta}$ is an irreducible $\mathcal{V}$- module. So each maximal weight vector f is described as a solution of the system of linear differential equations

$$L_{2n}\, f = 0 \qquad \text{for } n \geqq 1 \;,$$

which are equivalent to

$$L_2\, f = 0 \quad \text{and} \quad L_4\, f = 0 \;.$$

It is known from the Sato's theory that maximal weight vectors cover the homogeneous polynomial solutions of the KdV-hierarchies. So we obtain the following theorem:

Theorem. A homogeneous polynomial f in $\mathbb{C}[x_1, x_3, x_5, \ldots]$ is a solution of the KdV-hierarchies if and only if $L_2 f = 0$ and $L_4 f = 0$, where

$$L_2 = \frac{1}{2}\left(\frac{\partial}{\partial x_1}\right)^2 + \sum_{\substack{j=1 \\ j=\mathrm{odd}}}^{\infty} j x_j \frac{\partial}{\partial x_{j+2}}$$

$$L_4 = \frac{\partial^2}{\partial x_1 \, \partial x_3} + \sum_{\substack{j=1 \\ j=\mathrm{odd}}}^{\infty} j x_j \frac{\partial}{\partial x_{j+4}} .$$

Owing to the $\mathcal{V}$-irreducibility of $\sum_{n \in \mathbb{Z}} L(\Lambda_0)_{\lambda + n\delta}$, all vectors in other weight spaces are obtained by iterated operations of L_{2n}'s $(n < 0)$ to those maximal weight vectors.

4. I want to point out here that in the Segal's picture or in the Frenkel's picture, $\sum_{n \in \mathbb{Z}} L(\Lambda_0)_{\lambda + n\delta}$ can never be expected to be irreducible under the action of $\mathcal{V}'$, because $\mathcal{V}'$ commutes with $\mathring{\underline{g}}$. Let me explain about this more precisely. For a maximal weight λ, we set

$$V_\lambda = \sum_{n=0}^{\infty} L(\Lambda_0)_{\lambda - n\delta} .$$

Next decompose $L(\Lambda_0)$ under the action of $\underline{\mathring{g}} = \underline{sl}(2, \mathbb{C})$, and for a positive odd integer n let W_n denote the sum of all n-dimensional irreducible $\underline{\mathring{g}}$-submodules of $L(\Lambda_0)$. Then it is easily seen from the weights diagram that for every $\lambda \in \mathrm{Max}(\Lambda_0)$ there exists a positive integer n_λ satisfying $V_\lambda \cap W_n \neq \{0\}$ for any odd integer n larger than n_λ. As to the decomposition $V_\lambda = \sum_{n \geqq 1} (V_\lambda \cap W_n)$ of V_λ, each $V_\lambda \cap W_n$ is stable under the action of $\mathcal{V}'$, because of $[\mathcal{V}', \underline{\mathring{g}}] = 0$. Thus we have proved that in the extension of the 1st type, each V_λ is not $\mathcal{V}'$-irreducible and so maximal weight vectors cannot be characterized as the highest weight vectors with respect to $\mathcal{V}'$.

5. The representation theory of extended affine Lie algebras suggests or induces some problems. Consider the group orbit through the highest weight vector 1 in the completion of $L(\Lambda_0)$. It is well known that $G(A)\cdot 1$ coincides with the set of τ-functions of the KdV-equation. What is the differential equation which characterizes $G(A)$-orbit through 1 ? Or, take a subgroup $G' \subset \tilde{G}(A)$. What is the differential equation characterizing G'-orbit through 1 ? Now I want to point out an interesting fact; take a subalgebra $\underline{g}' = \underline{s} + \mathcal{V}$, where $\underline{s}$ is the principal subalgebra, then we have

$$f \in G' \cdot 1 \Longrightarrow f \otimes f \in \underline{g}'\text{-highest component in } L(\Lambda_0) \otimes L(\Lambda_0)$$

$$\Longleftrightarrow \langle u, f \otimes f \rangle = 0 \quad \text{for every } u \text{ in lower } \underline{g}'\text{-components}$$

$$\Longleftrightarrow D(f \circ f) = 0 \quad \text{for } {}^{\forall}D = \text{Hirota's bilinear differential operator in (non-reduced) BKP-hierarchies.}$$

Thus the G'-orbit is related to the non-reduced BKP-hierarchies.

According to the theory of Sato, Kashiwara, Miwa, Jimbo and Date, solutions of the BKP-equations are described by the orbit of $O(\infty)$. I cannot tell why BKP-hierarchies appear within the framework of $A_1^{(1)}$. I want to conclude this section by noticing a fact which may be related to the above phenomenon. Consider the natural inclusions $A_1^{(1)} \subset D_4^{(2)}$ and $D_n^{(2)} \subset D_{2n}^{(2)}$ for $n \in \mathbb{N}$. Then the basic representation of $D_4^{(2)}$ is irreducible when restricted to the subalgebra $A_1^{(1)}$, and it is nothing but the basic representation of $A_1^{(1)}$; i.e.,

$$L(\Lambda_0;\ A_1^{(1)}) = L(\Lambda_0;\ D_4^{(2)}) \ .$$

In a similar way, one has

$$L(\Lambda_0;\ D_n^{(2)}) = L(\Lambda_0;\ D_{2n}^{(2)}) \ .$$

So, under the sequence of inclusions

$$A_1^{(1)} \subset D_4^{(2)} \subset D_8^{(2)} \subset \cdots \subset D_{2^n}^{(2)} \subset \cdots \subset D_{2^\infty}^{(2)} ,$$

one has

$$L(\Lambda_0;\ A_1^{(1)}) = L(\Lambda_0;\ D_4^{(2)}) = L(\Lambda_0;\ D_8^{(2)})$$
$$= \ldots\ldots\ldots\ldots = L(\Lambda_0;\ D_{2^\infty}^{(2)}) \quad .$$

6. In this section, we take $V = \mathbb{C}[x_1, x_2, x_3, \ldots]$ and set

$$a_k = \frac{\partial}{\partial x_k} , \qquad a_{-k} = k x_k \qquad (\text{for } k \in \mathbb{N}),$$

$$a_0 = 0$$

and $$L_n = \frac{1}{2} \sum_{k \in \mathbb{Z}} : a_k a_{n-k} : \qquad (\text{for } n \in \mathbb{N}).$$

We denote by $\mathfrak{g}$ the Lie algebra spanned by $\{1, a_k, L_k; k \in \mathbb{Z}\}$, and consider the representation of $\mathfrak{g}$ on V. Then we can prove that

$f \in H\cdot 1 \implies f \otimes f \in$ the highest component of $V \otimes V$

$\iff \langle u, f \otimes f \rangle = 0$ for every u in lower components

$\iff D(f \circ f) = 0$ for $^\forall D$ = Hirota's bilinear differential operator in KP-hierarchies.

These facts suggest that the Virasoro algebra is deeply related to hierarchies. Further discussions as to the relation between the Virasoro algebra and modified KP-hierarchies will be shown in the joint work [15] with H. Yamada.

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