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Some new results on $L^2(\Gamma\backslash G)$ multiplicities

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Abstract. We announce in this paper the solution of a conjecture of Langlands on the multiplicity of an integrable discrete series representation in $L^2(\Gamma\backslash G)$. We also give an alternating sum formula for $L^2(\Gamma\backslash G)$ multiplicities (when $\Gamma\backslash G$ is compact), and we extend Moscovici's result on the geometric interpretation of discrete series multiplicities for Γ with finite co-volume.

1. Introduction. Let G be a connected non-compact linear semi-simple Lie group and let Γ be co-compact discrete subgroup of G . The regular representation of G on $L^2(\Gamma\backslash G)$ decomposes as a direct sum of irreducible unitary representations $\pi \in \hat{G}$ (= the unitary dual of G) where each π has a finite multiplicity $m_\pi(\Gamma)$. We shall assume that G has the rank of a maximal compact subgroup K of G . Then G admits $\pi \in \hat{G}$ with $L^2(G)$ matrix coefficients. If in fact $\pi \in \hat{G}$ has $L^1(G)$ matrix coefficients (i.e. π is an integrable discrete series representation of G) then in [9], [10] Langlands has worked out methods (based on results of Selberg [18], [19] and Harish-Chandra [2], [3]) for computing the multiplicity

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$m_\pi(\Gamma)$. In particular if 1 is the only elliptic element of Γ he has shown that

$$(1.1) \qquad m_\pi(\Gamma) = (\text{volume of } \Gamma\backslash G)\ (\text{formal degree of } \pi)$$

Equation (1.1) shows in particular that π actually occurs in $L^2(\Gamma\backslash G)$. In [6], [7] Hotta and Parthasarathy showed that with some restrictions on π , equation (1.1) holds in fact for infinitely many non-integrable discrete series representations π . These "restrictions" were recently removed by the author who thus obtained in [21] the most general multiplicity result possible for the discrete series. There, as in [6], [7], $m_\pi(\Gamma)$ was expressed as the (explicitly computable) index of a twisted Dirac operator on $\Gamma\backslash G/K$. Such an expression is also possible when $\Gamma\backslash G$ is non-compact. See Theorem 4.4 below. This geometric interpretation of $m_\pi(\Gamma)$ is analogous to the original geometric interpretation proposed by Langlands. We recall, briefly, Langlands conjecture.

Let $H \subset K$ be a Cartan subgroup of G and fix a G-invariant holomorphic structure on G/H . Given a non-singular parameter Λ by which the discrete series representation $\pi = \pi_\Lambda$ is determined (modulo the action of the Weyl group of (H,K)) let L_Λ be the corresponding holomorphic homogeneous line bundle over G/H , and for $X = \Gamma\backslash G/H$ let $H^q(X,L_\Lambda)$ be the space Γ-invariant, L_Λ-valued harmonic C^∞ forms of type (ρ,q) $(0,q)$ on G/H (relative to some G-invariant Hermitian metrics on G/H and L_Λ) . Alternatively $H^q(X,L_\Lambda)$ is the q^{th}-dimensional cohomology of the sheaf of local Γ-invariant holomorphic sections of L_Λ on pre-images of open sets in X. Suppose that

(1.2) Λ is sufficiently far away from Weyl chamber walls.

Then a theorem of Griffiths [5], [10] says that $H^q(X,L_\Lambda) = o$ except for $q = q_\Lambda$, where q_Λ is a distinguished integer (cf. (2.8) below where $\Lambda = \lambda+\delta$) completely determined by Λ. In [10] Langlands conjectured that

$$\dim H^{q_\Lambda}(X,L_\Lambda) = m_{\pi_\Lambda}(\Gamma) \tag{1.3}$$

for π_Λ integrable. Schmid in [15] proved (1.3) for Λ subject to (1.2). Recently the author [22] was able to by-pass condition (1.2) altogether and established (1.3) not only for all integrable π_Λ, but also for infinitely many non-integrable discrete series representations. Our result, which at the same times improves Griffiths' vanishing theorem for the cohomology spaces $H^q(X,L_\Lambda)$, is presented in Theorem 2.7 below. On the other hand Theorem 2.7 depends very much on strong results of Schmid in [15], [16], [17].

In the case when G/K is Hermitian symmetric we give a general alternating sum formula for the multiplicities $m_\pi(\Gamma)$. This new formula (formula (3.11) below) includes and extends some classical results found in [6], [8], [9].

In section 4 we drop the restriction of the co-compactness of Γ and assume only that Γ has a co-finite volume. Using a recent theorem of Moscovici we express the multiplicity of discrete series representations in the discrete spectrum of $L^2(\Gamma\backslash G)$ as the (finite) L^2-index of a twisted Dirac operator. This result (Theorem 4.4 below) also holds for all integrable discrete series representations of G and for infinitely many non-integrable discrete series. Thus it extends a result of Moscovici (Theorem 3.2 of [12]) and it extends the main result of [21] (Theorem 3.3 there) to non-compact $\Gamma\backslash G$. Moreover, up to computing L^2-indicies in the special rank one case, Theorem 4.4 implies in particular the Osborne-Warner formula [4], [13].

2. The complexifications of the Lie algebras g_o, k_o, h_o of G, K, H will be denoted by g, k, h, respectively. Let Δ be the set of non-zero roots of (g,h) and fix an arbitrary system of positive roots $\Delta^+ \subset \Delta$. If g_β is the root space of $\beta \in \Delta$ we set

(2.1) $$n = \sum_{\alpha \in \Delta^+} g_{-\alpha}$$

The quotient G/H can be assigned a unique G-invariant complex structure such that n is the space of anti-holomorphic tangent vectors at the origin. Let L be the lattice of differentials of characters of H ; $\Delta \subset L \subset h_R^* \stackrel{\text{def.}}{=\!=}$ the linear functionals on h with real-valued restriction to $\sqrt{-1}\, h_o$. If $\lambda \in L$, λ is integral (since G is linear) and λ induces a holomorphic homogeneous line bundle

L_λ over G/H . Let S_λ be the sheaf of germs of local Γ-invariant holomorphic sections of L_λ on the inverse images of open sets in $X = \Gamma\backslash G/H$ under the map $G/H \to X$ (as in section 1). Given $\pi \in \hat{G}$ let H_π denote the Hilbert space of π and also the space of K-finite vectors in H_π . Then the sheaf cohomology $H^q(X,S_\lambda)$ and the Lie algebra cohomology $H^q(n,H_\pi)$ are related by the following

<u>Theorem</u> 2.2.
$$H^q(X,S_\lambda) = \sum_{\substack{\pi \in \hat{G} \\ \pi(\Omega) = (\lambda,\lambda+2\delta)1}} m_\pi(\Gamma)\, H^q(n,H_\pi)_{-\lambda}$$

for $q \geq o$, $\lambda \in L$, where Ω = the Casimir operator of G , $2\delta = \sum_{\alpha \in \Delta^+} \alpha$ (,) = the Killing form of G , and where $H^q(n,H_\pi)_{-\lambda}$ is the subspace of vectors in $H^q(n,H_\pi)$ transforming according to the character $e^{-\lambda}$ of H .

This is proved in [22]. The Hochschild-Serre spectral sequence generated by the subalgebra ~~$k \subset n$~~ $k \cap n \subset n$ of n can be used (as in [17]) to compute $H^q(n,H_\pi)_{-\lambda}$. Its E_1 term is given by

(2.3)
$$E_1^{rs} = H^s\left(k \cap n, H_\pi \otimes \Lambda^r\, p/p \cap n\right)_{-\lambda}$$

where $g = k+p$ is a Cartan decomposition of g . Assume that $\lambda+\delta$ is regular and let

(2.4)
$$P^{(\lambda)} = \{\alpha \in \Delta \mid (\lambda+\delta,\alpha) > o\}$$

be the corresponding system of positive roots; let $2\delta^{(\lambda)} = \sum_{\alpha\varepsilon p(\lambda)} \alpha$.

Let Δ_k, Δ_n denote the set of compact, non-compact roots, respectively. Extending the arguments in section 4 of [17] we can show, using Theorem 2.6 of [21] and Schmid's lowest K type Theorem [16]:

Theorem 2.5. Let $\lambda \varepsilon L$ such that $\lambda+\delta$ is regular and such that $(\lambda+\delta-\delta^{(\lambda)},\alpha)>o$ for every non-compact root α in $P^{(\lambda)}$. (Note that $\lambda+\delta-\delta^{(\lambda)}$ is $P^{(\lambda)}$ - dominant.) If $\pi \varepsilon \hat{G}$ such that $\pi(\Omega) = (\lambda,\lambda+2\delta)1$, then in (2.3), $E_1^{rs} = o$ unless (i) π is the contragradient $\pi^*_{\lambda+\delta}$ of Harish-Chandra's discrete series representation $\pi_{\lambda+\delta}$ corresponding to the regular element $\lambda+\delta$ [3], (ii) $r = |\{\alpha\varepsilon\Delta^+\cap\Delta_n|(\lambda+\delta,\alpha)>o\}|$, where $|S|$ denotes the cardinality of a set S , (iii) $s = |\{\alpha\varepsilon\Delta^+\cap\Delta_k|(\lambda+\delta,\alpha)<o\}|$. Conversely (i), (ii), (iii) imply $\dim E_1^{rs} = 1$.

Thus if λ satisfies the condition

$$(2.6) \qquad (\lambda+\delta-\delta^{(\lambda)},\alpha) > o \quad \text{for every } \alpha\varepsilon P^{(\lambda)}\cap \Delta_n$$

the above spectral sequence degenerates and Theorems 2.2, 2.5 give (noting that $m_{\pi^*}(\Gamma) = m_\pi(\Gamma)$)

Theorem 2.7. (Solution of Langlands' conjecture [22]). Suppose $\lambda \varepsilon L$ such that $\lambda+\delta$ is regular. Let $\pi_{\lambda+\delta}$ be the corresponding Harish-Chandra discrete series representation. Define

(2.8) $$q_{\lambda+\delta} = |\{\alpha\varepsilon\Delta^+\cap\Delta_k \mid (\lambda+\delta,\alpha)<o\}| + |\{\alpha\varepsilon\Delta^+\cap\Delta_n \mid (\lambda+\delta,\alpha)>o\}| \quad .$$

If λ satisfies condition (2.6) (which is automatically the case if $\pi_{\lambda+\delta}$ is integrable, by Theorem 8.2 of [20]) then the cohomology groups $H^q(X,S_\lambda)$ vanish for $q \neq q_{\lambda+\delta}$. Moreover $\dim H^{q_{\lambda+\delta}}(X,S_\lambda) = m_{\pi_{\lambda+\delta}}(\Gamma)$.

3. Now we define

(3.1) $F_o' = \{$integral $\Lambda \varepsilon h^* \mid \Lambda+\delta$ is regular and

$(\Lambda+\delta,\alpha)>o$ for every α in $\Delta_k^+ \overset{\text{def.}}{=} \Delta^+\cap\Delta_k\}$.

For $\Lambda \varepsilon F_o'$ let

(3.2) $P^{(\Lambda)} = \{\alpha\varepsilon\Delta \mid (\Lambda+\delta,\alpha)>o\}$, $P_n^{(\Lambda)} = P^{(\Lambda)}\cap\Delta_n$,

as in (2.4). Let $2\delta^{(\Lambda)} = \sum_{\alpha\varepsilon P^{(\Lambda)}} \alpha$, and let

(3.3) $b_\Lambda = h + \sum_{\alpha\varepsilon P^{(\Lambda)}} g_\alpha$ (= a Borel subalgebra)

If θ is a parabolic subalgebra of g containing b_Λ we shall write $\theta = m + u$ for a Levi decomposition of θ where u is the unipotent radical of θ and m is a reductive complement. Let $\Delta(m)$, $\Delta(u)$ denote the set of roots of m,u respectively, and let $\theta_{u,n}$ = the set of non-compact roots in $\Delta(u)$. Thus

(3.4) $$m = h + \sum_{\alpha\varepsilon\Delta(m)} g_\alpha \qquad u = \sum_{\alpha\varepsilon P^{(\Lambda)}-\Delta(m)} g_\alpha$$

and $$\theta_{u,n} = P_n^{(\Lambda)}-\Delta(m) \quad .$$

Write $\langle Q\rangle = \sum_{\alpha\varepsilon Q}\alpha$ for $Q\subset\Delta$. A set of parabolic subalgebras $\{\theta_i\}_{i=1}^t$ containing b_Λ is a <u>representative set</u> if

(3.5) $$\langle\theta_{u_{i,n}}\rangle \neq \langle\theta_{u_{j,n}}\rangle \quad \text{for } i \neq j \text{ and}$$

$$\langle\theta_{u,n}\rangle = \langle\theta_{u_{i,n}}\rangle \quad \text{for some } i$$

for any given parabolic subalgebra $\theta = m+u\supset b_\Lambda$.

For $\Lambda \varepsilon F_o'$ we set

(3.6) $$Q_\Lambda = \{\alpha\varepsilon\Delta_n^+ \overset{def.}{=} \Delta^+\cap\Delta_n \mid (\Lambda+\delta,\alpha)>o\} \ , \ Q_\Lambda' = \Delta_n^+ - Q_\Lambda \ .$$

We note that if $\lambda \varepsilon h^*$ is integral such that $\lambda+\delta$ is regular, and if we choose w in the Weyl group of (k,h) such that $(w(-\lambda-\delta), \Delta_k^+)>o$ and set $\Lambda = \Lambda_\lambda = w(-\lambda-\delta) - \delta$ then $\Lambda \varepsilon F_o'$ and

(3.7) $$\lambda \text{ satisfies (2.6)} \iff (\Lambda+\delta\ \delta^{(\Lambda)},\alpha\ )>o$$

$$\text{for every } \alpha\,\varepsilon\, P^{(\Lambda)}\cap\Delta_n = P_n^{(\Lambda)} \ (= -w\, P_n^{(\lambda)}) \ .$$

Moreover, the discrete series representations $\pi_{\lambda+\delta}$, $\pi_{\Lambda+\delta}$ are related by: $\pi_{\Lambda+\delta} = \pi^*_{\lambda+\delta}$. If we drop the condition (2.6) (or equivalently condition (3.7)) then in contrast to statement (i) of Theorem 2.5 above, many non-discrete series representations as well will contribute to the formula of Theorem 2.2. And although the computation of dim $H^q(X,S_\lambda)$ might not be feasible one can compute the arithmetic genus $\chi(X,S_\lambda) = \sum_q (-1)^q \dim H^q(X,S_\lambda)$ of the sheaf S_λ ; this is done in [15]. Then a good knowledge of all the unitary representations π for which $H^q(n,H_\pi)_{-\lambda} = o$ would yield with the knowledge of $\chi(X,S_\lambda)$ an explicit <u>alternating sum formula</u> relating the multiplicities $m_\pi(\Gamma)$. In the case G/K is Hermitian symmetric we have carried out such a program to relate multiplicities, using an analogous sheaf S_Λ over $\Gamma\backslash G/K$ defined for $\Lambda \in F_o'$. The result is the following. We choose the system of positive roots Δ^+ in section 2 to be compatible with the G-invariant complex structure on G/K which is now assumed to be Hermitian. That is

(3.8)
$$p^- \overset{\text{def.}}{=} \sum_{\alpha\in\Delta_n^+} g_{-\alpha}$$

is the space of anti-holomorphic tangent vectors at the origin in G/K ; cf. (2.1). Up to a finite covering of G we may assume G has a simply-connected complexification. Given a representative set $\{\theta_i\}_{i=1}^t$ of parabolic subalgebras containing b_Λ (see (3.5)) we define

(3.9) $S(\Lambda,q) = \{ i \mid 2 \mid \theta_{u_{i,n}} \cap Q_\Lambda \mid + \mid Q'_\Lambda \mid - \mid \theta_{u_{i,n}} \mid = q$

and $\theta_{u_{i,n}} = \theta_{u,n}$ for some parabolic subalgebra $\theta = m + n \supset b_\Lambda$ with $(\Lambda+\delta-\delta^{(\Lambda)}, \Delta(m)) = o$; $1 \le i \le t\}$

for $o \le q \le n$, $2n = \dim_R G/K$;

see (3.4), (3.6) .

<u>Theorem</u> 3.10. (Alternating sum formula) Suppose G is linear as above and G/K is Hermitian symmetric. Suppose that Γ acts freely on G/K . Given $\Lambda \varepsilon F'_o$ let $\{\theta_i\}_{i=1}^t$ be a representative set of parabolic subalgebras containing the Borel subalgebra b_Λ of (3.3). Suppose that system of positive roots $P^{(\Lambda)}$ in (3.2) is also compatible with a G-invariant complex structure on G/K (as Δ^+ is by assumption); let $\bar{P}^{(\Lambda)} = \Delta^+_k \cup -P_n^{(\Lambda)}$ be the conjugate system. Define $\mu_i = \Lambda+\delta-\delta^{(\Lambda)} + \langle\theta_{u_{i,n}}\rangle$ and let π_{μ_i} be the irreducible G module with $\bar{P}^{(\Lambda)}$ - highest weight μ_i ; $1 \le i \le t$. Then for $i \varepsilon S(\Lambda,q)$ in (3.9) π_{μ_i} is unitarizable and the multiplicities $m_{\pi_{\mu_i}}(\Gamma)$ satisfy

$$(3.11) \quad (-1)^{|Q_\Lambda|} \sum_{q=o}^{n} \sum_{i \,\varepsilon\, S(\Lambda,q)} (-1)^q \, m_{\pi_{\mu_i}}(\Gamma) =$$

$$\mathrm{vol}\,(\Gamma\backslash G) \left| \frac{\prod_{\alpha\varepsilon\Delta^+} (\Lambda+\delta,\alpha)}{\prod_{\alpha\varepsilon\Delta^+} (\delta,\alpha)} \right| \frac{1}{|W|}$$

where W is the Weyl group of (g,h).

The coefficient of the volume of $\Gamma\backslash G$ in (3.11) is the formal degree of $\pi_{\Lambda+\delta}$; cf. equation (1.1). We note that if i_o is the unique index for which $< P_n^{(\Lambda)} > = < \theta_{u_{i_o,n}} >$, corresponding to the parabolic subalgebra $\theta = b_\Lambda$ following (3.5), then $\theta_{u_{i_o,n}} = P_n^{(\Lambda)}$ and hence

$$2 \mid \theta_{u_{i_o,n}} \cap Q_\Lambda \mid + \mid Q_\Lambda' \mid - \mid \theta_{u_{i_o,n}} \mid = \mid Q_\Lambda \mid \text{ ; i.e.}$$

$i_o \in S(\Lambda,|Q_\Lambda|) \rightarrow S(\Lambda,|Q_\Lambda|) \neq \phi$ in (3.11). Also

$\mu_{i_o} = \Lambda + \delta_n + \delta_n^{(\Lambda)}$ so that $\pi_{\mu_{i_o}}$ is just the holomorphic discrete series representation $\pi_{\Lambda+\delta}$ (corresponding to the G-invariant complex structure on G/K compatible with the positive system $\overline{P}^{(\Lambda)}$) with lowest Δ_k^+ - highest weight $\Lambda+\delta_n+\delta_n^{(\Lambda)}$. If Λ satisfies (3.7) in particular (which is the case when $\pi_{\Lambda+\delta}$ is integrable) then the left hand side of (3.11) reduces simply to $(-1)^{|Q_\Lambda|} m_{\Lambda+\delta}(\Gamma)$; i.e. $m_{\Lambda+\delta}(\Gamma) = (-1)^{|Q_\Lambda|}$ right hand side of (3.11). Theorem 3.10 is proved in [24], where various applications are given.

4. In this section we continue to assume that G is linear, but not that G/K is Hermitian, nor that $\Gamma\backslash G$ is compact. Again Δ^+ will denote an arbitrary system of positive roots. We assume that Γ is

torsion free and that $\Gamma\backslash G$ has a finite invariant volume. Let $S^{\pm}$ denote the ½ spin modules for k . Given $\Lambda \in F_0'$ (see (3.1)) let $V_{\Lambda+\delta_n}$ be the irreducible k module with Δ^+_k - highest weight $\Lambda+\delta_n$ where $2\delta_n = <\Delta^+_n>$. Then $V_{\Lambda+\delta_n} \otimes S^{\pm}$ is in fact a K module and one can therefore form the homogeneous ½ spinor bundles $E_\Lambda^{\pm} \to G/K$ over G/K induced by these modules. Replacing G by a double covering if necessary, we may assume G/K has a spin structure. Then we can consider the (twisted) Dirac operator $D_\Lambda^+ : \Gamma^\infty E_\Lambda^+ \to \Gamma^\infty E_\Lambda^-$ acting of C^∞ sections [14]. D_Λ^+ is a G-invariant elliptic first order differential operator. Being Γ-invariant in particular, D^+_Λ is the lift of an elliptic operator ${}_\Gamma D^+_\Lambda$ on the quotient bundle $\Gamma\backslash E_\Lambda^+$. In other words ${}_\Gamma D^+_\Lambda$ is <u>locally invariant</u> in the sense of [12] , and being elliptic it has a finite L^2 - <u>index</u> by Moscovici's Theorem 2.1 of [12]. This index is defined by

$$(4.1) \qquad \text{ind}\ {}_\Gamma D^+_\Lambda = \dim\ker {}_\Gamma D^+_\Lambda - \dim\ker ({}_\Gamma D_\Lambda^+)^*$$

where $({}_\Gamma D^+_\Lambda)^*$ is the formal adjoint of ${}_\Gamma D_\Lambda^+$.
Here we assume Γ satisfies the conditions imposed in [11]. For bundles over the locally symmetric space $\Gamma\backslash G/K$, Moscovici shows that the L^2 - index of a locally invariant elliptic operator is independent of that operator. In the specific case at hand one has

$$(4.2) \qquad \mathrm{ind}\ {}_\Gamma D^+_\Lambda = \sum_{\substack{\pi \in L^2_d(\Gamma\backslash G) \\ \pi(\Omega) = (\Lambda,\Lambda+2\delta)1}} m_\pi(\Gamma)\ d(\pi,\Lambda)$$

where $m_\pi(\Gamma)$ is the multiplicity of $\pi \in \hat{G}$ in the discrete spectrum $L^2_d(\Gamma\backslash G)$ of the regular representation of G on $L^2(\Gamma\backslash G)$, and where $d(\pi,\Lambda) = \dim \mathrm{Hom}_K (H_\pi, V_{\Lambda+\delta_n} \otimes S^+) - \dim \mathrm{Hom}_K (H_\pi, V_{\Lambda+\delta_n} \otimes S^-)$. Thus $\mathrm{ind}\ {}_\Gamma D^+_\Lambda$ is clearly independent of ${}_\Gamma D^+_\Lambda$, and is dependent on the inducing modules $V_{\Lambda+\delta_n} \otimes S^\pm$. Now in [21] (see Theorem 2.13 there) the following is proved.

<u>Theorem</u> 4.3. Suppose $\Lambda \in F_o'$ satisfies condition (3.7) above (i.e. $(\Lambda+\delta-\delta^{(\Lambda)}, P_n^{(\Lambda)}) > o$, where we note that $\Lambda+\delta-\delta^{(\Lambda)}$ is $P^{(\Lambda)}$ - dominant). Then if $\pi \in \hat{G}$ such that $\pi(\Omega) = (\Lambda,\Lambda+2\delta)1$ and such that $\mathrm{Hom}_K (H_\pi, V_{\Lambda+\delta n} \otimes S^\pm) = o$, π is unitarily equivalent to the Harish-Chandra discrete series representation $\pi_{\Lambda+\delta}$ defined by the regular element $\Lambda+\delta$. Moreover if σ is the unique element in the Weyl group W of (g,h) such that $\sigma\Delta^+ = P^{(\Lambda)}$, we must have $(-1)^{\ell(\sigma)} = \pm 1$ where $\ell(\sigma) = |\sigma(-\Delta^+) \cap \Delta^+|$ is the length of σ.

Thus under condition (3.7) on Λ, Theorem 4.3 says that only the discrete series representation $\pi_{\Lambda+\delta}$ can contribute to the L^2-index in (4.2). In this way we obtain

Theorem 4.4. (Theorem 2.7 of [23]) Let $\Gamma \subset G$ be a discrete torsion-free subgroup such that $\Gamma\backslash G$ has a finite invariant volume and such that Γ satisfies the assumption on page 16 of [11]. For $\Lambda \varepsilon F_o'$ let $\pi_{\Lambda+\delta}$ be the Harish-Chandra discrete series representation corresponding to the regular element $\Lambda+\delta$. Let $\sigma \varepsilon W$ be the unique Weyl group element such that $\sigma\Delta^+ = P^{(\Lambda)}$; see (3.1), (3.2). If Λ satisfies the condition

$$(4.5) \qquad (\Lambda+\delta-\delta^{(\Lambda)},\alpha)>o \quad \text{for every } \alpha \text{ in } P_n^{(\Lambda)}$$

(cf. (2.6), (3.7)) then the multiplicity of $\pi_{\Lambda+\delta}$ in the discrete spectrum $L^2_d(\Gamma\backslash G)$ of $L^2(\Gamma\backslash G)$ is given by

$$(4.6) \qquad m_{\pi_{\Lambda+\delta}}(\Gamma) = (-1)^{\ell(\sigma)} \text{ ind } _\Gamma D^+_\Lambda$$

where $\ell(\sigma)$ is the length of σ , $_\Gamma D^+_\Lambda$ as above is the twisted Dirac operator, and where the L^2 - index is defined in (4.1). As in Theorem 2.7 condition (4.5) is automatically satisfied (by the Trombi-Varadarajan result [20]) if $\pi_{\Lambda+\delta}$ is integrable; i.e. (4.6) holds for all integrable $\pi_{\Lambda+\delta}$.

Theorem 4.4 extends the main result in [21] (Theorem 3.3 there) to non-compact $\Gamma\backslash G$. A version of it was first obtained in Theorem 3.2 of [12]. However we have removed the strong restriction imposed on Λ in [12]. It is already clear even in the co-compact case (by equation (3.11) for example) that condition (4.5) cannot be relaxed if (4.6) is to be maintained. Thus Theorem 4.4 (like Theorem 2.7) represents the best kind of result possible. In the special case when the symmetric space G/K is rank one, Barbasch and Moscovici have computed the L^2-index ind ${}_{\Gamma}D^{+}_{\Lambda}$ in [1]. This coupled with (4.6) gives an extension of the Osborne-Warner multiplicity formula [13] (some errors in [13] are corrected by DeGeorge in [4]) to non-integrable discrete classes.

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