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MOVABILITY IN FIBER SHAPE THEORY

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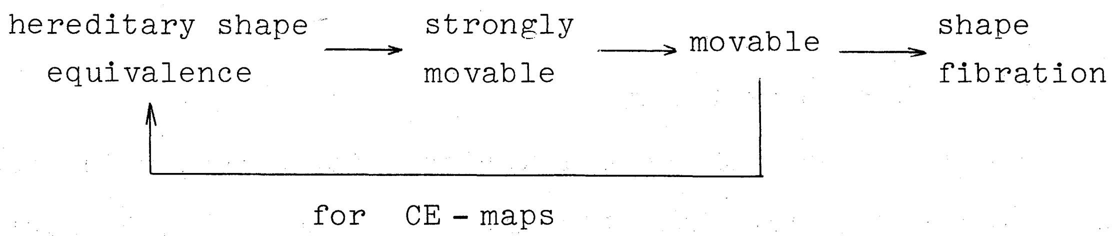
§1. Introduction.

In the shape theory ([Bo,MS]), the notion of movability is one of the most important notions and is very useful. In this article, we will study the <u>movability</u> in the <u>fiber shape theory</u> and give some results which are related with hereditary shape equivalences and shape fibrations. For this purpose, in §2, we give a brief description of the fiber shape theory according to the method of S. Mardešić and J. Segal ([MS]). Then the notion of (strong) movability in the fiber shape theory is defined in the same way as in the shape theory. In §3, we study the relation between the movability and hereditary shape equivalences or shape fibrations. As a summary of this section, we have the following diagram:



In [CD], D. Coram and P. F. Duvall introduced the notions of complete movability and k - movability for maps between ANR's and characterized the approximate fibrations in these terms. Compared with our movability condition, these movability can be regarded as local ones. In §4, we will generalize the notions of these local movabilities so that we can deal with maps between metric spaces, and then we will study their relation with the global movability and shape fibrations. In particular, we have the following result.

Every completely movable map $f : X \to Y$ is a weak shape fibration and in addition, if $\dim Y < \infty$, then f is a shape fibration.

We will also obtain some results concerning strongly regular maps with ANR fibers ([F,Ka_2]).

Throughout this paper, spaces are assumed to be metrizable. ANR's are ones for the metric spaces ([Hu]). A map $f : X \to Y$ is proper if $f^{-1}(B)$ is compact for each compact subset B of Y.

§2. Fiber shape theory.

The fiber shape theory has various approaches, which correspond to those in the shape theory. In [Ka_1], Borsuk's and Chapman's approaches are taken. [CM] is based on fibered

ANR sequences over a base space. Here we will give a much simpler and fairly general description of the fiber shape theory according to the method of S. Mardešić and J. Segal ([MS]). We will use fibered ANR systems instead of ANR systems. Now we go into the details.

Let Y be a metric space and be fixed. Let $f : X \to Y$ $g : Z \to Y$ be two maps. A map $\phi : X \to Z$ is said to be a fiber preserving map from f to g over Y if $g\phi = f$. In this case, $[\phi]_{g,f}$ denotes the fiber homotopy class of ϕ from f to g. By $\mathcal{FH}_Y$, we denote the fiber homotopy category over Y, that is, the objects of $\mathcal{FH}_Y$ are the maps from metric spaces to Y and the morphisms are the fiber homotopy classes over Y.

We say a map $f : X \to Y$ is a fibered ANR over Y ([CM]) if there exists an ANR M and an open set U of $Y \times M$ and maps $X \overset{i}{\to} U \overset{r}{\to} X$ such that $ri = id_X$ and i, r are fiber preserving (i.e., $pi = f$, $fr = p$, where $p : Y \times M \to Y$ is the projection). By $\mathcal{FA}_Y$, we denote the full subcategory of $\mathcal{FH}_Y$ consisting of the objects which are dominated (in $\mathcal{FH}_Y$) by a fibered ANR.

Let $f : X \to Y$ be a map. Then a $\mathcal{FA}_Y$-expansion of f in the category pro-$\mathcal{FH}_Y$ ([MS, p. 18]) is obtained as follows. Take a closed embedding of X into an ANR M ([BP, p.49]) and let $f^{-1} = \cup\{y \times f^{-1}(y) ; y \in Y\} \subset Y \times M$. Note that the projection $p : f^{-1} \to Y$ corresponds to f by the identification $X \overset{\approx}{\to} f^{-1} : x \to (f(x),x)$. Let $\{U_\lambda\}_{\lambda \in \Lambda}$ be an open neighborhood base of f^{-1} in $Y \times M$. Then Λ is directed by the order defined by $\lambda \le \lambda'$ iff $U_\lambda \supset U_{\lambda'}$. For each $\lambda \in \Lambda$, let $p_\lambda : U_\lambda \to Y$ be the restriction of the projection $p : Y \times M \to Y$

and $i_\lambda : X \approx f^{-1} \subset U_\lambda$ the inclusion and for each $\lambda \leq \lambda'$, let $i_{\lambda,\lambda'} : U_{\lambda'} \subset U_\lambda$ be the inclusion. We get an inverse system $\underline{p} = \{p_\lambda, [i_{\lambda,\lambda'}]_{p_\lambda, p_{\lambda'}}; \Lambda\}$ in FA_Y (i.e., an object of pro-FA_Y) and a system $\underline{i} = \{[i_\lambda]_{p_\lambda f}; \Lambda\} : f \to \underline{p}$ of morphisms in FH_Y (i.e., a morphism of pro-FH_Y) (see [MS, p.3]). For the simplicity, we denote $[i_\lambda]_{p_\lambda, f}$, $[i_{\lambda,\lambda'}]_{p_\lambda, p_{\lambda'}}$ by i_λ, $i_{\lambda,\lambda'}$, resp.

We can easily verify the following property of $\underline{i} : f \to \underline{p}$.

(i) If $k : f \to q$ is a morphism in FH_Y to a FA_Y-object q, then there exist a $\lambda \in \Lambda$ and a morphism $j_\lambda : p_\lambda \to q$ such that $k = j_\lambda i_\lambda$.

(ii) If $\lambda \in \Lambda$ and $k_\lambda, l_\lambda : p_\lambda \to q$ are morphisms to a FA_Y-object q and $k_\lambda i_\lambda = l_\lambda i_\lambda$, then there exists $\lambda' \geq \lambda$ in Λ such that $k_\lambda i_{\lambda\lambda'} = l_\lambda i_{\lambda\lambda'}$.

In fact, by the definition of FA_Y, we may assume that q is the projection from an open set U of $Y \times M$ to Y, where M is an ANR. Then, (i) and (ii) follows from the defining property of ANR's ([Hu]). By [MS, p.20, Theorem 1], (i) and (ii) implies that $\underline{i} : f \to \underline{p}$ is a FA_Y-expansion of f in pro-FH_Y, thst is,

(iii) For each morphism $\underline{k} : f \to \underline{q}$ in pro-FH_Y to a pro-FA_Y-object $\underline{q}$, there exists a unique morphism $\underline{j} : \underline{p} \to \underline{q}$ in pro-FH_Y such that $\underline{k} = \underline{j} \cdot \underline{i}$.

We have just proved the following.

Proposition 2.1. Every object of FH_Y has a FA_Y-expansion. In other words, FA_Y is a dense subcategory of FH_Y ([MS,p.22]).

Therefore, according to [MS, Chapter I, §2·3], we have a shape category FSh_Y and a shape functor $S : FH_Y \to FSh_Y$ based on (FH_Y, FA_Y). We call FSh_Y the fiber shape category over Y. The next proposition justifies our definition of FSh_Y and its proof is similar to that of [MS, Appendix 2, Theorem 1].

Let B be a compactum (compact metric space) and let FSh_B^c denote the full subcategory of FSh_B consisting of the maps from compacta to B.

<u>Proposition</u> 2.2. FSh_B^c is canonically isomorphic to the fiber shape category M_B (or R_B) given in [Ka_1].

The following proposition is the literal translation of [MS, p.27, Theorem 4, Corollary 2] to the fiber shape theory.

<u>Proposition</u> 2.3. Let f , g be objects of FH_Y .

(i) If g is an object of FA_Y , then the function

$$S : [f, g]_{FH_Y} \to [f, g]_{FSh_Y} \quad \text{is bijective.}$$

($[f, g]_{FH_Y}$, $[f, g]_{FSh_Y}$ are the sets of morphisms from f to g in FH_Y , FSh_Y , resp.)

(ii) If f , g are objects of FA_Y , then a morphism $\phi : f \to g$ in FH_Y is an isomorphism in FH_Y iff $S(\phi)$ is an isomorphism in FSh_Y .

Now we consider the movability in this shape category. Once we gave the description of the fiber shape theory in the term of expansions, we are automatically led to the notion of movability in the fiber shape theory (see [MS, Chapter II, §§ 6,7])

We say an object $\underline{p} = \{p_\lambda, i_{\lambda,\lambda'}; \Lambda\}$ of pro-FH_Y is <u>movable</u> if for each $\lambda \in \Lambda$, there exists $\lambda' \geq \lambda$ in Λ such that for each $\lambda'' \geq \lambda'$, there exists a morphism $j : p_{\lambda'} \to p_{\lambda''}$ in FH_Y such that $i_{\lambda\lambda''} j = i_{\lambda\lambda'}$. A map $f : X \to Y$ (or an object of FH_Y) is said to be <u>movable</u> if some (eq., any) FA_Y-expansion $\underline{i} : f \to \underline{p}$ of f, $\underline{p}$ is movable.

By our construction of a FA_Y-expansion of f, the movability of f is reduced to the following simple form.

<u>Proposition</u> 2.4. Let $f : X \to Y$ be a map and M an ANR which contains X as a closed subset. Then f is movable iff for each neighborhood U of f^{-1} in $Y \times M$, there exists a neighborhood V of f^{-1} in U such that for each neighborhood W of f^{-1} in V, there exists a homotopy $\phi : V \times [0,1] \to U$ such that $\phi_0 = id$, $\phi_1(W) \subset V$, $p\phi_t = p$ $(0 \leq t \leq 1)$, where $p : Y \times M \to Y$ is the projection.

In Proposition 2.4, if we can take ϕ so that $\phi_t|_{f^{-1}} = id$ $(0 \leq t \leq 1)$, then we say f is <u>strongly movable</u>. It is easily verified that this definition depends only on f and is independent of the choice of M.

We give some examples.

<u>Example</u> 2.5. Let $f : X \to Y$ be a proper onto map. If f satisfies one of the following conditions, then f is strongly movable.

(i) f is a hereditary shape equivalence.

(ii) f is an approximate fibration.

(iii) f is a bundle map with a FANR fiber and Y is locally compact.

(iv) f is completely movable and $\dim Y < \infty$.

As for (i),(ii) and (iv) , see §§ 3, 4. (iii) is reduced to the case of a trivial bundle by Proposition 2.6 (see below) and this case is obvious since every compact FANR is strongly movable ([Dy, HH]).

The next proposition is a fiber version of the sum theorem for FANR's (or strongly movable compacta).

Proposition 2.6. Let $f : X \to Y$ be a proper onto map. If each $y \in Y$ admits a neighborhood U_y in Y such that $f|_{f^{-1}(U_y)} : f^{-1}(U_y) \to U_y$ is strongly movable, then f is strongly movable.

§3. Movability, hereditary shape equivalences and shape fibrations.

In this section, we study the relation between the movability and hereditary shape equivalences (HSE's) or shape fibrations.

First we are concerned with CE - maps and HSE's. A CE - map is a proper onto map whose fibers have the trivial shape and a HSE is a proper onto map such that for each closed subset B of Y, $f|_{f^{-1}(B)} : f^{-1}(B) \to B$ is a shape equivalence. For the detail, we refer to [An, Ko]. In particular, [An] gives a characterization of HSE's in a relation theoretic term.

Theorem 3.1. ([An, Theorem 4.5]) A proper onto map $f : X \to Y$ is a HSE iff for some (eq., any) closed embedding of X into an ANR M, the relation $f^{-1} : Y \to M$ is slice trivial, i.e., for each neighborhood U of f^{-1} in $Y \times M$, there exist a neighborhood V of f^{-1} in U, a map $g : Y \to M$ and a homotopy $\phi : V \times [0,1] \to U$ such that $g \subset U$, $\phi_0 = id$, $\phi_1(V) = g$, $p\phi_t = p$ $(0 \le t \le 1)$. (g is identified with the graph of g in $Y \times M$ and $p : Y \times M \to Y$ is the projection.)

An easy homotopy construction based on Theorem 3.1 shows that every HSE is strongly movable. Furthermore, for CE-maps, we have the converse.

Theorem 3.2. A CE-map is a HSE iff it is movable.

Note that a cell-like shape fibration (see below) is not necessarily a HSE (see [MR_2, Remark 5]).

We turn our attention to the shape fibrations, for which we refer to [Ma, $MR_{1,2,3}$, R].

Let $f : X \to Y$ be a proper map and M, N be ANR's which contains X, Y as a closed set resp. Let $p : N \times M \to N$ denote the projection. Note that $X \approx f^{-1}$ is closed in the ANR $N \times M$. By [Ma, §4] and [MR_1, Proposition 2, Theorem 2], we have,

Proposition 3.3. The map f is a shape fibration iff

(*) for each neighborhood U of f^{-1} in $N \times M$, there exist a neighborhood V of f^{-1} in U and a neighborhood Y_0 of Y in N such that if $g : Z \to V$, $H : Z \times [0,1] \to Y_0$ are maps with $pg = H_0$, then there exists a map $G : Z \times [0,1] \to U$ with $G_0 = g$, $pG = H$.

A space Z is said to be an approximate ANR (AANR) ([Ma, §2]) if for each open cover $\mathcal{U}$ of Z, there exist an ANR P and maps $Z \overset{i}{\to} P \overset{r}{\to} Z$ such that ri and id_Z are $\mathcal{U}$-near (i.e., each $z \in Z$ admits a $U \in \mathcal{U}$ with $ri(z), z \in U$).

Proposition 3.4. If the space Y is an AANR, then the map f is a shape fibration iff

(**) for each neighborhood U of f^{-1} in $Y \times M$, there exists a neighborhood V of f^{-1} in U such that if $g : Z \to V$ $H : Z \times [0,1] \to Y$ are maps with $pg = H_0$, then there exists a map $G : Z \times [0,1] \to U$ with $G_0 = g$, $pG = H$.

The next proposition shows that the movability implies the approximate homotopy lifting property (AHLP) (*), (**).

Proposition 3.5. (i) If f is movable then (**) holds.

(ii) If f, M, N satisfies the following condition (#), then (*) holds:

(#) For each neighborhood U of f^{-1} in $N \times M$, there exists a neighborhood V of f^{-1} in U such that for each neighborhood W of f^{-1} in V, there exist a neighborhood Y_0 of Y in N and a homotopy $\phi : (V \cap (Y_0 \times M)) \times [0,1] \to U$ such that $\phi_0 = id$, $\phi_1(V \cap (Y_0 \times M)) \subset W$, $p\phi_t = p$ $(0 \leq t \leq 1)$.

If the map f is movable and Y (eq., X) is separable, then we can prove that (#) holds. Therefore we have

Theorem 3.6. Let $f : X \to Y$ be a proper onto movable map. If Y is separable or an AANR then f is a shape fibration.

An approximate fibration ([CD]) is just a shape fibration between ANR's ([MR_1, Corollary 1]). If the map $f : X \to Y$ is an approximate fibration, then by taking an approximate regular lift of a local equiconnecting function ([Du, p.334]) of the ANR Y, one can easily show that f is strongly movable.

Proposition 3.7. A proper onto map between ANR's is an approximate fibration iff it is movable.

Remark 3.8. By Theorem 3.1, it is easily verified that every HSE satisfies the condition (#) of Proposition 3.5 ( ). Therefor every HSE is a shape fibration ([R, Theorem 9]).

§4. Local movability and its uniformization.

In this section, we will discuss the local movabilities. The notions of complete movability and k-movability ([CD]) have the following generalization.

Let $f : X \to Y$ be a proper onto map and M an ANR which contains X as a closed subset. We say f is complete movable if for each $y \in Y$ and each neighborhood U of $f^{-1}(y)$, there exists a neighborhood V of $f^{-1}(y)$ in U such that for each fiber $f^{-1}(z)$ in V and each neighborhood W of $f^{-1}(z)$ in V, there exists a homotopy $\phi : V \times [0,1] \to U$ with $\phi_0 = \mathrm{id}$, $\phi_1(V) \subset W$ and $\phi_t|_{f^{-1}(z)} = \mathrm{id}$ $(0 \le t \le 1)$. This property is independent of the choice of M. We say f is k-movable if for each $y \in Y$ and each neighborhood U_0 of $f^{-1}(y)$ in M, there exist neighborhoods $U \supset V$ of $f^{-1}(y)$ in U_0 such that for each fiber $f^{-1}(z) \subset V$ and each $x \in f^{-1}(z)$, the projection homomorphism $\check{\pi}_i(f^{-1}(z),x) \longrightarrow \pi_i(U,x)$ is an isomorphism for $0 \le i \le k-1$ and an epimorphism for $i = k$ onto the image of the inclusion induced homomorphism $\pi_i(V,x) \longrightarrow \pi_i(U,x)$. As for the shape group $\check{\pi}_i$, see [CD, MS].

Proposition 4.1. (i) Any (proper onto) strongly movable maps and CE-maps are completely movable.

(ii) Each fiber of a completely movable map is a FANR (or strongly movable).

(iii) ([CD, Proposition 3.6]) Every completely movable map is k-movable for each $k \ge 0$.

As for the AHLP, we have the following. The proof is the same as that of [CD].

Proposition 4.2. (i) ([CD, Theorem 3.3]) Every n-movable map $f : X \to Y$ is an n-shape fibration, that is, f has the AHLP for cells $[0,1]^i$ $(0 \le i \le n)$ (see [MR_3, §5]).

(ii) ([CD, Proposition 3.6]) Every completely movable map f is a weak shape fibration, that is, f is an n-shape fibration for each $n \ge 0$ (see [MR_3, §6]).

In [MR_1, Example 6], it is shown that the Taylor's CE-map is not a shape fibration. Therefore in general, the complete movability does not imply the AHLP for all spaces. However, note that in this example, the range is infinite dimensional. In fact, if we require the finite dimensionality of ranges, then we have

Theorem 4.3. Let $f : X \to Y$ be a completely movable map. If $\dim Y < \infty$ then f is strongly movable.

The proof of Theorem 4.3 is based on the homotopy construction concerning the sum of strongly movable compacta, combined with the usual argument on the nerves of covers. Furthermore, we can prove directly that the condition (#) in Proposition 3.5 (ii) holds. Therefore we have a local condition for maps to be shape fibrations.

Theorem 4.4. Every completely movable map with a finite dimensional range is a shape fibration.

Next we consider the estimated complete movability. It turns out that this notion joins with the strongly regularity ([Ad, F]).

Let $f : X \to Y$ be a proper onto map and d a metric on X. The map f is said to be strongly regular with respect to d ([Ad]) if for each $y \in Y$ and $\varepsilon > 0$, there exists a neighborhood W of y in Y such that for each $z \in W$ there exist ε-maps $\phi : f^{-1}(y) \to f^{-1}(z)$, $\psi : f^{-1}(z) \to f^{-1}(y)$ such that $\phi\psi$ and $id_{f^{-1}(z)}$ are ε-homotopic. (This means that $d(x,\phi(x)) < \varepsilon$ $(x \in f^{-1}(y))$, $d(x',\psi(x')) < \varepsilon$ $(x' \in f^{-1}(z))$ and there exists a homotopy $H : f^{-1}(z) \times [0,1] \to f^{-1}(z)$ such that $H_0 = \phi\psi$, $H_1 = id_{f^{-1}(z)}$ and $\mathrm{diam}\, H(x' \times [0,1]) < \varepsilon$ $(x' \in f^{-1}(z))$.)

The estimated complete movability is defined as follows. Let M be an ANR containing X as a closed subset and ρ be a metric on M. We say f is completely movable in M with estimation with respect to ρ if for each $y \in Y$, each neighborhood U of $f^{-1}(y)$ in M and $\varepsilon > 0$, there exists a neighborhood V of $f^{-1}(y)$ in U such that for each fiber $f^{-1}(z) \subset V$ and each neighborhood W of $f^{-1}(z)$ in V, there exists an ε-homotopy $\phi : V \times [0,1] \to U$ with $\phi_0 = id$, $\phi_1(V) \subset W$ and $\phi_t|_{f^{-1}(z)} = id$ $(0 \le t \le 1)$.

The next lemma joins the above two notions.

Lemma 4.5. Under the above notations, the following conditions are equivalent.

(i) f is strongly regular with respect to $\rho|_X$ and each fiber of f is an ANR.

(ii) For each $y \in Y$ and $\varepsilon > 0$, there exists a neighborhood V of $f^{-1}(y)$ in M such that for each fiber $f^{-1}(z)$ $\subset V$, there exists an ε-retraction $r : V \to f^{-1}(z)$.

(iii) f is completely movable in M with estimation with respect to ρ and each fiber of f is an ANR.

By the estimated version of the proof of Theorem 4.3, we have the following result, which was partially proved in [F, Proposition 3.1].

Theorem 4.6. Suppose $f:X \to Y$ is a proper onto map and $\dim Y < \infty$. Then the following are equivalent.

(i) f is strongly regular with respect to some (eq., any) metric on X and each fiber of f is an ANR.

(ii) f is a fibered ANR over Y (see §2).

This gives another proof of the following fact which was proved in [F, Theorem 1], using the Michael's selection theorem ([Mi]), under the assumption of separability and completeness for domains.

Corollary 4.7. If $f:X \to Y$ is a strongly regular map with ANR fibers and $\dim Y < \infty$, then f is a Hurewicz fibration.

In fact, by Theorem 4.6, the map f is (strongly) movable, hence f has the AHLP (*) (see Proposition 3.4, 3.5 (i)) and again by Theorem 4.6, the AHLP of f turns out the homotopy lifting property (HLP) of f .

We conclude this section with a question. By Theorem 3.2, one can regard Theorem 4.3 as a generalization of the results in [An] (at least for the case of finite dimensional ranges) in another direction. In [An, Ko], it is shown that if $f : X \to Y$ is a CE-map and Y is countable dimensional, then f is a HSE. However we have no answer to the following question.

Question 4.8. Are Theorems 4.3, 4.4, 4.6 and Corollary 4.7 still true even if Y is countable dimensional?

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