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# HYPERSPACES AND WHITNEY MAPS

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Throughout this note, the word compactum means a compact metric space. A connected compactum is a continuum. A Peano continuum is a locally connected continuum. If x and y are points of a metric space, d(x,y) denotes the distance from x to y. For any subsets A and B of a metric space, let $d(A,B)=\inf\{d(a,b) \mid a\in A,\ b\in B\}$. Also, let $d_H(A,B)=\max\{\sup_{a\in A} d(a,B),\ \sup_{b\in B} d(b,A)\}$. $d_H$ is called the Hausdorff metric. The hyperspaces of a continuum are the spaces $2^X=\{A\subset X \mid A \text{ is compact and nonempty}\}$ and $C(X)=\{A\in 2^X \mid A \text{ is connected}\}$ which are metrized with the Hausdorff metric $d_H$. Let $F_1(X)=\{\{x\} \mid x\in X\}$. A Whitney map for a hyperspace H of a continuum X is a continuous function $w: H \longrightarrow [0,w(X)]$ such that $w(\{x\})=0$ for each $\{x\}\in F_1(X)$, anf if A, $B\in H$ and $A\subsetneq B$, then $w(A)<w(B)$ (see [9]). The notion of Whitney map is a important and convenient tool for hyperspace theory. If w is a Whitney map for H and $t\in[0,w(X))$, then $w^{-1}(t)$ is called a Whitney level. Whitney levels are coverings of X which, as t gets closed to zero, converge to $w^{-1}(0)=F_1(X)\cong X$. It is of interest to obtain information about the structure of Whitney levels and determine those properties which are preserved by the convergence of positive Whitney levels to zero level. In [1] and [8], Curtis, Schori and West proved that for any Peano continuum (locally connected continuum) X, $2^X$ is a Hilbert cube $Q=\prod_{i=1}^{\infty}[-1,1]$ and if X contains no free arc, C(X) is a Hilbert cube Q. Recently, Goodykoontz and Nadler introduced the notion "admissible Whitney map" and they proved the following

Theorem (Goodykoontz and Nadler). Let X be a Peano continuum and let w be an admissible Whitney map for $H=2^X$ or $C(X)$. If $H=C(X)$, assume that X contains no free arc. Then for any $t \in (0,w(X))$, $w^{-1}(t)$ is a Hilbert cube and w is an open map.

Let X be a continuum. A Whitney map w for $H=2^X$ or $C(X)$ is an admissible Whitney map for H [2] if there is a homotopy $h: H\times[0,1] \longrightarrow H$ satisfying the following conditions:

(1) $h(A,1)=A$, $h(A,0)\in F_1(X)$ for each $A \in H$, and

(2) if $w(h(A,t)) > 0$ for some $A \in H$ and $t \in (0,1]$, then $w(h(A,s)) < w(h(A,t))$ for each $0 \leq s < t \leq 1$.

Moreover, in [4] we proved the following

Theorem [4]. Under the same hypotheses as in above theorem, the restriction $w|w^{-1}((0,w(X)): w^{-1}((0,w(X))) \longrightarrow (0,w(X))$ of w to $w^{-1}((0,w(X)))$ is a trivial bundle map with Hilbert cube fibers. If X is the Hilbert cube Q, there is a Whitney map w for H such that $w|w^{-1}[0,w(X))$ is a trivial bundle map with Hilbert cube fibers. Also, if X is the n-sphere $(n \geq 1)$, then there is a Whitney map w for $H=2^{S^n}$ $(n \geq 1)$ or $C(S^n)$ $(n \geq 2)$ such that for some $t \in (0,w(X))$, $w|w^{-1}((0,t])$ is a trivial bundle map with $S^n \times Q$ fibers.

Also, in [5] we showed the following

Theorem [5]. Let $P_i$ be a 1 or 2 dimensional connected polytope for each i=1,2,..,n. Then there is a Whitney map w for $H=2^{\prod_{i=1}^n P_i}$ or $C(\prod_{i=1}^n P_i)$ $(n \geq 2)$ such that for some $t \in (0,w(\prod_{i=1}^n P_i))$,

$w|w^{-1}((0,t])$ is a trivial bundle map with $\prod_{i=1}^{n} P_i \times Q$ fibers.

In relation to above theorems, we have the following

Proposition [5]. Let X be a compact ANR but not AR. Let $H=2^X$ or $C(X)$. If $H=C(X)$, assume that X contains no free arc. If w is any Whitney map for H, there is a point $t_0 \in (0,w(X))$ such that $w|w^{-1}((0,t_0])$ is not a trivial bundle map.

Example [5]. Let X=S be the unit circle. Let $A \in H=2^X$ or $C(X)$. For each $n \geq 2$, let $F_n(A)=\{K \subset A \mid K \neq \emptyset$ and the cardinality of K is $\leq n\}$, define $\lambda_n: F_n(A) \to [0,\infty)$ by letting $\lambda_n(\{a_1,a_2,..,a_n\})= \min\{d(a_i,a_j) \mid i \neq j\}$ for each $\{a_i\} \in F_n(A)$, where d is the arc length metric for S. Also, let $w_n(A)= \sup \lambda_n(F_n(A))$ and let $w(A)=\sum_{n=2}^{\infty} w_n(A)/2^{n-1}$ for each $A \in H$. Then w is a Whitney map for H. Then $w|w^{-1}((0,\pi/2)): w^{-1}((0,\pi/2)) \to (0,\pi/2)$ is a trivial bundle map with $S \times Q$ fibers, but $w|w^{-1}((0,\pi/2])$ is not a trivial bundle map. In fact, $w|w^{-1}((0,\pi/2])$ is not an open map.

Example [5]. There is a Whitney map w for $H=2^{[0,1]}$ such that for every $t \in (0,w([0,1]))$, $w|w^{-1}((0,t])$ is not a trivial bundle map.

Question [5]. Is it true that if P is a n-dimensional ($n \geq 3$) polytope, there is a Whitney map w for $H=2^P$ or $C(P)$ such that for some $t \in (0,w(P))$, $w|w^{-1}((0,t])$ is a trivial bundle map with $P \times Q$ fibers ? (If $H=C(P)$, assume that P contains no free arc).

As an application of hyperspace theory, we obtain the following

Theorem [6]. If a compactum X has a scalene metric, then X is an absolute retract. Moreover, if a locally compact space X has a locally scalene metric, then X is an absolute neighborhood retract.

A metric d defined in a space X is a scalene metric [6] if $x_1$ and $x_2$ are different two points of X, then there is a point $x_0$ of X such that for each point x of X either $d(x,x_1) > d(x,x_0)$ or $d(x,x_2) > d(x,x_0)$ holds. This notion is a generalization of norm of a linear space. A metric d defined in a space X is a locally scalene metric [6] if for each point $x \in X$ there is a neighborhood U of x in X such that $d|U \times U$ is a scalene metric.

Remark [6]. Every 1-dimensional AR has a scalene metric but there is a 2-dimensional AR which does not admit a scalene metric.

Question [6]. Is it true that every locally compact polytope has a locally scalene metric ?

Question [6]. Is it true that every compact strongly convex metric space admits a scalene metric ? Is it true that every compact scalene metric space admits a strongly convex metric ?

Proposition [6]. If d is a scalene metric and convex, then d is a strongly convex metric.

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