数理解析研究所講究録
第 509 巻 1984 年 31-40



# On a generalization of homotopy groups

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In this note we shall introduce generalized homotopy groups for arbitary spaces. These groups are homotopy invariants. These are coinceded with usual homotopy groups for ANR spaces, but do not in general.

§ 1. Spaces and maps are topological spaces and continous maps, respectively. Let $I = [0,1]$ be the unit interval. For each integer $n \geqq 1$, we define

$$I^n = \{(t_1,t_2,\ldots,t_n): 0 \leqq t_i \leqq 1 \text{ for all } i\},$$

$$I^{n-1} = \{(t_1,t_2,\ldots,t_n) \in I^n: t_n = 0\},$$

$$J^{n-1} = \{(t_1,t_2,\ldots,t_n) \in I^n: t_i = 0 \text{ for some } i,\ 1 \leqq i \leqq n-1\}.$$

$I^{n-1} \cup J^{n-1}$ is the boundary of $I^n$. Let $(X,A,x_0)$ be a pair of spaces with base point. By $\Omega_n(X,A,x_0)$ we mean the set consisting of all maps $r : (I^n,I^{n-1},J^{n-1}) \longrightarrow (X,A,x_0)$. By $\mathrm{Cov}(X)$ we mean the set of all normal open coverings of X.

Let $f,g:X \longrightarrow Y$ be maps and $\underline{V} \in \mathrm{Cov}(Y)$. We say that f and g are $\underline{V}$-near, in notation $(f,g) \leqq \underline{V}$, provided that for each $x \in X$ there exists a $V \in \underline{V}$ with $f(x), g(x) \in V$.

For each $\underline{U} \in \mathrm{Cov}(X)$ we define a relation on $\Omega_n(X,A,x_0)$ as follows: Let $r,s \in \Omega_n(X,A,x_0)$. We say that r and s are $\underline{U}$-chainable in $\Omega_n(X,A,x_0)$, in notation $r \underset{\underline{U}}{\equiv} s$, provided that there exists a finite subset $\{h_1, h_2, \ldots, h_k\}$ of $\Omega_n(X,A,x_0)$ such that $h_1 = r$, $h_k = s$ and $(h_i, h_{i+1}) \leqq \underline{U}$ for each i, $1 \leqq i \leqq k-1$. It is easy to show the followings:

Lemma 1. For each $\underline{U} \in \mathrm{Cov}(X)$, $\underset{\underline{U}}{\equiv}$ forms an equivalence relation on $\Omega_n(X,A,x_0)$.

Lemma 2. Let $\underline{U}_1, \underline{U}_2 \in \mathrm{Cov}(X)$. If $\underline{U}_1$ is a refinement of $\underline{U}_2$ ( in notation, $\underline{U}_1 \leqq \underline{U}_2$), then $r \underset{\underline{U}_1}{\equiv} s$ implies $r \underset{\underline{U}_2}{\equiv} s$ for $r,s \in \Omega_n(X,A,x_0)$.

Lemma 3. Let $r,s \in \Omega_n(X,A,x_0)$. If r and s are homotopic ( in notation $r \simeq s$), then $r \underset{\underline{U}}{\equiv} s$ for each $\underline{U} \in \mathrm{Cov}(X)$.

Proof of Lemma 3. Since r and s are homotopic, there exists a map $F: (I^n, I^{n-1}, J^{n-1}) \times I \longrightarrow (X,A,x_0)$ such that

(1) $F(x,0) = r(x)$ and $F(x,1) = s(x)$ for each $x \in I^n$.

Take any $\underline{U} \in \mathrm{Cov}(X)$. Then $F^{-1}(\underline{U})$ is a covering of $I^n \times I$. Since $I^n$ and I are compact, there exist a $\underline{V} \in \mathrm{Cov}(I^n)$ and an integer k such that

(2) $\underline{V} \times \underline{K} \leqq F^{-1}(\underline{U})$.

Here $\underline{K} = \{[i/k, (i+1)/k]: i = 0,1,\ldots,k-1\}$. Now, we define maps $h_i \in \Omega_n(X,A,x_0)$ for $i = 0,1,\ldots,k$ as follows: For each $x \in I^n$ $h_i(x) = F(x, i/k)$. By (1) we have that $h_0 = r$ and $h_k = s$. Take any $x \in I^n$. Since $\underline{V} \in \mathrm{Cov}(I^n)$ there exists a $V \in \underline{V}$ with $x \in V$. Take any $i$, $0 \leqq i \leqq k-1$. By (2) there exists a $U \in \underline{U}$ such that $V \times [i/k, (i+1)/k] \subset F^{-1}(U)$. Hence we have that $h_i(x), h_{i+1}(x) \in U$. This means that $(h_i, h_{i+1}) \leqq \underline{U}$. Hence $r$ and $s$ are $\underline{U}$-chainable in $\Omega_n(X,A,x_0)$. This completes the proof of Lemma 3.

Now, we define the usual product on $\Omega_n(X,A,x_0)$ as follows: Let $r,s \in \Omega_n(X,A,x_0)$ . We defie $r*s \in \Omega_n(X,A,x_0)$ by

$$(r*s)(t_1,t_2,\ldots,t_n) = \begin{cases} r(2t_1,t_2,\ldots,t_n) & \text{for } 0 \leqq t_1 \leqq 1/2 \\ s(2t_1-1,t_2,\ldots,t_n) & \text{for } 1/2 \leqq t_1 \leqq 1. \end{cases}$$

From the definition it is easy to show the following:

Lemma 4. For each $\underline{U} \in \mathrm{Cov}(X)$, if $r_1 \equiv_{\underline{U}} r_2$ and $s_1 \equiv_{\underline{U}} s_2$, then $r_1 * s_1 \equiv_{\underline{U}} r_2 * s_2$.

By Lemma 1 $\equiv_{\underline{U}}$ is an equivalence relation on $\Omega_n(X,A,x_0)$. Let $[r]_{\underline{U}}$ be the equivalence class of $r$ by $\equiv_{\underline{U}}$ . By $W_n(X,A,x_0:\underline{U})$ we mean the set $\{[r]_{\underline{U}}: r \in \Omega_n(X,A,x_0)\}$. By Lemma 4 we can define the product in $W_n(X,A,x_0:\underline{U})$ by $[r]_{\underline{U}}*[s]_{\underline{U}} = [r*s]_{\underline{U}}$ for $r,s \in \Omega_n(X,A,x_0)$. Concerning this product we show the following:

Theorem 5. For each $\underline{U} \in Cov(X)$ $W_n(X,A,x_0:\underline{U})$ forms a group.

Proof. (i) Let $c_{x_0}:I^n \to X$ be the constant map to $x_o$. Then it is well known in the homotopy groups that

(1) $r*c_{x_0} \simeq r$ and $c_{x_0}*r \simeq r$ for each $r \in \Omega_n(X,A,x_0)$.

By Lemma 3 and (1) we have that $[r]_{\underline{U}}*[c_{x_0}]_{\underline{U}} = [r*c_{x_0}]_{\underline{U}} = [r]_{\underline{U}}$ and $[c_{x_0}]_{\underline{U}}*[r]_{\underline{U}} = [c_{x_0}*r]_{\underline{U}} = [r]_{\underline{U}}$, that is,

(2) $[r]_{\underline{U}}*[c_{x_0}]_{\underline{U}} = [r]_{\underline{U}}$ and $[c_{x_0}]_{\underline{U}}*[r]_{\underline{U}} = [r]_{\underline{U}}$ for each $r \in \Omega_n(X,A,x_0)$.

(2) means that $[c_{x_0}]_{\underline{U}}$ is the identity element of $W_n(X,A,x_0:\underline{U})$.

(ii) Take any $r \in \Omega_n(X,A,x_0)$ and define the map $\bar{r} \in \Omega_n(X,A,x_0)$ by $\bar{r}(t_1,t_2,\ldots,t_n) = r(1-t_1,t_2,\ldots,t_n)$ for each $(t_1,t_2,\ldots,t_n) \in I^n$. It is well known in homotopy theory that

(3) $r*\bar{r} \simeq c_{x_0}$ and $\bar{r}*r \simeq c_{x_0}$.

From Lemma 3 and (3) we have that

(4) $[r]_{\underline{U}}*[\bar{r}]_{\underline{U}} = [c_{x_0}]_{\underline{U}}$ and $[\bar{r}]_{\underline{U}}*[r]_{\underline{U}} = [c_{x_0}]_{\underline{U}}$.

(4) means that $[\bar{r}]_{\underline{U}}$ is the inverse of $[r]_{\underline{U}}$.

(iii) Take any $r,s,u \in \Omega_n(X,A,x_0)$. It is well known in homotopy theory that

(5) $(r*s)*u \simeq r*(s*u)$.

By Lemma 3 and (5) we have that

(6) $([r]_{\underline{U}}*[s]_{\underline{U}})*[u]_{\underline{U}} = [r]_{\underline{U}}*([s]_{\underline{U}}*[u]_{\underline{U}})$.

(6) means the associative law. By (i),(ii) and (iii) we prove the required one.

By $\pi_n(X,A,x_0)$ we mean the n-th homotopy group of $(X,A,x_0)$. By $[r]$ we mean the homotopy class of r for $r \in \Omega_n(X,A,x_0)$. By Lemmas 3 and 4 we have the well-defined functions $\rho_{\underline{U}}$ : $\pi_n(X,A,x_0) \longrightarrow W_n(X,A,x_0:\underline{U})$ and $\rho_{\underline{U}_1,\underline{U}_2}: W_n(X,A,x_0:\underline{U}_1) \longrightarrow W_n(X,A,x_0:\underline{U}_2)$ which are defined by $\rho_{\underline{U}}([r]) = [r]_{\underline{U}}$ and $\rho_{\underline{U}_1,\underline{U}_2}([r]_{\underline{U}_1}) = [r]_{\underline{U}_2}$ for $\underline{U},\underline{U}_1,\underline{U}_2 \in Cov(X)$ with $\underline{U}_1 \leqq \underline{U}_2$ and for each $r \in \Omega_n(X,A,x_0)$. From the definitions of group structures in $\pi_n(X,A,x_0)$ and $W_n(X,A,x_0:\underline{U})$, it is easy to show the following:

Lemma 6. $\rho_{\underline{U}_1}$ and $\rho_{\underline{U}_1,\underline{U}_2}$ are group homomorphisms and $\rho_{\underline{U}_2} = \rho_{\underline{U}_1,\underline{U}_2}\ \rho_{\underline{U}_1}$, $\rho_{\underline{U}_1,\underline{U}_3} = \rho_{\underline{U}_2,\underline{U}_3}\ \rho_{\underline{U}_1,\underline{U}_2}$ for $\underline{U}_1 \leqq \underline{U}_2 \leqq \underline{U}_3$.

By Lemma 6 $\{\underline{W}_n(X,A,x_0:\underline{U}),\ \rho_{\underline{U}_1,\underline{U}_2},\ Cov(X)\}$ forms an inverse system in the category of groups. We denote it by pro-$W_n(X,A,x_0)$. Also $\rho(X,A,x_0) = \{\rho_{\underline{U}}:\underline{U} \in Cov(X)\} : \pi_n(X,A,x_0) \longrightarrow$ pro-$W_n(X,A,x_0)$ forms a system map. Let $\check{W}_n(X,A,x_0)$ be the inverse limit group of pro-$W_n(X,A,x_0)$ and let $\tau_{\underline{U}} : \check{W}_n(X,A,x_0) \longrightarrow W_n(X,A,x_0:\underline{U})$ be the natural projection. Then $\tau_{(X,A,x_0)} = \{\tau_{\underline{U}}:\underline{U} \in \mathrm{Cov}(X)\} : \check{W}_n(X,A,x_0) \longrightarrow$ pro-$W_n(X,A,x_0)$ makes an inverse limit. Trivially $\rho_{(X,A,x_0)}$ induces the homomorphism $\check{\rho}_{(X,A,x_0)} : \pi_n(X,A,x_0) \longrightarrow \check{W}_n(X,A,x_0)$ with $\rho_{(X,A,x_0)} = \tau_{(X,A,x_0)}\check{\rho}_{(X,A,x_0)}$.

By $W_n(X,x_0:\underline{U})$ we mean $W_n(X,\{x_0\},x_0:\underline{U})$. Similarly we define pro-$W_n(X,x_0)$ and $\check{W}_n(X,x_0)$. By the standard fact in homotopy theory and the same way as Theorem 5 we can prove the following :

Lemma 7. For each $\underline{U} \in \mathrm{Cov}(X)$, $W_n(X,A,x_0:\underline{U})$ are abelian for $n \geqq 3$ and $W_n(X,x_0:\underline{U})$ are abelian for $n \geqq 2$.

Let $f :(X,A,x_0) \longrightarrow (Y,B,y_0)$ be a map. Take any $\underline{V} \in \mathrm{Cov}(Y)$. Then $f^{-1}(\underline{V}) \in \mathrm{Cov}(X)$. Clearly f induces the homomorphism $f_{*\underline{V}}: W_n(X,A,x_0:f^{-1}(\underline{V})) \longrightarrow W_n(Y,B,y_0:\underline{V})$ which is defined by $f_{*\underline{V}}([r]_{f^{-1}(\underline{V})}) = [fr]_{\underline{V}}$ for each $r \in \Omega_n(X,A,x_0)$. Moreover, for $\underline{V}_1, \underline{V}_2 \in \mathrm{Cov}(Y)$ with $\underline{V}_1 \leqq \underline{V}_2$ we have that $f_{*\underline{V}_2}\rho_{f^{-1}(\underline{V}_1),f^{-1}(\underline{V}_2)} = \rho_{\underline{V}_1,\underline{V}_2} f_{*\underline{V}_1}$. These means that $f_* = \{f_{*\underline{V}}, f^{-1}\}$ forms a system map from pro-$W_n(X,A,x_0)$ to pro-$W_n(Y,B,y_0)$. Here $f^{-1}$: $\mathrm{Cov}(Y) \longrightarrow \mathrm{Cov}(X)$ is the function defined by $\underline{V} \longrightarrow f^{-1}(\underline{V})$. Then $f_*$ induces the homomorphism

$\check{f}_* : \check{W}_n(X,A,x_0) \longrightarrow \check{W}_n(Y,B,y_0)$ with $f_* \tau_{(X,A,x_0)} =$

$= \tau_{(Y,B,y_0)} \check{f}_*$.

Lemma 8. Let $f,g: (X,A,x_0) \longrightarrow (Y,B,y_0)$ be maps. If $f \simeq g$, then $f_*$, $g_*$: pro-$W_n(X,A,x_0) \longrightarrow$ pro-$W_n(Y,B,y_0)$ are equivalent in pro-groups.

Proof. Take any $\underline{V} \in \mathrm{Cov}(Y)$. Take any $\underline{U} \in \mathrm{Cov}(X)$ which refies $f^{-1}(\underline{V})$ and $g^{-1}(\underline{V})$. By Lemma 3 we can easily show that $f_{*\underline{V}} \rho_{\underline{U},f^{-1}(\underline{V})} = g_* \rho_{\underline{U},g^{-1}(\underline{V})}$. Hence we completes the proof of Lemma 8.

Lemma 9. Let $f :(X,A,x_0) \longrightarrow (Y,B,y_0)$ and $g: (Y,B,y_0) \longrightarrow (Z,C,z_0)$ be maps. Then $(gf)_* = g_* f_*$ and $1_{(X,A,x_0)*} =$

$= 1_{\text{pro-}W_n(X,A,x_0)}$.

Corollary 10. pro-$W_n(X,A,x_0)$ and $\check{W}_n(X,A,x_0)$ are homotopy invariants.

Remark 11. In [/] Hurewicz introduced groups $\pi_{n\varepsilon}(X,x)$ for each real number $\varepsilon > 0$. His groups are coincided with the kernel of $\rho_{\underline{U}}: \pi_n(X,x) \longrightarrow W_n(X,x:\underline{U})$ in our sense. Since all homomorphism $\rho_{\underline{U}}$ are onto, our group $W_n(X,x:\underline{U})$ is isomorphic to $\pi_n(X,x)/\mathrm{Ker}(\rho_{\underline{U}})$, that is, $W_n(X,x:\underline{U})$ is isomorphic to $\pi_n(X,x)/\pi_{n\varepsilon}(X,x)$ in the sense of Hurewicz. Therefore, we have to say that our groups $W_n(X,x:\underline{U})$ are defined in essential by Hurewicz. However he did not consider pro-$W_n(X,x)$ and

$\check{W}_n(X,x)$.

2. In this section we shall state results on our groups without proofs. The detailed proofs are contained in Watanabe [5].

Theorem 12. Let X be a compact Hausdorff space. Then $\check{W}_0(X,x)$ corresponds to the set consisting of all connected components of X.

Theorem 12 means that in general $\check{W}_0(X,x)$ and $\pi_0(X,x)$ are different.

Theorem 13. Let X and A be paracompact Hausdorff spaces with $A \subset X$, and let A is P-embeded in X. If X is $LC^n$ and A is $LC^{n-1}$, then $\rho_{(X,A,x)}: \pi_n(X,A,x) \longrightarrow \text{pro-}W_n(X,A,x)$ is isomorphic, and hence $\check{\rho}(X,A,x) : \pi_n(X,A,x) \longrightarrow \check{W}_n(X,A,x)$ is an isomorphism.

Corollary 14. Let X and A be ANR spaces and let A be a closed subset of X. Then $\rho_{(X,A,x)} : \pi_n(X,A,x) \longrightarrow \text{pro-}W_n(X,A,x)$ and $\check{\rho}_{(X,A,x)}: \pi_n(X,A,x) \longrightarrow \check{W}_n(X,A,x)$ are isomorphisms for each n.

Theorem 15. Let X be a compact metric space. Then pro-$W_n(X,A,x)$ is stable if and only if $\check{W}_n(X,A,x)$ is a countable group.

Corollary 16. Let X be a compact metric space. If $\check{W}_n(X,A,x)$ is a countable group, then $\check{\rho}_{(X,A,x)} : \pi_n(X,A,x) \longrightarrow \check{W}_n(X,A,x)$ is surjective.

Theorem 17. If $(X,x)$ is a pointed $AANR_N$( see [4]), then pro-$W_n(X,x)$ is stable, for each n.

3. In this section we define a system map from pro-$W_n(X,A,x)$ to pro-$\pi_n(X,A,x)$. Let $(X,A,x)$ be a pair of spaces with base point. By using Theorem 6 of Mardesic[3] and a method of Watanabe[4] we can construct an approximative resolution $\underline{p} = \{p_b : b \in B\} : (X,A,x) \longrightarrow (\underline{(X,A,x)},\underline{U}) = \{((X_b,A_b,x_b),\underline{U}_b),\ p_{b',b},\ B\}$ satisfying the following condition:

(1) $X_b$ and $A_b$ are ANR spaces and $A_b$ are closed subset of $X_b$ for each $b \in B$.

(2) $\underline{U}_b$ is a covering of $X_b$ such that for any closed metrizable pair $(Z,C,z)$ , if any maps $f,g:(Z,C,z) \longrightarrow (X_b,A_b,x_b)$ with $(f,g) \leqq \underline{U}_b$, then f and g are homotopic.

From (2) we have that

(3) $W_n(X_b,A_b,x_b:\underline{U}_b) = \pi_n(X_b,A_b,x_b)$ for each n.

Since $p_b$ induces a homomorphism $p_{b*}: W_n(X,A,x: p_b^{-1}(\underline{U}_b)) \longrightarrow W_n(X_b,A_b,x_b:\underline{U}_b)$ for each $b \in B$, it is easy to shoe that $\underline{p}_* = \{p_{b*},p^{-1}\}$ : pro-$W_n(X,A,x) \longrightarrow \{W_n(X_b,A_b,x_b:\underline{U}_b),\ p_{b',b*},\ B\} = \{\pi_n(X_b,A_b,x_b),p_{b',b*},B\} =$ pro-$\pi_n(X,A,x)$ forms a system map by (3). Here $p^{-1}: B \longrightarrow Cov(X)$ is a function defined by

$p^{-1}(b) = p_b^{-1}(\underline{U}_b)$ for each $b \in B$.

Theorem 18. If $(X,x)$ is a pointed $AANR_C$ ( see [4]), then $\underline{p}_* : \text{pro-}W_n(X,x) \longrightarrow \text{pro-}\pi_n(X,x)$ is an isomorphism for each $n$ .

Corollary 19. If $(X,x)$ is a pointed $AANR_C$, then $\check{W}_n(X,x)$ is isomorphic to shape group $\check{\pi}_n(X,x)$ for each n.

Addendum 20. By Corollary 10 we show that pro-$W_n(X,A,x)$ and $\check{W}_n(X,A,x)$ are homotopy invariants. However, these are not shape invariants. For example, let X be the circle and Y be a Warsaw circle. Then X and Y have the same shape . However pro-$W_1(X,x)$ is isomorphic to Z and pro-$W_1(Y,y)$ is isomorphic to 0.

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