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Manifolds Modeled on the Direct Limits of Euclidian Spaces and Hilbert cubes : A Survey

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Let $\mathbb{R}$ denote the real line and Q the Hilbert cube which we regard as the countable infinite product of the unit closed interval $I = [0,1]$. By $\mathbb{R}^n$ and Q^n , we denote the products of n copies of $\mathbb{R}$ and Q respectively and we identify $\mathbb{R}^n$ and Q^n with $\mathbb{R}^n \times \{0\} \subset \mathbb{R}^{n+1}$ and $Q^n \times \{0\} \subset Q^{n+1}$ respectively. Then let $\mathbb{R}^\infty = \text{dir lim}\,\mathbb{R}^n$ and $Q^\infty = \text{dir lim}\, Q^n$, that is, $\mathbb{R}^\infty = \cup_{n\in\mathbb{N}} \mathbb{R}^n$ and $Q^\infty = \cup_{n\in\mathbb{N}} Q^n$ with the weak topologies determined by the families $\{ \mathbb{R}^n \mid n \in \mathbb{N} \}$ and $\{Q^n \mid n \in \mathbb{N}\}$ respectively. These spaces are regular, separable, and hereditarily Lindelöf, so hereditarily paracompact [1], [4]. The space Q^∞ is homeomorphic to ($\cong$) $B^*(b^*)$, the conjugate (or dual) B^* of a separable infinite-dimensional Banach space B with its bounded weak-* topology [2, Theorem II-6]. Paracompact, topological manifolds modeled on $\mathbb{R}^\infty$ and Q^∞ are called $\mathbb{R}^\infty$-manifolds and Q^∞-manifolds respectively. These manifolds are separable and Lindelöf if thy are connected [3, Proposition III. 4]. Any separable Q^∞-manifold ($\mathbb{R}^\infty$-manifold) M is a countable direct limit of (finite dimensional) compact metric spaces, that is, $M = \text{dir lim } X_n$ where each X_n is a (finite dimensional) compact metric subspace of X_{n+1} [4,Proposition III-2]. In this note, we assume separability of all these manifolds.

Aspects of these manifolds have been studied by R.E.Heisey and V.T.Liem after the pattern of Hilbert (space) or Hilbert cube manifold theory.[a,c] One of central importance was the problem of whether or not manifolds are stable with respect to multiplication by the model space (see the results of [4]). This was proved by Heisey.

STABILITY THEOREM [8],[12] : For any $\mathbb{R}^\infty$-manifold (resp. Q^∞-manifold) M, the projection $p : M \times \mathbb{R}^\infty \to M$ (resp. $p : M \times Q^\infty \to M$) is a near homeomorphism, that is, arbitrarily closely approximated by homeomorphisms.

Many of the arguments used in Hilbert or Hilbert cube manifold theory could not apply to the above because $\mathbb{R}^\infty$ and Q^∞ are not countable products. Non-existence of normal covers was also one of troubles [9]. Combining the results of [4] with the Stability Theorem, one can obtain the followings. The approach in [4] is similar to Henderson's one in Hilbert manifold theory.[b]

OPEN EMBEDDING THEOREM [6],[12] : Any $\mathbb{R}^\infty$-manifold (resp. Q^∞-manifold) can be embedded in $\mathbb{R}^\infty$ (resp. Q^∞) as an open set.

CLASSIFICATION THEOREM [6],[12] : Two $\mathbb{R}^\infty$-manifolds (or Q^∞-manifolds) are homeomorphic if and only if they have the same homotopy type.

D.W.Henderson [14] proved that for any neighborhood retract X of $\mathbb{R}^\infty$ the product space $X \times \mathbb{R}^\infty$ is homeomorphic to an open subset of $\mathbb{R}^\infty$, in particular, for any countable simplicial complex K the product space $|K| \times \mathbb{R}^\infty$ is an $\mathbb{R}^\infty$-manifold. This was generalized by Heisey and Toruńczyk [13]. The similar result for Q^∞ is also follows from [5]. By the Open Embedding Theorem, $\mathbb{R}^\infty$-manifolds and Q^∞-manifolds are direct limits of compact ANR's. Then they have the homotopy type of ANR's by [1, Corollary 6.4] (cf. [4, Theorem II.10]) and hence, of countable locally finite simplicial complex by the Milnor's theorem. Thus the following follows from the Classification Theorem.

TRIANGULATION THEOREM [8],[12] : Each $\mathbb{R}^\infty$-manifold (resp. Q^∞-manifold) is homeomorphic to $|K| \times \mathbb{R}^\infty$ (resp. $|K| \times Q^\infty$) where K is a countable locally finite simplicial complex.

A closed subset A of a space X is said to be $\mathbb{R}^\infty$-deficient (resp. Q^∞-deficient) in X if there is a homeomorphism $h : X \to X \times \mathbb{R}^\infty$ (resp. $h : X \to X \times Q^\infty$) such that $h(A) \subset X \times \{0\}$. The following Unknotting Theorem was establised by Liem.

UNKNOTTING THEOREM [16],[18] : Let A be an $\mathbb{R}^\infty$-deficient (resp. Q^∞-deficient) set in an $\mathbb{R}^\infty$-manifold (resp. a Q^∞-manifold) M and let α and β be open covers of M. If an $\mathbb{R}^\infty$-deficient (resp. Q^∞-deficient) embedding $f : A \to M$ is α-homotopic to the inclusion $A \subset M$, then f extends to a homeomorphism $\tilde{f} : M \to M$ which is ambiently invertibly $st(\alpha,\beta)$-isotopic to the identity.

A closed set A of a space X is said to be a Z-set in X provided there is a map $f : X \to X \setminus A$ arbitrarily close to the identity. Liem [16] showed that the Unknotting Theorem for Z-sets in Q^∞ or $\mathbb{R}^\infty$ is false.

The Liem's proof of the Unknotting Theorem is based on the fact the union of two $\mathbb{R}^\infty$-deficient (resp. Q^∞-deficient) sets in an $\mathbb{R}^\infty$-manifold (resp. a Q^∞-manifold) is also $\mathbb{R}^\infty$-deficient (resp. Q^∞-deficient) [16], [18] and the following. These heavily depend on theory of PL-manifolds or Q-manifolds.

APPROXIMATION THEOREM [15], [17] : Let M be an $\mathbb{R}^\infty$-manifold (resp. a Q^∞-manifold) and $f : X \to M$ a map of a closed set in $\mathbb{R}^\infty$ (resp. Q^∞) that restricts to an $\mathbb{R}^\infty$-deficient (resp. Q^∞-deficient) embedding on a closed subset A of X. Then for each open cover α of M, f is α-homotopic to an $\mathbb{R}^\infty$-deficient (resp. Q^∞-deficient) embedding $g : X \to M$ stationarily on A.

The below is also due to Liem.

COLLARING THEOREM [18] : Let N be a closed submanifold of an $\mathbb{R}^\infty$-manifold (resp. a Q^∞-manifold) M. Then N is $\mathbb{R}^\infty$-deficient (resp. Q^∞-deficient) in M if and only if N is collared in M.

The α-Approximation Theorem for $\mathbb{R}^\infty$-manifolds or Q^∞-manifolds can be proved similarly as the theorem for ℓ_2-manifols due to Ferry.[f] Let α be an open cover of a space Y. A map $f : X \to Y$ said to be an α-equivalence if there is a map $g : Y \to X$ such that fg is α-homotopic the identity of Y and gf is

$f^{-1}(\alpha)$-homotopic to the identity of X .

α-APPROXIMATION THEOREM [15],[17] : For any open cover α of an $\mathbb{R}^\infty$-manifold (or a Q^∞-manifold) N , there exists an open cover β of N such that every β-equivalence $f : M \to N$ from an $\mathbb{R}^\infty$-manifold (or a Q^∞-manifold) M to N is α-homotopic to a homeomorphism.

Recently, the author established the following characterization of these manifolds.

CHARACTERIZATION THEOREM [19] : A space X is an $\mathbb{R}^\infty$-manifold if and only if it satisfies the following conditions:

(a) X is a direct limit of finite dimensional compact metric spaces, and

(b) each embedding from a closed subset A of any finite dimensional compact metric space B into X can be extended to an embedding from some neighborhood of A in B into X .

A space X is a Q^∞-manifold if and only if it satisfies the conditions (a) and (b) with the phrase "finite dimensional" deleted from each.

The proof of this theorem is elementary and containes the proof of the Open Embedding Theorem which is elementary and easy. Using this characterization, we have an alternative proof of the result of Heisey and Toruńczyk [13]. As a corollary, we can prove the next Sum Theorem:

SUM THEOREM [20] : Let X_1 and X_2 be closed subsets of a space X with $X = X_1 \cup X_2$ and $X_0 = X_1 \cap X_2$. If X_0 , X_1 and X_2 are $\mathbb{R}^\infty$-manifolds (or Q^∞-manifolds) then so is X .

A map $f : X \to Y$ is a fine homotopy equivalence provided for each open cover α of Y , f is an α-equivalence. Although the following is a corollary of the α-Approximation Theorem, this has an elementary and short proof which is appeared in [19].

THEOREM [15],[17] : Each fine homotopy equivalence between $\mathbb{R}^\infty$-manifolds (or Q^∞-manifolds) is a near homeomorphism.

The Stability Theorem can be obtained as a corollary of this theorem. The Classification Theorem can be also proved elementarily and easily as this theorem.

In [20], the author also gave an elementary and short proofs of the Unknotting Theorem, the Approximation Theorem and the Collaring Theorem, where $\mathbb{R}^\infty$-manifolds and Q^∞-manifolds are under the same treatments. The author introduced in [20] the notions of D-sets and D*-sets. A closed ssubset A of a space X is called a D-set in X if it satisfies the following condition:

(*) For each compact sets $C \subset C_0$ in X and each open cover α of X , there exists an embedding $h : C \to X$ α-near to id with $h|C_0 = \text{id}$ and $h(C \setminus C_0) \subset X \setminus A$.

We call A a D*-set in X if it satisfies the condition (*) with compact sets C and C_0 replaced by the whole space X and any closed set X_0 respectively. A closed set contained in a collared set is a D*-set and a D*-set is a D-set. First the

Unknotting Theorem for D-sets is proved and then it is proved that $\mathbb{R}^\infty$-deficient (or Q^∞-deficient) sets, closed sets contained in collared sets, D*-sets and D-set in an $\mathbb{R}^\infty$-manifold (or Q^∞-manifold) are equivalent to each other. This characterization of deficiency is very convenient to see some fundamental properties of deficient sets.

M.Brown and B.Cassler[d] proved that each compact connected n-manifold M can be obtained from the n-cube I^n by making identifications on the boundary ∂I^n, that is, there is a map $\Phi : I^n \to M$ such that $\Phi|\mathring{I}^n$ is an open embedding and $\Phi(\partial I^n) = M \setminus \Phi(\mathring{I}^n)$ is nowhere dense in M. V.S.Prasad[e] established the Q-manifold version, that is, for any compact connected Q-manifold M, there is a map $\Phi : Q \times [0,1] \to M$ such that $\Phi|Q \times [0,1)$ is an open embedding and $\Phi(Q \times \{1\}) = M \setminus \Phi(Q \times [0,1))$ is nowhere dense in M. The author obtained the similar result for connected manifolds modeled on $\mathbb{R}^\infty$, Q^∞, σ, Σ or ℓ_2. (Here σ is the linear span of the usual orthonormal basis for ℓ_2 and Σ is the linear span of the Hilbert cube $\Pi_{n \in \mathbb{N}} [-2^{-n}, 2^{-n}]$ in ℓ_2.

MAPPING THEOREM [21] : Let E be one of $\mathbb{R}^\infty$, Q^∞, σ, Σ and ℓ_2. For any connected E-manifold M, there exists a perfect map $\Phi : E \times [0,1] \to M$ such that $\Phi|E \times [0,1)$ is an open embedding and $\Phi(E \times \{1\}) = M \setminus \Phi(E \times [0,1))$ is nowhere dense in M.

As seen in [19], [20] and [21], we feel that $\mathbb{R}^\infty$-manifolds and Q^∞-manifolds are handy in contrast to ℓ_2-manifolds or Q-manifolds.

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