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On Galois module structure of the cohomology groups of an algebraic variety

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§1. Main results

Let k be a field and $f : X \longrightarrow Y$ a finite Galois covering between projective varieties X and Y over k. Put $G = \mathrm{Gal}(X/Y)$. A coherent sheaf $\mathcal{F}$ on X is called a coherent G-sheaf when the group G acts on $\mathcal{F}$ in a way compatible with its action on X. For a coherent G-sheaf $\mathcal{F}$ on X, the group G naturally acts on the cohomology groups $H^i(X,\mathcal{F})$ ($i \geqq 0$). We are concerned with the $k[G]$-module structure of $H^i(X,\mathcal{F})$ ($i \geqq 0$).
Our general result is the following two theorems.

Theorem 1. Assume that $f : X \longrightarrow Y$ is tamely ramified, i.e. for any geometric point $\bar{\eta}$ of X, the order of its stabilizer group $G_{\bar{\eta}}$ in G is prime to char k. Then for a coherent G-sheaf $\mathcal{F}$ on X, we have a finite complex

$$(M) : \quad 0 \longrightarrow M^0 \longrightarrow M^1 \longrightarrow \cdots \longrightarrow M^m \longrightarrow 0$$

of $k[G]$-modules with the following properties :

(1) Each M^i is a finitely generated projective $k[G]$-module.
(2) The i-th cohomology group of the complex (M) is isomorphic to $H^i(X,\mathcal{F})$ as a $k[G]$-module ($i = 0,1,2,\dots$) .

Theorem 2. In the situation of Theorem 1 , assume further that f is unramified, i.e. $G_{\bar{\eta}} = \{1\}$ for any geometric point $\bar{\eta}$ of X . Then, concerning the complex (M) in Theorem 1 , we can replace the condition (1) by the following stronger condition (1)' :

(1)' Each M^i is a finitely generated free $k[G]$-module.

As a direct consequence of Theorems 1 and 2 , we give Corollary below, which seems most useful in application.

Corollary. For a coherent G-sheaf $\mathcal{F}$ on X , assume that $H^i(X,\mathcal{F}) = 0$ for all i except one value, say i_o , of i . Then, when f is tamely ramified, the remaining cohomology group $H^{i_o}(X,\mathcal{F})$ is a projective $k[G]$-module. Further, if f is unramified, $H^{i_o}(X,\mathcal{F})$ is a free $k[G]$-module.

Remarks. (1) When the algebra $k[G]$ is semi-simple (i.e. when char k does not divide the order of G) , all $k[G]$-modules are projective and the property (1) of the complex (M) in Theorem 1 is trivial. However, when $k[G]$ is not semi-simple, it is a good information about a $k[G]$-module that it is $k[G]$-projective.

(2) The complex (M) in Theorems 1 or 2 is never unique. We can change the modules M^i ($i \geqq 0$) in many ways keeping the conditions (1) (or (1)') and (2) . For example, in Theorem 1 , we can take a complex (M) for which each M^i is a free $k[G]$-module for $i \geqq 1$ (i.e. except M^0) .

(3) Theorem 2 is proved in [3] . Although Theorem 1 is not written explicitly in [3] , it is essentially proved in the course of the proof of Theorem 2 .

The motivation of this work was to generalize Chevalley-Weil's theorem to arbitrary characteristic. Chevalley-Weil [1] treats a finite Galois covering $f : X \longrightarrow Y$ of connected complete non-singular algebraic curves X and Y over the complex number field $\mathbb{C}$ (i.e. X and Y are compact Riemann surfaces) , and determines the $\mathbb{C}[G]$-module structure ($G = Gal(X/Y)$) of $H^0(X, \mathcal{K})$, where $\mathcal{K}$ is the canonical sheaf of X . ($H^0(X, \mathcal{K})$ is the space of holomorphic differentials on X .) Their theorem is obtained by calculating, using the Riemann-Roch theorem, the character of the $\mathbb{C}[G]$-module $H^0(X, \mathcal{K})$. If one tries to generalize Chevalley-Weil's theorem to arbitrary characteristic, i.e. to arbitrary base field k , difficulties arise as to the case where $p = \text{char}\, k$ divides $|G|$, the order of G . Even in this case we can calculate the Brauer character of the $k[G]$-module $H^0(X, \mathcal{K})$ by use of the Riemann-Roch theorem, but, when p divides $|G|$, the knowledge of the Brauer character does not suffice to determine the $k[G]$-module structure of $H^0(X, \mathcal{K})$ since the algebra $k[G]$ is no longer semi-simple. The author found that this difficulty can be overcome by making good use of (*) mentioned below (for (*) , see [4] § 16 or [2] chap. 2) . (Although $H^0(X, \mathcal{K})$ itself is not necessarily $k[G]$-projective, we can determine its $k[G]$-module structure by embedding it into some other $k[G]$-module whose structure is determined by virtue of (*) .)

(*) ... If two finitely generated <u>projective</u> $k[G]$-modules have the same Brauer character, then they are isomorphic as $k[G]$-modules.

The generalization of Chevalley-Weil's theorem to the case of arbitrary base field is given in the next section.
It was pointed out by Professor J-P. Serre that the method adopted in the case of curves is also effective in higher dimensional cases, and Theorems 1 and 2 were obtained following his suggestion.

§ 2. The case of curves

In this section we apply the general results of § 1 to the special case where X and Y are of dimension one and the G-sheaf $\mathcal{F}$ is locally free.

Let k be an algebraically closed field of characteristic p ($p \geqq 0$) , and let $f : X \longrightarrow Y$ be a finite Galois covering of connected complete non-singular curves over k . Put $G = \mathrm{Gal}(X/Y)$. We assume that f is <u>tamely ramified</u> . To state

our result, some notations are necessary. For each point P of X, put $G_P = \{ \sigma \in G \mid \sigma \cdot P = P \}$ (the stabilizer of P) and $e_P = |G_P|$. We have $e_P = 1$ except for a finite number of P . Since f is tamely ramified, G_P is a cyclic group of order ($= e_P$) prime to $p = \mathrm{char}\, k$.
Define $\theta_P \in \mathrm{Hom}(G_P, k^\times)$ by

$$\theta_P(\sigma) = \frac{\sigma \cdot \pi}{\pi} \quad (\mathrm{mod}(\pi)) \quad \text{for} \quad \sigma \in G_P ,$$

where $\pi = \pi_P$ denotes a local uniformizing parameter at P . Then θ_P is a generator of the group $\mathrm{Hom}(G_P, k^\times)$. For $d \in \mathbb{Z}$, we also use the symbol θ_P^d to denote the one-dimensional $k[G]$-module corresponding to $\theta_P^d \in \mathrm{Hom}(G_P, k^\times)$. Suppose that a locally free G-sheaf $\mathcal{E}$ of rank r on X is given. Then for $P \in X$ the group G_P acts on the stalk $\mathcal{E}_P$ of $\mathcal{E}$ at P . The $k[G_P]$-module $\mathcal{E}_P \otimes_{\mathcal{O}_P} k$ ($\mathcal{O}_P$ is the stalk at P of the structure sheaf $\mathcal{O}$ of X) decomposes into a direct sum of one-dimensional $k[G_P]$-modules because $p \nmid e_P$ and G_P is cyclic. Define integers $\ell_{P,i}$ ($0 \leqq \ell_{P,i} \leqq e_P - 1$) for $P \in X$ and $i = 1, \ldots, r$ so that $\mathcal{E}_P \otimes_{\mathcal{O}_P} k$ is isomorphic to $\bigoplus_{i=1}^{r} \theta_P^{-\ell_{P,i}}$ as a $k[G]$-module. (We have $\ell_{P,i} = 0$ for all P and i if $\mathcal{E}$ is of the form $f^*(\mathcal{G})$ for a locally free sheaf $\mathcal{G}$ on Y .)

With the above notation, the result is as follows. (For a $k[G]$-module L and $n \in \mathbb{N}$, nL means the n-times direct sum of L .)

Theorem 3. (i) There exists a $k[G]$-module N such that

$$|G|\cdot N \cong \bigoplus_{P\in X} \mathrm{Ind}_{G_P}^{G} \Big(\bigoplus_{d=0}^{e_P-1} d\cdot \theta_P^{d} \Big)$$

as $k[G]$-modules.

(ii) Let $\mathcal{E}$ be a locally free G-sheaf of rank r on X and let $\ell_{P,i}$ be as defined above. Then we have two finitely generated $k[G]$-modules M^0 and M^1 with the following properties :

(a) We have an exact sequence

$$0 \longrightarrow H^0(X,\mathcal{E}) \longrightarrow M^0 \longrightarrow M^1 \longrightarrow H^1(X,\mathcal{E}) \longrightarrow 0$$

of $k[G]$-modules.

(b) The modules M^0 and M^1 are $k[G]$-projective and $M^0 \oplus r\cdot N$ (N is the module defined in (i)) is stably isomorphic to $M^1 \oplus \bigoplus_{Q\in Y} \bigoplus_{i=1}^{r} \bigoplus_{d=1}^{\ell_{P,i}} \mathrm{Ind}_{G_{\tilde{Q}}}^{G} (\theta_{\tilde{Q}}^{-d})$ as a $k[G]$-module, where for $Q \in Y$, $\tilde{Q}$ denotes an (arbitrarily fixed) point of X such that $f(\tilde{Q}) = Q$ holds, and the sum denoted by $\bigoplus_{i=1}^{\ell_{P,i}}$ means empty sum when $\ell_{P,i} = 0$.

Remarks. (1) The $k[G]$-module N in Theorem 3 (i) is determined by the covering $f : X \longrightarrow Y$ (independent of sheaves) , and reflects the state of ramification of f . The module N can be defined only "globally" . This means that each $k[G]$-module $\mathrm{Ind}_{G_P}^{G}\left(\bigoplus_{d=0}^{e_P-1} d\cdot\Theta_P^d\right)$ for $P \in X$ is <u>not</u> divisible by $|G|$ in general while their sum over all $P \in X$ is always divisible by $|G|$.

(2) If the $k[G]$-module structure of either of the two modules $H^0(X,\mathcal{E})$ and $H^1(X,\mathcal{E})$ is known, we can determine by Theorem 3 the $k[G]$-module structure of the other (cf. Schanuel's lemma, [2] (2.24)) . In this sense, Theorem 3 gives a "relation" between $k[G]$-modules $H^0(X,\mathcal{E})$ and $H^1(X,\mathcal{E})$.

As for the canonical sheaf $\mathcal{K}$ of X , we know that $H^1(X,\mathcal{K}) \cong k$ as $k[G]$-modules, where k denotes the trivial $k[G]$-modules. Hence, by Theorem 3 , the $k[G]$-module structure of $H^0(X,\mathcal{K})$ can be determined. (We have $\ell_{P,i} = e_P - 1$ in this case.) For the sake of simplicity, we restrict ourselves to the case where f is unramified, i.e. $e_P = 1$ for all $P \in X$. Denote by I_G the augmentation ideal of $k[G]$; $I_G = \{ \sum_{\sigma\in G} a_\sigma\cdot\sigma \in k[G] \mid \sum_{\sigma\in G} a_\sigma = 0 \}$. Then we have

Theorem 4. <u>Assume that</u> $f : X \longrightarrow Y$ <u>is unramified and let</u> $\mathcal{K}$ <u>be the canonical sheaf of</u> X . <u>Then we have an exact sequence of</u> $k[G]$-<u>modules</u>

$$0 \longrightarrow H^0(X,\mathcal{K}) \longrightarrow k[G]^g \longrightarrow I_G \longrightarrow 0 ,$$

<u>where</u> g <u>is the genus of</u> Y . <u>This exact sequence uniquely determines the</u> $k[G]$-<u>module structure of</u> $H^0(X,\mathcal{K})$.

When char $k = 0$, the exact sequence in Theorem 4 splits for every group G , and consequently we have $H^0(X, \mathcal{K}) \cong k \oplus k[G]^{g-1}$ as $k[G]$-modules. (This isomorphism is proved in Chevalley-Weil [1].) However, when char $k > 0$, that exact sequence does not necessarily split. Hence we find that, when char $k > 0$, the $k[G]$-module $H^0(X, \mathcal{K})$ is not isomorphic to $k \oplus k[G]^{g-1}$ in general, in contrast to the case char $k = 0$.

References

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