Specialized Characters and Power Series Identities

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1. Introduction —— Weyl-Kac character formula:

Let me start my talk with a skip review on representation theory of Kac-Moody Lie algebras. Details would be available on many good references [3][6][7][10][19][20] and so on.

A realization of an $n \times n$-generalized Cartan matrix (= GCM) $A = (a_{ij})_{i,j=1,\ldots,n}$ is a triple $(\mathcal{J}, \Pi, \Pi^{\vee})$ such that

    i) $\mathcal{J}$ is an $(n + \text{corank } A)$-dimensional complex vector space,

    ii) $\Pi = \{\alpha_1, \ldots, \alpha_n\}$ (resp. $\Pi^{\vee} = \{\alpha_1^{\vee}, \ldots, \alpha_n^{\vee}\}$)

        is a linearly independent subset in $\mathcal{J}^*$ (resp. $\mathcal{J}$),

    iii) $\langle \alpha_i^{\vee}, \alpha_j \rangle = a_{ij}$ for every i and j.

The corresponding Lie algebra $\mathcal{g}(A)$ is generated by $\mathcal{J}$, $e_1, \ldots, e_n, f_1, \ldots, f_n$ and satisfies

    i) $[e_i, f_j] = \delta_{ij} \alpha_i^{\vee}$

ii) $[h, e_i] = \alpha_i(h)e_i$ and $[h, f_i] = -\alpha_i(h)f_i$

for every $h \in \mathfrak{h}$ and $i = 1, \ldots, n$,

iii) $[\mathfrak{h}, \mathfrak{h}] = \{0\}$,

iv) $\mathfrak{g}(A)$ contains no proper ideals which intersect trivially with $\mathfrak{h}$.

Let Z, N or N_0 denote the set of all, all positive or all non-negative integers respectively, and we set

$W = W(A) =$ the Weyl group of $(\mathfrak{g}(A), \mathfrak{h})$,

$\Delta = \Delta(A) = \{\text{non-zero roots}\}$,

$\Delta_+ = \Delta_+(A) = \{\text{positive roots}\}$,

$P = P(A) = \{\text{integral forms on } \mathfrak{h}\}$

$= \{\lambda \in \mathfrak{h}^* \; ; \; \langle \alpha_i^\vee, \lambda \rangle \in Z \text{ for every } i\}$,

$P_+ = P_+(A) = \{\text{dominant integral forms on } \mathfrak{h}\}$

$= \{\lambda \in \mathfrak{h}^* \; ; \; \langle \alpha_i^\vee, \lambda \rangle \in N_0 \text{ for every } i\}$.

It is known that for each $\Lambda \in P_+$ there exists a unique irreducible integrable highest weight module $L(\Lambda)$ with highest weight Λ and that the character ch $L(\Lambda)$ is given by the celebrated Weyl-Kac character formula:

Proposition 1.1. Assume that A is symmetrizable and $\Lambda \in P_+$, then

$$\text{ch } L(\Lambda) = \frac{\displaystyle\sum_{w \in W} (\text{sgn } w)\, e^{w(\Lambda + \rho) - \rho}}{\displaystyle\prod_{\alpha \in \Delta_+} (1 - e^{-\alpha})^{\text{mult } \alpha}}$$

where $\rho \in \mathfrak{h}^*$ is chosen such that $\langle \alpha_i^\vee, \rho \rangle = 1$ for every i.

The denominator formula

$$\sum_{w \in W} (\text{sgn } w) \, e^{w\rho - \rho} = \prod_{\alpha \in \Delta_+} (1 - e^{-\alpha})^{\text{mult } \alpha}$$

follows immediately with the special choice of $\Lambda = 0$.

For an n-tuple $s = (s_1, \ldots, s_n)$ of positive integers, the specialization of type s is the ring homomorphism

$$F_s : \mathbb{C}[[e^{-\alpha_1}, \ldots, e^{-\alpha_n}]] \longrightarrow \mathbb{C}[[q]] \quad ,$$
$$\cup \qquad\qquad\qquad\qquad \cup$$
$$e^{-\alpha_i} \longmapsto q^{s_i}$$

and

$$d_\Lambda^s(q) = d_\Lambda^{(s; A)}(q) = F_s(e^{-\Lambda} \text{ ch } L(\Lambda))$$

is called the specialized character of $L(\Lambda)$ of type s.
We set

$$Q(s; A) = F_s \left(\prod_{\alpha \in \Delta_+(A)} (1 - e^{-\alpha})^{\text{mult } \alpha} \right) .$$

Especially in case of $s = 1 = (1, \ldots, 1)$, $d_\Lambda(q) = d_\Lambda^1(q)$ is called the principally specialized character, whose beautiful expression was found by Lepowsky:

Proposition 1.2. Assume that A is symmetrizable and $\Lambda \in P_+$, then

$$d_\Lambda(q) = \frac{Q(s_\Lambda + 1; \, {}^t A)}{Q(1; A)}$$

where $s_\Lambda = (\langle \alpha_1^\vee, \Lambda \rangle, \ldots, \langle \alpha_n^\vee, \Lambda \rangle)$.

2. Specialized characters:

The problem we shall concerned with now is to compute specialized characters $d_\Lambda^s(q)$ for arbitrary specialization s. It seems very difficult to obtain similar general results as in Proposition 1.2 for arbitrary s. However, for some Λ and for some s ($\neq$ 1), we can compute $d_\Lambda^s(q)$ by using the following two theorems (cf. [22]):

Theorem 2.1. Let A = (a_{ij}) be a symmetrizable GCM of degree n, s = $(s_1,\ldots,s_n)$ an n-tuple of positive integers such that $\tilde{A} = (s_i a_{ij} s_j^{-1})$ is a GCM. Then, for every $\Lambda \in P_+$,

$$d_\Lambda^s(q) = \frac{Q(s_1(\langle \alpha_1^\vee, \Lambda \rangle +1),\ldots,s_n(\langle \alpha_n^\vee, \Lambda \rangle +1);\ {}^t\tilde{A})}{Q(s;\ A)}.$$

Theorem 2.2. Assume that A is a symmetrizable GCM and $\Lambda \in P_+(A)$ and $\Lambda' \in P_+({}^tA)$. Then

$$d_\Lambda^{(s_{\Lambda'}+1;\ A)}(q)\ Q(s_{\Lambda'}+1;\ A)$$

$$= d_{\Lambda'}^{(s_\Lambda +1;\ {}^tA)}(q)\ Q(s_\Lambda +1;\ {}^tA)\ .$$

Table 1 is a list of specialized characters of affine Lie algebras obtained from the above theorems, where we shall use following notations:

   i)   s = $(s_0,\ldots,s_\ell)$ is an (ℓ+1)-tuple of positive integers,

   ii) $\Lambda = (n_0,\ldots,n_\ell)$ is a dominant integral form

such that $\langle \alpha_i^\vee, \Lambda \rangle = n_i$ $(i=0,\dots,\ell)$,

iii) n is a positive integer,

iv) $\varphi(q) = \prod\limits_{j=1}^{\infty} (1 - q^j)$.

<u>Table 1</u>

1) $A_1^{(1)}$:

$$d_\Lambda^{(12)}(q) = \frac{Q(n_0+1,\ 2(n_1+1);\ A_2^{(2)})}{\varphi(q)}$$

$$d_\Lambda^{(21)}(q) = \frac{Q(2(n_0+1),\ n_1+1;\ A_2^{(2)})}{\varphi(q)}$$

$$d_{(n-1,2n-1)}^{s}(q) = \frac{Q(ns_0,\ 2ns_1;\ A_2^{(2)})}{Q(s;\ A_1^{(1)})}$$

$$d_{(2n-1,n-1)}^{s}(q) = \frac{Q(ns_1,\ 2ns_0;\ A_2^{(2)})}{Q(s;\ A_1^{(1)})}$$

2) $A_2^{(2)}$:

$$d_\Lambda^{(12)}(q) = \frac{Q(n_0+1,\ 2(n_1+1);\ A_1^{(1)})}{\varphi(q)}$$

$$d_\Lambda^{(14)}(q) = \frac{Q(n_0+1,\ 4(n_1+1);\ A_2^{(2)})}{Q(1,4;\ A_2^{(2)})}$$

$$= \frac{\mathcal{G}(q^2)\,\mathcal{G}(q^3)\,\mathcal{G}(q^{12})\,Q(n_0+1,\ 4(n_1+1);\ A_2^{(2)})}{\mathcal{G}(q)\,\mathcal{G}(q^4)\,\mathcal{G}(q^6)^2}$$

$$d^s_{(2n-1,n-1)}(q) = \frac{Q(2ns_0,\ ns_1;\ A_1^{(1)})}{Q(\ s;\ A_2^{(2)})}$$

$$d^s_{(4n-1,n-1)}(q) = \frac{Q(ns_1,\ 4ns_0;\ A_2^{(2)})}{Q(s;\ A_2^{(2)})}$$

3) $B_\ell^{(1)}$:

$$d_\Lambda^{(2\ldots21)}(q) = \frac{Q(2(n_0+1),\ldots,2(n_{\ell-1}+1),n_\ell+1;\ B_\ell^{(1)})}{Q(2\ldots21;\ B_\ell^{(1)})}$$

$$d^s_{(n-1,\ldots,n-1,2n-1)}(q) = \frac{Q(ns_0,\ldots,ns_{\ell-1},2ns_\ell;\ A_{2\ell-1}^{(2)})}{Q(s;\ B_\ell^{(1)})}$$

4) $C_\ell^{(1)}$:

$$d_\Lambda^{(21\ldots1)}(q) = \frac{Q(n_\ell+1,\ldots,n_1+1,2(n_0+1);\ A_{2\ell}^{(2)})}{\mathcal{G}(q)^\ell}$$

$$d_\Lambda^{(1\ldots12)}(q) = \frac{Q(n_0+1,\ldots,n_{\ell-1}+1,2(n_\ell+1);\ A_{2\ell}^{(2)})}{\mathcal{G}(q)^\ell}$$

$$d_\Lambda^{(21\ldots12)}(q) = \frac{Q(2(n_0+1),n_1+1,\ldots,n_{\ell-1}+1,2(n_\ell+1);\ C_\ell^{(1)})}{Q(21\ldots12;\ C_\ell^{(1)})}$$

$$d^s_{(2n-1,\ldots,2n-1,n-1)}(q) = \frac{Q(ns_\ell,2ns_{\ell-1},\ldots,2ns_0;\ A_{2\ell}^{(2)})}{Q(s;\ C_\ell^{(1)})}$$

$$d^s_{(n-1,2n-1,\ldots,2n-1)}(q) = \frac{Q(ns_0,2ns_1,\ldots,2ns_\ell \; ; \; A^{(2)}_{2\ell})}{Q(s; \; C^{(1)}_\ell)}$$

$$d^s_{(n-1,2n-1,\ldots,2n-1,n-1)}(q)$$

$$= \frac{Q(ns_0,2ns_1,\ldots,2ns_{\ell-1},ns_\ell \; ; \; D^{(2)}_{\ell+1})}{Q(s; \; C^{(1)}_\ell)}$$

5) $F^{(1)}_4$:

$$d^{(22211)}(q) = \frac{Q(2(n_0+1),2(n_1+1),2(n_2+1),n_3+1,n_4+1; \; F^{(1)}_4)}{Q(22211; \; F^{(1)}_4)}$$

$$d^s_{(n-1,n-1,n-1,2n-1,2n-1)}(q)$$

$$= \frac{Q(ns_0,ns_1,ns_2,2ns_3,2ns_4; \; E^{(2)}_6)}{Q(s; \; F^{(1)}_4)}$$

6) $G^{(1)}_2$:

$$d^{(331)}(q) = \frac{Q(3(n_0+1),3(n_1+1),n_2+1; \; G^{(1)}_2)}{Q(331; \; G^{(1)}_2)}$$

$$= \frac{\mathcal{G}(q^2)\,\mathcal{G}(q^{12})\,Q(3(n_0+1),3(n_1+1),n_2+1; \; G^{(1)}_2)}{\mathcal{G}(q)\,\mathcal{G}(q^3)\,\mathcal{G}(q^4)\,\mathcal{G}(q^6)}$$

$$d^s_{(n-1,n-1,3n-1)}(q) = \frac{Q(ns_0,ns_1,3ns_2; \; D^{(3)}_4)}{Q(s; \; G^{(1)}_2)}$$

7) $A^{(2)}_{2\ell}$:

$$d^{(12\ldots2)}_\Lambda(q) = \frac{Q(n_0+1,2(n_1+1),\ldots,2(n_\ell+1); \; D^{(2)}_{\ell+1})}{Q(12\ldots2; \; A^{(2)}_{2\ell})}$$

$$d_\Lambda^{(1\ldots12)}(q) = \frac{Q(n_0+1,\ldots,n_{\ell-1}+1,2(n_\ell+1);\ C_\ell^{(1)})}{\mathcal{S}(q)^\ell}$$

$$d_\Lambda^{(12\ldots24)}(q)$$

$$= \frac{Q(n_0+1,2(n_1+1),\ldots,2(n_{\ell-1}+1),4(n_\ell+1);\ A_{2\ell}^{(2)})}{Q(12\ldots24;\ A_{2\ell}^{(2)})}$$

$$d_{(2n-1,n-1,\ldots,n-1)}^s(q) = \frac{Q(ns_\ell,\ldots,ns_1,2ns_0;\ C_\ell^{(1)})}{Q(s;\ A_{2\ell}^{(2)})}$$

$$d_{(2n-1,\ldots,2n-1,n-1)}^s(q) = \frac{Q(ns_\ell,2ns_{\ell-1},\ldots,2ns_0;\ D_{\ell+1}^{(2)})}{Q(s;\ A_{2\ell}^{(2)})}$$

$$d_{(4n-1,2n-1,\ldots,2n-1,n-1)}^s(q)$$

$$= \frac{Q(ns_\ell,2ns_{\ell-1},\ldots,2ns_1,4ns_0;\ A_{2\ell}^{(2)})}{Q(s;\ A_{2\ell}^{(2)})}$$

8) $A_{2\ell-1}^{(2)}$:

$$d_\Lambda^{(1\ldots12)}(q) = \frac{Q(n_0+1,\ldots,n_{\ell-1}+1,2(n_\ell+1);\ A_{2\ell-1}^{(2)})}{Q(1\ldots12;\ A_{2\ell-1}^{(2)})}$$

$$d_{(2n-1,\ldots,2n-1,n-1)}^s(q) = \frac{Q(2ns_0,\ldots,2ns_{\ell-1},ns_\ell\ ;\ B_\ell^{(1)})}{Q(s;\ A_{2\ell-1}^{(2)})}$$

9) $D_{\ell+1}^{(2)}$:

$$d_\Lambda^{(12\ldots2)}(q) = \frac{Q(n_0+1,2(n_1+1),\ldots,2(n_\ell+1);\ A_{2\ell}^{(2)})}{Q(12\ldots2;\ D_{\ell+1}^{(2)})}$$

$$d_\Lambda^{(2\ldots21)}(q) = \frac{Q(n_\ell+1,2(n_{\ell-1}+1),\ldots,2(n_0+1);\ A_{2\ell}^{(2)})}{Q(2\ldots21;\ D_{\ell+1}^{(2)})}$$

$$d_{\Lambda}^{(12\ldots 21)}(q) = \frac{Q(n_0+1, 2(n_1+1), \ldots, 2(n_{\ell-1}+1), n_\ell+1; \ D_{\ell+1}^{(2)})}{Q(12\ldots 21; \ D_{\ell+1}^{(2)})}$$

$$d_{(2n-1, n-1, \ldots, n-1)}^{s}(q) = \frac{Q(ns_\ell, \ldots, ns_1, 2ns_0; \ A_{2\ell}^{(2)})}{Q(s; \ D_{\ell+1}^{(2)})}$$

$$d_{(n-1, \ldots, n-1, 2n-1)}^{s}(q) = \frac{Q(ns_0, \ldots, ns_{\ell-1}, 2ns_\ell; \ A_{2\ell}^{(2)})}{Q(s; \ D_{\ell+1}^{(2)})}$$

$$d_{(2n-1, n-1, \ldots, n-1, 2n-1)}^{s}(q)$$

$$= \frac{Q(2ns_0, ns_1, \ldots, ns_{\ell-1}, 2ns_\ell; \ C_\ell^{(1)})}{Q(s; \ D_{\ell+1}^{(2)})}$$

10) $E_6^{(2)}$:

$$d_{\Lambda}^{(11122)}(q) = \frac{Q(n_0+1, n_1+1, n_2+1, 2(n_3+1), 2(n_4+1); \ E_6^{(2)})}{Q(11122; \ E_6^{(2)})}$$

$$d_{(2n-1, 2n-1, 2n-1, n-1, n-1)}^{s}(q)$$

$$= \frac{Q(2ns_0, 2ns_1, 2ns_2, ns_3, ns_4; \ F_4^{(1)})}{Q(s; \ E_6^{(2)})}$$

11) $D_4^{(3)}$:

$$d_{\Lambda}^{(113)}(q) = \frac{Q(n_0+1, n_1+1, 3(n_2+1); \ D_4^{(3)})}{Q(113; \ D_4^{(3)})}$$

$$= \frac{\varphi(q^2)\,\varphi(q^3)\,\varphi(q^{18})\ Q(n_0+1, n_1+1, 3(n_2+1); \ D_4^{(3)})}{\varphi(q)^2\,\varphi(q^6)^2\,\varphi(q^9)}$$

$$d_{(3n-1, 3n-1, n-1)}^{s}(q) = \frac{Q(3ns_0, 3ns_1, ns_2; \ G_2^{(1)})}{Q(s; \ D_4^{(3)})}$$

We want to note that there is a class of specialized characters which can be computed by making use of suitable inclusions of affine Lie algebras; some of them are shown in Table 2, where we shall use the following notations:

$$\prod_{j}^{(+)}(1-q^{m(j)}) = \prod_{\substack{j \in Z \\ m(j) > 0}}(1-q^{m(j)})$$

$$\prod_{j}^{(+)}[(1-q^{m_1(j)})\dots(1-q^{m_r(j)})] = \prod_{i=1}^{r}[\prod_{j}^{(+)}(1-q^{m_i(j)})].$$

Table 2

1) $A_2^{(1)}$:

i) $\Lambda = (n_0-1, n_1-1, n_1-1)$;

$$d_\Lambda^{(112)}(q) = \frac{\mathscr{G}(q^{|n|})\mathscr{G}(q^{3|n|})}{\mathscr{G}(q)^2}\prod_{j}^{(+)}[(1-q^{j|n|\pm n_0})(1-q^{j|n|\pm n_1}) \times (1-q^{3j|n|\pm 3n_1})]$$

where $|n| = n_0+2n_1$.

ii) $\Lambda = (2n-1, n-1, n-1)$ and $s = (s_0, s_0, s_2)$;

$$d_\Lambda^s(q) = \frac{\mathscr{G}(q^{n|s|})\mathscr{G}(q^{3n|s|})}{\mathscr{G}(q^{|s|})^2}$$

$$\times \prod_{j}^{(+)}\frac{(1-q^{n(j|s|\pm s_0)})(1-q^{n(j|s|\pm s_2)})(1-q^{3n(j|s|\pm s_0)})}{(1-q^{j|s|\pm s_0})^2(1-q^{j|s|\pm s_2})}$$

where $|s| = 2s_0+s_2$.

2) $A_3^{(1)}$:

i) $\Lambda = (2n-1, n-1, n-1, n-1)$;

$$d_{\Lambda}^{(1122)}(q) = \frac{\mathcal{G}(q^n)\,\mathcal{G}(q^{2n})\,\mathcal{G}(q^{8n})}{\mathcal{G}(q)^2\,\mathcal{G}(q^2)}$$

$$\times \prod_{j}^{(+)} [(1-q^{n(8j\pm3)})(1-q^{n(16j\pm2)})],$$

$$d_{\Lambda}^{(2211)}(q) = \frac{\mathcal{G}(q^n)\,\mathcal{G}(q^{2n})\,\mathcal{G}(q^{8n})}{\mathcal{G}(q)^2\,\mathcal{G}(q^2)}$$

$$\times \prod_{j}^{(+)} [(1-q^{n(8j\pm1)})(1-q^{n(16j\pm6)})],$$

$$d_{\Lambda}^{s}(q) = \frac{\mathcal{G}(q^{n|s|})\,\mathcal{G}(q^{2n|s|})^2}{\mathcal{G}(q^{|s|})^3} \prod_{i=1,3} \prod_{j}^{(+)} \frac{(1-q^{n(j|s|\pm s_i)})}{(1-q^{j|s|\pm s_i})}$$

$$\times \prod_{k=1}^{2} \prod_{j}^{(+)} \frac{(1-q^{kn(j|s|\pm s_0)})(1-q^{kn(j|s|\pm(s_0+s_1))})}{(1-q^{j|s|\pm s_0})(1-q^{j|s|\pm(s_0+s_1)})}$$

where $s = (s_0, s_1, s_0, s_3)$ and $|s| = 2s_0 + s_1 + s_3$.

ii) $\Lambda = (2n-1, 2n-1, n-1, n-1)$;

$$d_{\Lambda}^{(1112)}(q) = \frac{\mathcal{G}(q^n)\,\mathcal{G}(q^{2n})\,\mathcal{G}(q^{8n})}{\mathcal{G}(q)^3}$$

$$\times \prod_{j}^{(+)} [(1-q^{n(8j\pm3)})(1-q^{n(16j\pm2)})],$$

$$d_{\Lambda}^{(2111)}(q) = \frac{\mathcal{G}(q^n)\,\mathcal{G}(q^{2n})\,\mathcal{G}(q^{8n})}{\mathcal{G}(q)^3}$$

$$\times \prod_{j}^{(+)} [(1-q^{n(8j\pm1)})(1-q^{n(16j\pm6)})].$$

iii) $\Lambda = (n_0-1, n_0-1, n_2-1, n_2-1)$;

$$d_{\Lambda}^{(1212)}(q) = d_{\Lambda}^{(2121)}(q) = \frac{\mathcal{G}(q^6)\,\mathcal{G}(q^{2|n|})^2\,\mathcal{G}(q^{3|n|})}{\mathcal{G}(q)^2\,\mathcal{G}(q^3)^2}$$

$$\times \prod_{k=1}^{3} \prod_{j}^{(+)} (1-q^{k(j|n|\pm n_0)})$$

where $|n| = n_0 + n_2$.

iv) $\Lambda = (n_0-1, n_1-1, n_0-1, n_3-1)$;

$$d_{\Lambda}^{(2111)}(q) = \frac{\mathcal{G}(q^{|n|})\,\mathcal{G}(q^{2|n|})^2}{\mathcal{G}(q)^3} \left[\prod_{i=1,3} \prod_{j}^{(+)} (1-q^{j|n|\pm n_i})\right]$$

$$\times \prod_{k=1}^{2} \prod_{j}^{(+)} [(1-q^{k(j|n|\pm n_0)})(1-q^{k(j|n|\pm(n_0+n_1))})]$$

where $|n| = 2n_0 + n_1 + n_3$.

v) $\Lambda = (2n-1, n-1, 2n-1, n-1)$ and $s = (s_0, s_0, s_2, s_2)$;

$$d_{\Lambda}^{s}(q) = \frac{\mathcal{G}(q^{2n|s|})^2\,\mathcal{G}(q^{3n|s|})}{\mathcal{G}(q^{|s|})^2\,\mathcal{G}(q^{2|s|})}$$

$$\times \prod_{k=1}^{3} \prod_{j}^{(+)} (1-q^{kn(j|s|\pm s_0)})$$

$$\times \prod_{j}^{(+)} [(1-q^{j|s|\pm s_0})^2(1-q^{2j|s|\pm 2s_0})]$$

where $|s| = s_0 + s_2$.

3) $A_5^{(1)}$:

$$d_{\Lambda_0}^{(111122)}(q) = \frac{\mathcal{G}(q^2)\,\mathcal{G}(q^{12})^2}{\mathcal{G}(q)\,\mathcal{G}(q^6)\,\mathcal{G}(q^8)}$$

$$d_{\Lambda_0}^{(222211)}(q) = \frac{\mathcal{G}(q^6)^2\,\mathcal{G}(q^{12})}{\mathcal{G}(q^2)\,\mathcal{G}(q^3)}$$

$$d_{\Lambda_0}^{(112112)}(q) = \frac{\mathscr{S}(q^2)\,\mathscr{S}(q^3)\,\mathscr{S}(q^8)\,\mathscr{S}(q^{12})}{\mathscr{S}(q)\,\mathscr{S}(q^4)^2\,\mathscr{S}(q^6)}$$

$$d_{\Lambda_0}^{(211211)}(q) = \frac{\mathscr{S}(q^6)^5\,\mathscr{S}(q^8)}{\mathscr{S}(q^2)\,\mathscr{S}(q^3)^2\,\mathscr{S}(q^4)\,\mathscr{S}(q^{12})}$$

$$d_{\Lambda_0}^{(112222)}(q) = \frac{\mathscr{S}(q^{12})}{\mathscr{S}(q^2)} \prod_j^{(+)} (1+q^{12j\pm1})$$

$$d_{\Lambda_0}^{(221122)}(q) = \frac{\mathscr{S}(q^{12})}{\mathscr{S}(q^2)} \prod_j^{(+)} (1+q^{12j\pm5})$$

4) $A_7^{(1)}$:

For $s = (s_0, s_1, s_0, s_3, s_0, s_1, s_0, s_3)$,

$$d_{\Lambda_0}^{s}(q) = \frac{\mathscr{S}(q^{2|s|})^3}{\mathscr{S}(q^{|s|})^3} \prod_j^{(+)} [(1+q^{j|s|\pm s_0})(1+q^{j|s|\pm(s_0+s_1)})],$$

where $|s| = 2s_0 + s_1 + s_3$.

5) $D_\ell^{(1)}$:

For $s = (s_1, s_1, s_2, s_3, \ldots, s_\ell) \in \mathbb{N}^{\ell+1}$,

$$d_{\Lambda_0}^{s}(q) = \frac{\mathscr{S}(q^{2|s|})^2}{\mathscr{S}(q^{|s|})^2} \prod_{i=1}^{\ell-1} \prod_j^{(+)} (1+q^{j|s|\pm(s_1+\ldots+s_i)})$$

where $|s| = 2(s_1 + \ldots + s_{\ell-2}) + s_{\ell-1} + s_\ell$.

6) $E_6^{(1)}$:

For $s = \begin{pmatrix} s_0 & s_0 & s_3 & s_0 & s_0 \\ & & s_0 & & \\ & & s_0 & & \end{pmatrix}$,

$$d_{\Lambda_0}^{s}(q) = \frac{\mathscr{S}(q^{3|s|})}{\mathscr{S}(q^{|s|})} \prod_j^{(+)} \frac{(1-q^{3j|s|\pm 3s_0})}{(1-q^{j|s|\pm s_0})}$$

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where $|s| = 3s_0 + s_3$.

7) $E_7^{(1)}$:

For $s = \begin{pmatrix} s_0 & s_1 & s_0 & s_3 & s_0 & s_1 & s_0 \\ & & & s_7 & & & \end{pmatrix}$,

$$d_{\Lambda_0}^s(q) = \mathscr{G}(q^{2|s|}) \prod_j^{(+)} [(1+q^{j|s|\pm s_0})(1+q^{j|s|\pm(s_0+s_1)})$$
$$\times (1+q^{j|s|\pm(2s_0+s_1+s_3)})]$$

where $|s| = 4s_0 + 2s_1 + 2s_3 + s_7$.

8) $F_4^{(1)}$:

For $\Lambda = (n_0-1, n_1-1, n_0-1, n_3-1, n_4-1)$,

$$d_{\Lambda}^{(21111)}(q) = \frac{\mathscr{G}(q^{|n|})^6}{\mathscr{G}(q)^4} [\prod_{i=1}^4 \prod_j^{(+)} (1-q^{j|n|\pm n_i})]$$

$$\times [\prod_{i=1}^3 \prod_j^{(+)} (1-q^{j|n|\pm(n_0+n_i)})][\prod_j^{(+)} (1-q^{j|n|\pm 2(n_0+n_1)})]$$

$$\times [\prod_{i=1,3} \prod_j^{(+)} (1-q^{j|n|\pm(2n_0+n_i)})]$$

$$\times [\prod_{k=1}^3 \prod_j^{(+)} (1-q^{j|n|\pm(kn_0+n_1+n_3)})(1-q^{j|n|\pm((k+1)n_0+2n_1+n_3)})]$$

where $n_2 = n_0$ and $|n| = 4n_0 + 2n_1 + 2n_3 + n_4$.

9) $G_2^{(1)}$:

i) $s = (s_0, s_1, s_0)$;

$$d_{\Lambda_0}^s(q) = \prod_j^{(+)} \frac{(1+q^{j|s|\pm s_0})}{(1-q^{|s|(5j\pm 2)})} \quad ,$$

$$d^s_{\Lambda_2}(q) = \prod_j^{(+)} \frac{(1+q^{j|s|\pm s_0})}{(1-q^{|s|(5j\pm1)})} \ ,$$

where $|s| = 2s_0+s_1$.

ii) $\Lambda = (n_0-1, n_0-1, n_2-1)$;

$$d^{(211)}_{\Lambda}(q) = \frac{\mathscr{S}(q^{|n|})}{\mathscr{S}(q)^2} [\prod_{k=1}^{3} \prod_j^{(+)} (1-q^{j|n|\pm kn_0})]$$

$$\times \ [\prod_j^{(+)} (1-q^{3j|n|\pm 3n_0})]$$

where $|n| = 3n_0+n_2$.

10) $A^{(2)}_{2\ell-1}$:

For $s = (s_1, s_1, s_2, s_3, \ldots, s_\ell) \in N^{\ell+1}$,

$$d^s_{\Lambda_0}(q) = \prod_{i=0}^{\ell-1} \prod_j^{(+)} (1+q^{j|s|\pm(s_1+\ldots+s_i)})$$

where $|s| = 2(s_1+\ldots+s_{\ell-1})+s_\ell$.

11) $D^{(2)}_3$:

For $\Lambda = (n_0-1, n_1-1, n_0-1)$,

$$d^{(211)}_{\Lambda}(q) = \frac{\mathscr{S}(q^4)\,\mathscr{S}(q^{|n|})\,\mathscr{S}(q^{4|n|})}{\mathscr{S}(q)^2\,\mathscr{S}(q^8)}$$

$$\times \ \prod_j^{(+)} [(1-q^{j|n|\pm n_0})(1-q^{4j|n|\pm 2n_0})]$$

where $|n| = n_0+n_1$.

12) $E^{(2)}_6$:

For $s = (s_0, s_1, s_0, s_3, s_4)$,

$$d^s_{\Lambda_0}(q) = \mathcal{G}(q^{|s|})^2 \prod_j^{(+)} [(1+q^{j|s|\pm s_0})(1+q^{j|s|\pm(s_0+s_1)})$$

$$\times (1+q^{j|s|\pm(2s_0+s_1+s_3)})]$$

where $|s| = 4s_0+2s_1+2s_3+s_4$.

13) $D_4^{(3)}$:

i) $\Lambda = (2n-1,n-1,n-1)$ and $s = (s_0,s_0,s_2)$;

$$d^s_{\Lambda}(q) = \frac{\mathcal{G}(q^{n|s|})}{\mathcal{G}(q^{|s|})} \left[\prod_j^{(+)} \frac{(1-q^{3n(j|s|\pm s_0)})}{(1-q^{j|s|\pm s_0})} \right]$$

$$\times \left[\prod_{k=1}^3 \prod_j^{(+)} \frac{(1-q^{n(j|s|\pm k s_0)})}{(1-q^{j|s|\pm k s_0})} \right]$$

where $|s| = 3s_0+s_2$.

ii) $\Lambda = (n_0-1,n_1-1,n_0-1)$;

$$d^{(211)}_{\Lambda}(q) = \frac{\mathcal{G}(q^{|n|})^2}{\mathcal{G}(q)^2} \prod_j^{(+)} \left[\frac{(1-q^{3(5j\pm2)})}{(1-q^{|n|(5j\pm2)})} \right.$$

$$\left. \times \prod_{i=0,1} (1-q^{j|n|\pm n_i}) \right]$$

$$d^{(112)}_{\Lambda}(q) = \frac{\mathcal{G}(q^{|n|})^2}{\mathcal{G}(q)^2} \prod_j^{(+)} \left[\frac{(1-q^{3(5j\pm1)})}{(1-q^{|n|(5j\pm1)})} \right.$$

$$\left. \times \prod_{i=0,1} (1-q^{j|n|\pm n_i}) \right]$$

where $|n| = 2n_0+n_1$.

3. Power series identities:

In this section, we are going to compute specialized characters of an integrable highest weight module of an affine Lie algebra $\mathfrak{g}(A)$ in terms of weight system. It may be expected that many things will be comparatively simple if A and Λ satisfy the condition

(*) $\mathrm{Max}_0(\Lambda)$ contains only one element Λ,

where $\mathrm{Max}(\Lambda) = \{$ maximal weights of $L(\Lambda)\}$ and $\mathrm{Max}_0(\Lambda) = P_+ \cap \mathrm{Max}(\Lambda)$.

For such $L(\Lambda)$, the weight system $P(\Lambda)$ is given by

$$P(\Lambda) = \left\{ \lambda \in P; \begin{array}{l} |\lambda|^2 \leqq |\Lambda|^2 \\ \langle \lambda, c \rangle = \langle \Lambda, c \rangle \end{array} \right\} ,$$

where c is the canonical central element and $|\ |$ is the norm defined by the standard bilinear form on $\mathfrak{g}(A)$.

Table 3 is the complete list of $L(\Lambda; A)$'s enjoying the property (*) (see also [5]):

Table 3

1) $A_1^{(1)}$: Λ_0, Λ_1, $\Lambda_0 + \Lambda_1$.

   $A_\ell^{(1)}$: Λ_i $(i = 0,\ldots,\ell)$.

2) $B_\ell^{(1)}$: Λ_ℓ.

3) $C_2^{(1)}$: Λ_1.

4) $D_\ell^{(1)}$: Λ_i $(i = 0,1,\ell-1,\ell)$.

5) $E_6^{(1)}$: Λ_i $(i = 0,4,6)$.

$E_7^{(1)}$: $\quad \Lambda_0$, Λ_6, $\Lambda_0 + \Lambda_6$.

$E_8^{(1)}$: $\quad \Lambda_0$.

6) $A_{2\ell}^{(2)}$ ($\ell \geqq 1$): $\quad \Lambda_0$.

7) $A_{2\ell-1}^{(2)}$: $\quad \Lambda_0$, Λ_1.

8) $D_{\ell+1}^{(2)}$: $\quad \Lambda_0$, Λ_ℓ, $\Lambda_0 + \Lambda_\ell$.

9) $E_6^{(2)}$: $\quad \Lambda_0$.

10) $D_4^{(3)}$: $\quad \Lambda_0$.

Now look at the weight system of $L(\Lambda_0; A_1^{(1)})$;

$$P(\Lambda_0) = \{ \lambda(m,n) = \Lambda_0 + m\alpha_1 + n\delta \ ; \ m,n \in Z \text{ and } n \leqq -m^2 \},$$

where $\delta = \alpha_0 + \alpha_1$ is the fundamental imaginary root. Since two elements $\lambda(m_1,n_1)$ and $\lambda(m_2,n_2)$ in $P(\Lambda_0)$ are W-conjugate if and only if $n_1 + m_1^2 = n_2 + m_2^2$, we can put

$$\xi(k) = \text{multiplicity of } \lambda(-m^2-k,m)$$

for $k \in N_0$ and $m \in Z$. Then the character of $L(\Lambda_0; A_1^{(1)})$ is written

$$\text{ch } L(\Lambda_0) = e^{\Lambda_0} \sum_{k=0}^{\infty} \sum_{m \in Z} \xi(k) \, e^{m\alpha_1 - (m^2+k)\delta}$$

$$= e^{\Lambda_0} [\sum_{k=0}^{\infty} \xi(k) \, e^{-k\delta}][\sum_{m \in Z} e^{-m\alpha_1 - m^2\delta}] \quad \ldots (1).$$

First, in order to compute the multiplicity $\xi(k)$, let us take the principal specialization;

$$d_{\Lambda_0}(q) = [\sum_{k=0}^{\infty} \xi(k)q^{2k}][\sum_{m \in Z} q^{2m^2+m}].$$

Noting that
$$d_{\Lambda_0}(q) = \frac{\mathscr{S}(q^2)}{\mathscr{S}(q)}$$

and
$$\sum_{m \in Z} q^{2m^2+m} = \frac{\mathscr{S}(q^2)^2}{\mathscr{S}(q)} \qquad \text{(Gauss 1866)},$$

one obtains

$$\sum_{k=0}^{\infty} \xi(k)q^{2k} = \frac{1}{\mathscr{S}(q^2)}$$

i.e.,

$$\sum_{k=0}^{\infty} \xi(k)q^{k} = \frac{1}{\mathscr{S}(q)} \qquad \qquad \dots \ (2).$$

Thus it is proved that the string function of $L(\Lambda_0; A_1^{(1)})$ is equal to the generating function of the partition functions (this is a well-known fact in some literatures).

Next, the specialization of (1) of type $s = (s_0, s_1)$ gives

$$d^s_{\Lambda_0}(q) = [\sum_{k=0}^{\infty} \xi(k)q^{|s|k}][\sum_{m \in Z} q^{s_1 m + |s|m^2}]$$

$$= \frac{1}{\mathscr{S}(q^{|s|})} \sum_{m \in Z} q^{|s|m^2 + s_1 m} \qquad \dots \ (3),$$

where $|s| = s_0 + s_1$.

Now combining (3) with a formula

$$d^s_{\Lambda_0}(q) = \prod_{j}^{(+)} [(1+q^{|s|j})(1+q^{2|s|j \pm s_0})]$$

in Table 1, one obtains an identity

$$\sum_{m \in Z} q^{|s|m^2 + s_1 m} = \mathcal{G}(q^{2|s|}) \prod_j^{(+)} (1 + q^{2|s|j \pm s_0}).$$

Note that this formula is equivalent to the Jacobi triple product identity.

Similar arguments are valid for highest weight modules in Table 3, and one can deduce power series identities, some of which are shown in Table 4;

Table 4

1) $L(\Lambda_0; A_1^{(1)})$, $L(\Lambda_0; A_2^{(2)})$ or $L(\rho; A_1^{(1)})$:

For $(s_0, s_1) \in N \times \frac{1}{2} N_0$,

$$\sum_{n \in Z} q^{|s|n^2 + s_1 n} = \mathcal{G}(q^{2|s|}) \prod_j^{(+)} (1 + q^{2j|s| \pm s_0})$$

where $|s| = s_0 + s_1$.

2) $L(\Lambda_1; C_2^{(1)})$:

For $s = (s_0, s_1, s_2) \in N^3$,

$$\sum_{n \in Z^2} q^{X_s(n)} = \mathcal{G}(q^{|s|})^2 \prod_j^{(+)} [(1 + q^{j|s| \pm s_1})(1 + q^{j|s| \pm (s_0 + s_1)})],$$

where $X_s(n) = X_s(n_1, n_2)$

$$= |s| \left[\frac{n_1^2 - n_1}{2} + n_2^2 - n_1 n_2 \right] + \sum_{i=1}^{2} s_i n_i$$

and $|s| = s_0 + 2s_1 + s_2$.

3) $L(\Lambda_\ell; B_\ell^{(1)})$ $(\ell \geqq 3)$:

For $s = (s_0, \ldots, s_\ell) \in \mathbb{N}^{\ell+1}$,

$$\sum_{n \in \mathbb{Z}^\ell} q^{X_s(n)} = \mathscr{S}(q^{|s|})^\ell \prod_{i=1}^{\ell} \prod_{j}^{(+)} (1 + q^{j|s| \pm (s_i + \ldots + s_\ell)}),$$

where $X_s(n) = X_s(n_1, \ldots, n_\ell)$

$$= |s| \left[\sum_{i=1}^{\ell-1} (n_i^2 - n_i n_{i+1}) + \frac{n_\ell^2 - n_\ell}{2} \right] + \sum_{i=1}^{\ell} s_i n_i$$

and $|s| = s_0 + s_1 + 2(s_2 + \ldots + s_\ell)$.

4) $L(\Lambda_0; A_{2\ell}^{(2)})$ $(\ell \geqq 2)$:

For $s = (s_0, \ldots, s_\ell) \in \mathbb{N}^{\ell+1}$,

$$\sum_{n \in \mathbb{Z}^\ell} q^{X_s(n)} = \frac{1}{\mathscr{S}(q^{|s|})^\ell} \prod_{i=0}^{\ell-1} \prod_{j}^{(+)} (1 + q^{j|s| \pm (s_0 + \ldots + s_i)}),$$

where $X_s(n) = X_s(n_0, \ldots, n_{\ell-1})$

$$= |s| \left[\sum_{i=1}^{\ell-1} (n_i^2 - n_{i-1} n_i) + \frac{n_0^2 - n_0}{2} \right] + \sum_{i=0}^{\ell-1} s_i n_i$$

and $|s| = 2(s_0 + \ldots + s_{\ell-1}) + s_\ell$.

5) $L(\Lambda_0; D_{\ell+1}^{(2)})$ $(\ell \geqq 2)$:

For $s = (s_0, \ldots, s_\ell) \in \mathbb{N}^{\ell+1}$,

$$\sum_{n \in \mathbb{Z}^\ell} q^{X_s(n)} = \mathscr{S}(q^{2|s|})^\ell \prod_{i=0}^{\ell-1} \prod_{j}^{(+)} (1 + q^{2j|s| \pm (s_0 + \ldots + s_i)}),$$

where $X_s(n) = X_s(n_1, \ldots, n_\ell)$

$$= |s|\,[\,2\sum_{i=1}^{\ell-1}(n_i^2-n_i n_{i+1}) + n_\ell^2\,] + \sum_{i=1}^{\ell}s_i n_i$$

and $|s| = s_0+\ldots+s_\ell$.

6) $L(\Lambda_0+\Lambda_\ell\,;\,D_{\ell+1}^{(2)})\ (\ell \geqq 2)$:

For $s = (s_0,\ldots,s_\ell)\in N^{\ell+1}$,

$$\sum_{n\in Z^\ell} q^{X_s(n)} = \mathscr{G}(q^{|s|})\prod_{i=0}^{\ell-1}\prod_{j}^{(+)}(1+q^{\,j|s|\pm(s_0+\ldots+s_i)}),$$

where $X_s(n) = X_s(n_1,\ldots,n_\ell)$

$$= |s|\,[\,\sum_{i=1}^{\ell-1}(n_i^2-n_i n_{i+1}) + \frac{n_\ell^2-n_\ell}{2}\,] + \sum_{i=1}^{\ell}s_i n_i$$

and $|s| = s_0+\ldots+s_\ell$.

7) $L(\Lambda_0\,;\,A_{2\ell-1}^{(2)})\ (\ell \geqq 3)$:

For $s = (s_1,\ldots,s_\ell)\in N^\ell$,

$$\sum_{n\in Z^\ell} q^{X_s(n)} = \mathscr{G}(q^{|s|})^{\ell-2}\mathscr{G}(q^{2|s|})^2\prod_{i=1}^{\ell-1}\prod_{j}^{(+)}(1+q^{\,j|s|\pm(s_1+\ldots+s_i)}),$$

where $X_s(n) = X_s(n_1,\ldots,n_\ell)$

$$= |s|\,[\,\sum_{i=1}^{\ell}n_i^2 - \sum_{i=1}^{\ell-2}n_i n_{i+1} - n_{\ell-2}n_\ell\,] + \sum_{i=1}^{\ell}s_i n_i$$

and $|s| = 2(s_1+\ldots+s_{\ell-2})+s_{\ell-1}+s_\ell$.

8) $L(\Lambda_0\,,\,A_\ell^{(1)})\ (\ell \geqq 1)$:

For $s = (s_0,\ldots,s_\ell)\in N^{\ell+1}$,

$$\sum_{n \in Z^{\ell}} q^{X_s(n)} = \mathcal{G}(q^{|s|}) \; d^{(s; A_{\ell}^{(1)})}_{\Lambda_0}(q),$$

where $X_s(n) = X_s(n_1, \ldots, n_{\ell})$

$$= |s| [\sum_{i=1}^{\ell-1} (n_i^2 - n_i n_{i+1}) + n_{\ell}^2] + \sum_{i=1}^{\ell} s_i n_i$$

and $|s| = s_0 + \ldots + s_{\ell}$.

Example 1. $s = 1 = (1, \ldots, 1)$;

$$\sum_{n \in Z^{\ell}} q^{X_1(n)} = \frac{\mathcal{G}(q^{\ell+1})^{\ell+1}}{\mathcal{G}(q)},$$

where $X_1(n) = X_1(n_1, \ldots, n_{\ell})$

$$= (\ell+1) [\sum_{i=1}^{\ell-1} (n_i^2 - n_i n_{i+1}) + n_{\ell}^2] + \sum_{i=1}^{\ell} n_i.$$

Example 2. $\ell = 2$ and $s = (s_1, s_1, s_2)$;

$$\sum_{n \in Z^2} q^{X(n)} = \mathcal{G}(q^{|s|}) \mathcal{G}(q^{3|s|}) \prod_{j}^{(+)} \frac{(1 - q^{3(j|s| \pm s_1)})}{(1 - q^{j|s| \pm s_1})},$$

where $X(n) = X(n_1, n_2)$

$$= |s|(n_1^2 + n_2^2 - n_1 n_2) + \sum_{i=1}^{2} s_i n_i$$

and $|s| = 2s_1 + s_2$.

Example 3. $\ell = 3$ and $s = (s_2, s_1, s_2, s_3)$;

$$\sum_{n \in Z^3} q^{X(n)} = \mathcal{G}(q^{|s|}) \mathcal{G}(q^{2|s|})^2 \prod_{j}^{(+)} [(1 + q^{j|s| \pm s_2})(1 + q^{j|s| \pm (s_1 + s_2)})]$$

where $X(n) = X(n_1, n_2, n_3)$

$$= |s| \left[\sum_{i=1}^{3} n_i^2 - n_2(n_1 + n_3) \right] + \sum_{i=1}^{3} s_i n_i$$

and $|s| = s_1 + 2s_2 + s_3$.

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