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# A Subsystem of Classical Analysis proper to Takeuti's Reduction Method for $\Pi_1^1$-Analysis

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After Gentzen's works for the pure number theory, G.Takeuti gave consistency proofs of some impredicative subsystems of classical analysis in [7], [9], [10] and [11] ( [11] with M.Yasugi ). In these proofs, the only 'Überschreitung' beyond the finitist standpoint in the Hilbert's sense is accessibility of some systems of o.d.'s ( ordinal diagrams ) which were also introduced by Takeuti in [6] and [8]. Thus these works may be regarded as nice extensions of Gentzen's. But, unfortunately, the consistency proof for the formal system SINN'*) which is equivalent to ($\Pi_1^1$-CA) + (BI) is unsatisfactory for the following two reasons:

i) The proof consists of two ideas, i.e., $\gamma$-degree ( cf. [10, p.317] ) and use of the substitution rule (cf. [10, p.318]). And in the reduction of impredicative proof-figures, the latter plays an essential role, while the former can be deleted.

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*) Usually this proof is said to be one for SINN which is equivalent to ($\Pi_1^1$-CA), but as remarked in [10, footnote 2], it is at the same time one for SINN'.

ii) The system of o.d.'s $O(\omega+1,\omega^3)$ with respect to $<_0$ used in this proof was not shown to be optimal for the consistency proof for ($\Pi^1_1$-CA) + (BI).

In this paper, we will propose a subsystem of classical analysis AII which is convenient to the reduction method using the substitution rule and equivalent to $\overline{\text{SINN}}$', and prove the consistency of AII by the accessibility of the system $O(\omega+1,1)$ with respect to $<_0$, following Gentzen [2] and Takeuti [10]. Also in [1], we will show that the transfinite induction up to each o.d. from the system $O(\omega+1,1)$ with respect to $<_0$ is derivable in AII. Thus we will complement the Takeuti's consistency proof for ($\Pi^1_1$-CA) + (BI). (cf. i), ii) ).

In §1 the definition of AII and some preliminary definitions for a consistency proof will be given. In §2 the main lemma will be proved and from which together with the accessibility of the system $O(\omega+1,1)$ with respect to $<_0$, the consistency of AII follows immediately. A discussion for the significance of the consistency proof will be given in the Appendix.

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## §1. Preliminary Definitions

In this paper, we will use the terminology and notation in the same sense as those in [PT].

Definition 1.1.

The systems of second order arithmetic INN', AII and $AI\Pi_1^1$ are obtained from INN ( Definition 27.4, [PT, p.320] ) by modifying second order $\forall$:left

$$\frac{F(V),\ \Gamma \rightarrow \Delta}{\forall\phi F(\phi),\ \Gamma \rightarrow \Delta}$$

, as follows :

1.1.1. The system INN' ( , which was called SINN' in [10, footnote 2] ).

1.1.1.1. The principal formula $\forall\phi F(\phi)$ is isolated ( Definition 27.2.(5), [PT, p.322] ),

or,

1.1.1.2. the abstract V in the auxiliary formula ( Nebenformel ) F(V) is isolated.

1.1.2. The system AII.

1.1.2.1. The principal formula $\forall\phi F(\phi)$ is isolated,

or,

1.1.2.2. the abstract V in F(V) is a second order free variable.

1.1.3. The system $AI\Pi^1_1$.

1.1.3.1. The principal formula $\forall\phi F(\phi)$ is a $\Pi^1_1$-formula, or,

1.1.3.2. the abstract V in F(V) is a second order free variable.

AII ( $AI\Pi^1_1$ ) is an abbreviation of the Axiom of Instantiation $\forall\phi F(\phi) \supset F(V)$ with the Isolated formulae ( $\Pi^1_1$-formulae ) $\forall\phi F(\phi)$.

Observe that $AI\Pi^1_1$ contains (BI) +($\Pi^0_\infty$-CA), and (BI) contains $AI\Pi^1_1$, hence $AI\Pi^1_1$, (BI) + ($\Pi^0_\infty$-CA) and (BI) are equivalent each other. Also note that a formula of the form,

$$\exists\phi\forall x_1\ldots\forall x_n(\ \phi(x_1,\ldots,x_n) \equiv A(x_1,\ldots,x_n)),$$

is isolated provided that A is a $\Pi^1_1$-formula and the bound variable $\phi$ does not occur in A. Hence the $\Pi^1_1$-comprehension axioms are derivable in AII, and so, AII, INN' and ($\Pi^1_1$-CA) + (BI) are equivalent each other.

In the rest of this section, we will give some preliminary definitions for a consistency proof of AII.

Following the idea of Takeuti, we add the rule of substitution to AII.

Definition 1.2.

1.2.1. Rule of substitution.

$$\frac{A_1 \quad ,\dots,A_n \longrightarrow B_1 \quad ,\dots,B_m}{A_1(\begin{smallmatrix}\alpha\\ V\end{smallmatrix}),\dots,A_n(\begin{smallmatrix}\alpha\\ V\end{smallmatrix}) \longrightarrow B_1(\begin{smallmatrix}\alpha\\ V\end{smallmatrix}),\dots,B_m(\begin{smallmatrix}\alpha\\ V\end{smallmatrix})}$$

where $\alpha$ is a second order free variable, V is an arbitrary abstract with the same number of argument-places as $\alpha$, $A_1,\dots,A_n$, $B_1,\dots,B_m$ are arbitrary formulae, and $A_1(\begin{smallmatrix}\alpha\\ V\end{smallmatrix})$ is the formula obtained from $A_1$ by replacing every occurrence of $\alpha$ in $A_1$ by V, etc. Here $\alpha$ is called the eigenvariable of the substitution.

1.2.2. Rule of term-replacement.

$$\frac{\Gamma_1,F(s),\Gamma_2 \longrightarrow \Delta}{\Gamma_1,F(t),\Gamma_2 \longrightarrow \Delta} \; , \qquad \frac{\Gamma \longrightarrow \Delta_1,F(s),\Delta_2}{\Gamma \longrightarrow \Delta_1,F(t),\Delta_2}$$

where s and t are closed terms of the same numerical values, and F(t) is obtained from F(s) by replacing some occurrences of s in F(s) by t.

In what follows, a proof (-figure) will mean a proof tree which is locally correct with respect to the rules of AII, substitution and term-replacement.

The end-piece of a proof of $\longrightarrow$ contains the following inference rules only : cut, weakening, contraction, exchange, term-replacement, substitution and ind (induction rule).

Definition 1.3. Let P be a proof of $\longrightarrow$ and d a mapping (called an assignment of P ) from the set of substitutions in P to the set of positive integers, where the value d(J) is called the degree of J ( with respect to d ) for each substitution J in P.

We call the pair $\langle P,d\rangle$ a proof with degree if the following conditions are satisfied.

1.3.1. Every substitution is in the end-piece and there is no ind under a substitution.

1.3.2. Let A be a semi-formula in P. If we calculate the degree d(A) of A by the following clauses 1.3.2.1.-1.3.2.4., then we have

(*) $d(B) < d(J)$ for every substitution J in P and every formula B in the upper-sequent of J.

1.3.2.1. $d(A) = \omega$ if A is not isolated. Suppose A is isolated.

1.3.2.2. $d(A) = 0$ if A contains no logical symbol.

1.3.2.3. $d(\neg A) = d(A)$, $d(A_1 \wedge A_2) = \max\{d(A_1), d(A_2)\}$, $d(\forall x A(x)) = d(A(x))$.

1.3.2.4. $d(\forall \phi F(\phi)) = \max\{d(F(\phi)) + 1, d(J)\}$ where J ranges over substitutions which disturb $\forall \phi F(\phi)$.

Definition 1.4.

1.4.1. The grade of a formula A, denoted by g(A), is the number of occurrences of logical symbols in A.

1.4.2. Let P be a proof and S a sequent in P. The height of S in P, denoted by h(S;P) or simply h(S), is defined inductively 'from below to above', as follows :

1.4.2.1. h(S) = 0 if S is the end-sequent of P or S is the uppersequent of a substitution in P.

1.4.2.2. h(S) = h(S') where S is an uppersequent of an inference exept substitution, cut, ind and second order $\forall$:left, and S' is the lowersequent of the inference.

1.4.2.3. h(S) = max{h(S'),g(D) + 1} where S is an uppersequent of cut, ind or second order $\forall$:left, and D is the cut formula, induction formula or auxiliary formula of the inference, respectively, and S' is the lowersequent of the inference.

Next, we will assign an o.d. from $O(\omega+1,1)$ to each sequent in a proof with degree. For simplicity, we write $(i,\mu)$ for a non-zero connected o.d. $(i,0,\mu)$.

Definition 1.5. For each o.d. $\mu$ from $O(\omega+1,1)$ and natural number n, we define inductively an o.d. $\omega(n,\mu)$, as follows :

$\omega(0,\mu) = \mu$,

$\omega(n+1,\mu) = (\omega,\omega(n,\mu))$.

Definition 1.6. For each i such that $0 \leq i \leq \omega$, we define a relation $<<_i$ between two o.d.'s, as follows :

1.6.1. $\mu <<_i \nu$ iff

1.6.1.1. for each j such that $i \leq j < \omega$ and for each j-section $\rho$ of $\mu$, there exists a j-section $\tau$ of $\nu$ for which $\rho \leq_j \tau$,

and,

1.6.1.2. for each k such that $i \leq k \leq \omega$, $\mu <_k \nu$.

1.6.2. $\mu \underline{<<}_i \nu$ iff $\mu <<_i \nu$ or $\mu = \nu$.

By the definition, the following proposition is easily verified ( cf. Lemma 27.1, [PT, p.320] ).

Proposition 1.7.

1.7.1. $\mu <<_i \nu$ implies $\omega(n,\mu) <<_i \omega(n,\nu)$ and $\omega(n,\mu\#\theta) <<_i \omega(n,\nu\#\theta)$ for every natural number n and every o.d. $\theta$.

1.7.2. $\mu <<_i \nu$ and $i \leq j < \omega$ imply $(j,\mu) <<_i (j,\nu)$.

Definition 1.8. Let $\langle P,d \rangle$ be a proof with degree. To each sequent S in P, we will assign an o.d. $O(S;P,d)$ or simply $O(S)$ from $O(\omega+1,1)$ inductively 'from above to below', as follows :

1.8.1. $O(S) = 0$ if S is an initial sequent of P.

In the following, we will assume that S is the lowersequent of an inference J, and the o.d.('s) has (have) been assigned to the uppersequent(s) of J.

1.8.2. $\dfrac{S'}{S}$ : J is a weak structural rule or term-replacement.

$O(S) = O(S')$.

1.8.3. $\dfrac{S'}{S}$ : J is $\neg$, $\wedge$:left, first order $\forall$ or second order $\forall$: right.

$$O(S) = (\omega, O(S')).$$

1.8.4. $\dfrac{S_1 \quad S_2}{S}$ : J is $\wedge$:right.

$$O(S) = (\omega, O(S_1) \# O(S_2)).$$

1.8.5. $\dfrac{S'}{S}$ : J is second order $\forall$:left or ind.

$$O(S) = \omega(h(S') - h(S) + 1, O(S')).$$

1.8.6. $\dfrac{S_1 \quad S_2}{S}$ : J is cut.

$$O(S) = \omega(h(S_1) - h(S), O(S_1) \# O(S_2)).$$

1.8.7. $\dfrac{S'}{S}$ : J is substitution.

$$O(S) = (d(J), O(S')).$$

1.8.8. $O(P,d) = O(S;P,d)$ where S is the end-sequent of P.

The preliminary definitions has finished and now we can state the following main lemma.

Main Lemma.

If $\langle P,d \rangle$ is a proof with degree, then we can construct, primitive recursively, another proof with degree $\langle P',d' \rangle$ such that :

$$O(P',d') <_0 O(P,d).$$

Assume that the main lemma has been proved finitistically. Since for any proof of $\longrightarrow$ in AII and the empty assignment $\phi$, $\langle P,\phi\rangle$ is a proof with degree, the consistency of AII will follow from the accesibility of the system $O(\omega+1,1)$ with respect to $<_0$.

A proof of the main lemma will be given in the next section.

§2. Proof of the Main Lemma.

The reduction step from $\langle P,d\rangle$ to $\langle P',d'\rangle$ is almost the same as in [PT]. Up to (3) in [PT, p.328], the reduction steps are completely the same as in [PT], i.e., (1) substitution the individual constant 0 for redundant first order free variables, (2) 'VJ Reduktion' in [2] and (3) eliminating equality axioms in the end-piece of P.

(4) By virtue of the above, we may assume that there are no applications of ind and no equality axioms as initial sequents in the end-piece of P. Suppose that the end-piece of P contains logical initial sequents.

Suppose P is of the following form and $D \longrightarrow D$ is one of the initial sequents in the end-piece of P :

$$
\begin{array}{lcl}
 & \qquad\qquad\qquad D \xrightarrow{0} D & \\
P_0 \left\{ \begin{array}{ccc} \vdots & & \vdots \\ \Gamma \xrightarrow{\mu} \Delta, D' & & D', \Pi \xrightarrow{\nu} \Lambda_1, D'', \Lambda_2 \end{array} \right. & & l \\
\hline
\Gamma, \Pi \xrightarrow{\omega(l-m,\mu\#\nu)} \Delta, \Lambda_1, D'', \Lambda_2 & & m \\
\vdots & & \\
\longrightarrow & &
\end{array}
$$

where $D''$ is $D'$ up to term-replacement and $l$ is $h(\Gamma \longrightarrow \Delta, D'; P)$, etc.

We reduce $P$ to the following $P'$ ( and $d'$ is defined to be the restriction of $d$ to $P'$. ) :

$$
\begin{array}{lc}
P_0 \left\{ \begin{array}{c} \vdots \\ \Gamma \xrightarrow{\mu'} \Delta, D' \end{array} \right. & m \\
\hline\hline
\Gamma, \Pi \longrightarrow \Delta, \Lambda_1, D'', \Lambda_2 & m \\
\vdots & \\
\longrightarrow &
\end{array}
$$

We see easily that for every sequent $S$ in $P_0$

$$O(S;P',d') \underline{<<}_0 \ \omega(h(S;P)-h(S;P'),O(S;P,d)).$$

In particular $\mu' \underline{<<}_0 \ \omega(l-m,\mu)$, and so, $\mu' <<_0 \omega(l-m,\mu\#\nu)$. Thus by proposition 1.7. we have $O(P',d') <<_0 O(P,d)$.

(5) We may assume besides the conditions in (4) that the end-piece of $P$ contains no logical initial sequents. Then let $P^*$ be the proof obtained from $P$ by eliminating weakenings in the end

piece of P and d* be the restriction of d for P*. Similarly in the case (4) we have $O(P^*,d^*) \underline{\ll_0} O(P,d)$.

(6) Suppose that the end-piece of P contains no ind, weakening or axiom other than mathematical ones. Then the end-piece of P contains a suitable cut J. ( cf. Sublemma 12.9., [PT, p.105] )

(7) The case where the cut formula of J is of the form $\forall\phi F(\phi)$.

Case 1. $\forall\phi F(\phi)$ is isolated.

Let P be the following form :

$$
\begin{array}{llll}
 & \vdots & \vdots & \\
 & \Gamma_1 \xrightarrow{\lambda} \Delta_1, F_1(\alpha) & F_2(V), \Pi_1 \xrightarrow{\mu} \Lambda_1 & l_4 \\
\hline
 & \Gamma_1 \xrightarrow{(\omega,\lambda)} \Delta_1, \forall\phi F_1(\phi) & \forall\phi F_2(\phi), \Pi_1 \xrightarrow{\omega(l_4-l_3+1,\mu)} \Lambda_1 & l_3 \\
 & \vdots & \vdots & \\
 & \Gamma_2 \xrightarrow{\tau} \Delta_2, \forall\phi F(\phi) & \forall\phi F(\phi), \Pi_2 \xrightarrow{\rho} \Lambda_2 & l_2 \\
\hline
J & \multicolumn{2}{c}{\Gamma_2, \Pi_2 \xrightarrow{\omega(l_2-l_1,\tau\#\rho)} \Delta_2, \Lambda_2} & l_1 \\
 & \multicolumn{2}{c}{\vdots} & \\
 & \multicolumn{2}{c}{\Gamma_3 \xrightarrow{\nu} \Delta_3} & 0 \\
 & \multicolumn{2}{c}{\vdots} & \\
 & \multicolumn{2}{c}{\xrightarrow{\sigma}} & 0
\end{array}
$$

where both $\forall\phi F_1(\phi)$ and $\forall\phi F_2(\phi)$ are $\forall\phi F(\phi)$ up to term-replacement. Let i be $d(\forall\phi F_1(\phi))$. $\Gamma_3 \longrightarrow \Delta_3$ is the i-resolvent of $\Gamma_2, \Pi_2 \longrightarrow \Delta_2, \Lambda_2$. Let P' be the following :

$$\begin{array}{ccr} \vdots & \vdots & \\ \dfrac{\Gamma_1 \xrightarrow{\lambda} \Delta_1, F_1(\alpha)}{\Gamma_1 \longrightarrow F_1(\alpha), \Delta_1, \forall\phi F_1(\phi)} & \dfrac{F_2(V), \Pi_1 \xrightarrow{\mu} \Lambda_1}{\forall\phi F_2(\phi), \Pi_1 \xrightarrow{\omega(l_4 - l_3 + 1, \mu)} \Lambda_1} & \begin{array}{r} l_4 \\ l_3 \end{array} \\ \vdots & \vdots & \end{array}$$

$$\begin{array}{cr} \dfrac{\Gamma_2 \xrightarrow{\tau'} F(\alpha), \Delta_2, \forall\phi F(\phi) \qquad \forall\phi F(\phi), \Pi_2 \xrightarrow{\rho} \Lambda_2}{\Gamma_2, \Pi_2 \xrightarrow{\omega(l_2 - l_1, \tau' \# \rho)} F(\alpha), \Delta_2, \Lambda_2} & \begin{array}{r} l_2 \\ l_1 \end{array} \\ \vdots & \end{array}$$

$$\begin{array}{cr} J_1 \dfrac{\Gamma_3 \xrightarrow{\theta} \Delta_3, F(\alpha) \quad 0}{\Gamma_3 \xrightarrow{(i,\theta)} \Delta_3, F(V)} \qquad \begin{array}{c} \vdots \\ F(V), \Pi_1 \xrightarrow{\mu} \Lambda_1 \end{array} & l_4 \\ \dfrac{\Gamma_3, \Pi_1 \xrightarrow{\omega(l_4 - l_3, (i,\theta) \# \mu)} \Delta_3, \Lambda_1}{\forall\phi F_2(\phi), \Pi_1, \Gamma_3 \longrightarrow \Delta_3, \Lambda_1} & l_3 \end{array}$$

$$\begin{array}{ccr} \vdots & \vdots & \\ \Gamma_2 \xrightarrow{\tau} \Delta_2, \forall\phi F(\phi) & \forall\phi F(\phi), \Pi_2, \Gamma_3 \xrightarrow{\rho'} \Delta_3, \Lambda_2 & l_2 \end{array}$$

$$\begin{array}{cr} \dfrac{\dfrac{}{\Gamma_2, \Pi_2, \Gamma_3 \xrightarrow{\omega(l_2 - l_1, \tau \# \rho')} \Delta_2, \Delta_3, \Lambda_2}}{\Gamma_2, \Pi_2, \Gamma_3 \longrightarrow \Delta_3, \Delta_2, \Lambda_2} & l_1 \\ \vdots & \\ \dfrac{\Gamma_3, \Gamma_3 \xrightarrow{\nu'} \Delta_3, \Delta_3}{\Gamma_3 \longrightarrow \Delta_3} & 0 \\ \vdots & \\ \xrightarrow{\sigma'} & 0 \end{array}$$

where $J_1$ is a substitution with eigenvariable $\alpha$. $d'(J')$ for a substitution $J'$ except $J_1$ is defined to be $d(J'')$ where $J''$ is the corresponding substitution to $J'$ in P. $d'(J_1)$ is defined to be i.

Following propositions (7.1)-(7.5) are easily verified by Proposition 1.7. ( cf., [PT, pp.332-333] ) :

(7.1) $\tau' \ll_0 \tau$,

(7.2) $\theta \ll_0 \nu$,

(7.3) $\omega(l_4 - l_3, (i,\theta)\#\mu) \ll_{i+1} \omega(l_4 - l_3 + 1, \mu)$,

(7.4) $\rho' \ll_{i+1} \rho$,

(7.5) $\nu' <_j \nu$ for all $j \leq \omega$, and for each k such that $k < i$ and for each k-section $\pi'$ of $\nu'$, there exists a k-section $\pi$ of $\nu$ for which $\pi' \leq_k \pi$, and for each i-section $\eta$ of $\nu'$, $\eta <_i \nu$. ( Here note that $\nu$ and $\nu'$ are connected. )

It follows from (7.5) that $\sigma' <_0 \sigma$.

Case 2. $\forall\phi F(\phi)$ is not isolated.

Let P be the following form :

$$\dfrac{\Gamma_1 \xrightarrow{\lambda} \Delta_1, F_1(\alpha)}{\Gamma_1 \xrightarrow{(\omega,\lambda)} \Delta_1, \forall\phi F_1(\phi)} \quad m \qquad \dfrac{F_2(\beta), \Pi_1 \xrightarrow{\mu} \Lambda_1}{\forall\phi F_2(\phi), \Pi_1 \xrightarrow{(\omega,\mu)} \Lambda_1} \quad m$$

$$\vdots$$

$$J\;\dfrac{\Gamma_2 \longrightarrow \Delta_2, \forall\phi F(\phi) \qquad \forall\phi F(\phi), \Pi_2 \longrightarrow \Lambda_2 \quad l}{\Gamma_2, \Pi_2 \longrightarrow \Delta_2, \Lambda_2}$$

$$\vdots$$

$$\Phi \longrightarrow \Psi \qquad n$$

$$\vdots$$

$$\longrightarrow$$

where $\Phi \longrightarrow \Psi$ denotes the uppermost sequent below J whose height is less than l. Let P' be the following :

$$\vdots \qquad\qquad\qquad\qquad\qquad\qquad \vdots$$

$$\frac{\dfrac{\Gamma_1 \overset{\lambda}{\rightarrow} \Delta_1, F_1(\beta)}{\Gamma_1 \rightarrow F_1(\beta), \Delta_1, \forall\phi F_1(\phi)}}{} \qquad\qquad \frac{\dfrac{F_2(\beta), \Pi_1 \overset{\mu}{\rightarrow} \Lambda_1}{\forall\phi F_2(\phi), \Pi_1, F_2(\beta) \rightarrow \Lambda_1}}{} \quad m$$

$$\vdots \qquad\qquad \vdots \quad \vdots \qquad\qquad \vdots$$

$$\frac{\Gamma_2 \rightarrow F(\beta), \Delta_2, \forall\phi F(\phi) \quad \forall\phi F(\phi), \Pi_2 \rightarrow \Lambda_2}{\Gamma_2, \Pi_2 \longrightarrow F(\beta), \Delta_2, \Lambda_2} \quad \frac{\Gamma_2 \rightarrow \Delta_2, \forall\phi F(\phi) \quad \forall\phi F(\phi), \Pi_2, F(\beta) \rightarrow \Lambda_2}{\Gamma_2, \Pi_2, F(\beta) \longrightarrow \Delta_2, \Lambda_2} \quad l$$

$$\vdots \qquad\qquad\qquad \vdots$$

$$\frac{\dfrac{\Phi \longrightarrow F(\beta), \Psi}{\Phi \longrightarrow \Psi, F(\beta)} \qquad \dfrac{\Phi, F(\beta) \longrightarrow \Psi}{F(\beta), \Phi \longrightarrow \Psi}}{\dfrac{\Phi, \Phi \longrightarrow \Psi, \Psi}{\Phi \longrightarrow \Psi}} \quad l' \quad n$$

$$\vdots$$

$$\longrightarrow$$

And for every substitution J' in P', d'(J') is defined to be d(J") where J" is the corresponding substitution to J' in P.

From l' < l we see easily that $O(P',d') <_0 O(P,d)$.

(8) The cases where the cut formula of J is of the form $F_1 \wedge F_2$, $\neg F$ or $\forall x F(x)$ are treated in the same way as the Case 2. in (7).

This completes the proof of the main lemma.

Remark. A consistency proof of $AI\Pi^1_1$ by the accessibility of the system O(2,1) with respect to $<_0$ can be given similarly for the above consistency proof of AII.

Appendix.

If the works of Gentzen's, Takeuti's and mine are regarded as 'merely' computing the ordinals of the formal systems, then it is fair to say that the ordinal analysis ( cf. [5] ) by Schütte and Pohlers, etc. is superior to ours in itself.

Therefore we may ask what has been achieved by these consistency proofs. The question is whether they have the specific - sometimes called 'epistemologically interesting' - characters that one expects from a 'consistency proof'.

I must confess that I have my doubt about it, or rather, at the present time I cannot answer it. Here we are faced by Kreisel's view in [4, p.240] :

> the analysis of the significance of a consistency proof may be more difficult than the proof itself.

It seems to me that the analysis is a matter of paramount importance for a consistency proof. As a matter of fact, the o.d.'s give an appropriate representation of complexities of some impredicative (intuitive) proofs expressed by formal proof-figures, as seen above. But we should ask further, e.g., in the Case 1. of (7) of the proof of the main lemma, in what sense the proof expressed by P' is simpler than the proof expressed by P. Since we do not know whether any natural ordering among proofs exists, this question is also left open.

When we neglect the combinatorial aspect in these consistency proofs, though I think this aspect to be important, we can note a minor fact that follows immediately from 'optimal' consistency proofs.

Let T be the pure number theory, $A\Pi^1_1$ or AII, and $\prec$ the primitive recursive well-ordering obtained by a canonical arithmetization of the notation system whose accessibility is used in the consistency proof of T. And let c be an enumeration of variables in a formula A of the primitive recursive arithmetic PRA.

Then the accessibility of $\prec$ may be expressed in the language of PRA by the following rule $Acc_\prec$ :

$Acc_\prec$

infer $\neg A(c)$ from $A(c) \longrightarrow \psi(c,x') \prec \psi(c,x)$

where $\psi$ is an arbitrary primitive recursive function and x is a variable distinct from the variables in c.

For a formula F, if there is a derivation P of F in PRA + $Acc_\prec$ such that every sub-derivation ending with a premiss of $Acc_\prec$ in P is a derivation in PRA, we say that F is deducible from PRAwith one application of $Acc_\prec$. That is to say, when we require that the premiss of $Acc_\prec$ should be established finitistically, i.e., should be derivable in PRA, we have the deducibility from PRA with one application of $Acc_\prec$.

It follows from the consistency proof for T that $Consis_T$, i.e., $\neg Prov_T(x, \ulcorner 0 = 1 \urcorner)$ is deducible from PRA with one application of $Acc_\prec$, where $Prov_T$ is a canonical proof predicate for T. ( In the case that T is AII, $Prov_T$ contains the code of ' degree assignment function' d. )

Conversely, if $A(c) \longrightarrow \psi(c,x') \prec \psi(c,x)$ is derivable in PRA, then it follows from [3] or the remark 2. in [1] that $\neg A(c)$ is derivable in PRA + $Consis_T$. Thus we have the following fact :

one application of $Acc_\prec$ is equivalent to $Consis_T$ over PRA.

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