数理解析研究所講究録
第 516 巻 1984 年 132-141

# A method of axiomatizing fragments of intuitionistic theories

上江洲 忠弘
Tadahiro Uesu

Department of Mathematics
Faculty of Science
Kyushu University

The purpose of this paper is to present a method of axiomatizing fragments of intuitionistic theories. The main theorem is an extension of Motohashi's Axiomatization Theorem which concernes classical theories (N. Motohashi: An Axiomatization Theorem, J. Math. Soc. Japan 34 (1982), PP. 551-560).

Let Q be a set of predicate symbols. A formula is Q-atomic if it is an atomic formula with a predicate symbol in Q.

A sequent is a Q-clause if its sequent formulas are either Q-atomic or Q-free.

A set of Q-clauses is Q-closed if it is closed for substitutions, contractions, interchanges and the following inference rules:

$$\frac{\Gamma \to A \qquad A, \Gamma \to \Delta}{\Gamma \to \Delta},$$

where A is Q-atomic.

$$\frac{\Gamma \to P(\bar{s}) \qquad P(\bar{t}),\Gamma \to \Delta}{\bar{s}=\bar{t},\Gamma \to \Delta} ,$$

where $P \in Q$.

$$\frac{\mathbb{A}_1(s),\ldots,\mathbb{A}_m(s) \to \mathbb{B}_1(s),\ldots,\mathbb{B}_n(s)}{s=t,\ \mathbb{A}_1(t),\ldots,\mathbb{A}_m(t) \to \mathbb{B}_1(t),\ldots,\mathbb{B}_n(t)} .$$

Proposition 1. Let C be a finite set of Q-clauses whose sequent formulas are quantifier free. Then there is a Q-closed set $\bar{C}$ of Q-clauses such that $C \subseteq \bar{C}$, the theory $LJ_=[\bar{C}]$ is equivalent to the theory $LJ_=[C]$, and $\bar{C}$ is primitive recursive.

Let P, N and F be mutually disjoint sets of predicate symbols. A formula is a $(P,N,F)_+$ formula if it is P-positive, N-negative and F-free. A formula is a $(P,N,F)_-$ formula if it is $(N,P,F)_+$.

A (P,N,F)-basic sequent is a sequent of the form

$$A_1 \supset B_1,\ldots,A_n \supset B_n, \forall \bar{x} C \to \exists \bar{y} D,$$

where $A_1,\ldots,A_n$ and D are disjunctions of conjunctions of formulas which are either $P \cup F$-atomic or $P \cup N \cup F$-free, and $B_1,\ldots,B_n$ and C are conjunctions of formulas which are $N \cup F$-atomic or $P \cup N \cup F$-free.

A sequence of $(P,N,F)_+$ formulas is a $(P,N,F)_+$-sequence if it has the form

$$\langle A(\Sigma) \mid \Sigma \text{ is a finite sequence of } P \cup F\text{-atomic formulas}\rangle,$$

where each free variable in $A(\Sigma)$ occurs in $\Sigma$ for each $\Sigma$.

A sequence of $(P,N,F)_-$ formulas is a $(P,N,F)_-$-sequence if it has the form

$$\langle A(\Sigma;R) \mid \Sigma \text{ is a finite sequence of } P\cup F\text{-atomic formulas and } R \text{ is a } N\cup F\text{-atomic formula}\rangle,$$

where each free variable in $A(\Sigma;R)$ occurs in $\Sigma$ or $R$.

Let $A_+$ and $A_-$ be a $(P,N,F)_+$-sequence and $(P,N,F)_-$-sequence, respectively. For each finite sequence $\Sigma$ of $P\cup F$-atomic formulas, each disjunction $D$ of conjunctions of formulas which are either $P\cup F$-atomic or $P\cup N\cup F$-free, and each conjunction $C$ of formulas which are either $N\cup F$-atomic or $P\cup N\cup F$-free, we define formulas $A_+(\Sigma,D)$, $A_-(\Sigma;C)$ and $A_-(\Sigma,D;C)$ by the following way:

If $C$ is $P\cup N\cup F$-free, then $A_-(\Sigma;C)$ is the formula $A_+(\Sigma)\supset C$.

If $C$ is not $P\cup N\cup F$-free and has the form $C_1\wedge C_2$, then $A_-(\Sigma;C)$ is the forula $A_-(\Sigma;C_1)\wedge A_-(\Sigma;C_2)$.

If $D$ is not $P\cup N\cup F$-free and has the form $D_1\vee D_2$, then $A_+(\Sigma,D)$ and $A_-(\Sigma,D;C)$ are the formulas $A_+(\Sigma,D_1)\vee A_+(\Sigma,D_2)$ and $A_-(\Sigma,D_1;C)\wedge A_-(\Sigma,D_2;C)$, respectively.

If $D$ has the form $\bigwedge\!\!\!\bigwedge \Sigma_0 \wedge A_1 \wedge \bigwedge\!\!\!\bigwedge \Sigma_1 \wedge \cdots \wedge A_n \wedge \bigwedge\!\!\!\bigwedge \Sigma_n$, where $\Sigma_0,\Sigma_1,\ldots,\Sigma_n$ are sequences of $P\cup F$-atomic formulas and $A_1,\ldots,A_n$ are $P\cup N\cup F$-free, then $A_+(\Sigma,D)$ and $A_-(\Sigma,D;C)$ are

$$A_+(\Sigma,\Sigma_0,\Sigma_1,\ldots,\Sigma_n)\wedge A_1\wedge \cdots \wedge A_n$$

and

$$A_1\wedge\cdots\wedge A_n\supset A_-(\Sigma,\Sigma_0,\Sigma_1,\ldots,\Sigma_n;C),$$

respectively.

Let S be a (P,N,F)-basic sequent of the form

$\mathbb{A}_1(\bar{u}) \supset \mathbb{B}_1(\bar{u}), \ldots, \mathbb{A}_n(\bar{u}) \supset \mathbb{B}_n(\bar{u}),\ \forall\bar{x}\mathbb{C}(\bar{x},\bar{u}) \to \exists\bar{y}\mathbb{D}(\bar{y},\bar{u})$,

where $\bar{u}$ is the sequence (lined up by a fixed order) of all the variables occurring free in S. Then we denote $A_+(\Sigma:S)$ and $A_-(\Sigma;R:S)$ the following formulas, respectively:

$$\forall\bar{z}\Big(\bigwedge\!\!\!\bigwedge_{1\leq i\leq n} A_-(\Sigma,\mathbb{A}_i(z);\mathbb{B}_i(z)) \wedge \forall\bar{x}A_-(\Sigma;\mathbb{C}(\bar{x},\bar{z})) \supset \exists\bar{y}A_+(\Sigma,\mathbb{D}(\bar{y},\bar{z}))\Big),$$

where no variable in $\bar{z}$ occurs in $\Sigma$ or S.

$$\exists\bar{z}\Big(\bigwedge\!\!\!\bigwedge_{1\leq i\leq n} A_-(\Sigma,\mathbb{A}_i(\bar{z});\mathbb{B}_i(\bar{z})) \wedge \forall\bar{x}A_-(\Sigma;\mathbb{C}(\bar{x},\bar{z})) \wedge \forall\bar{y}A_-(\Sigma,\mathbb{D}(\bar{y},\bar{z});R)\Big),$$

where no variable in $\bar{z}$ occurs in $\Sigma$, R or S.

Let $\mathbb{S}$ be a finite set of (P,N,F)-basic sequents. Then we define the pair $(A_+^{\mathbb{S},k}, A_-^{\mathbb{S},k})$ of a $(P,N,F)_+$-sequence and a $(P,N,F)_-$-sequence for each natural number k as follows:

$A_+^{\mathbb{S},0}(\Sigma)$ is $A_+(\Sigma)$;

$A_-^{\mathbb{S},0}(\Sigma;R)$ is $A_-(\Sigma;R)$.

$A_+^{\mathbb{S},k+1}(\Sigma)$ is the formula

$$\bigwedge\!\!\!\bigwedge\{A_+^{\mathbb{S},k}(\Sigma:S) \mid S\in\mathbb{S}\} \wedge A_+^{\mathbb{S},k}(\Sigma);$$

$A_-^{\mathbb{S},k+1}(\Sigma;R)$ is the formula

$$\bigvee\!\!\!\bigvee\{A_-^{\mathbb{S},k}(\Sigma;R:S) \mid S\in\mathbb{S}\} \vee (A_+^{\mathbb{S},k}(\Sigma) \supset A_-^{\mathbb{S},k}(\Sigma;R)).$$

<u>Theorem</u> 2. Let $\mathbb{C}$ be a $P\cup N\cup F$-closed set of $P\cup N\cup F$-clauses and $\mathbb{S}$ a finite set of (P,N,F)-basic sequents. Let $A_+$ and $A_-$ be

a $(P,N,F)_+$-sequence and a $(P,N,F)_-$-sequence, respectively. Let T be a theory with the following properties:

(1) For each P-atomic formula A, $\vdash_T A_+(A) \rightarrow A$.

(2) For each N-atomic formula A, $\vdash_T A \rightarrow A_-(\varepsilon;A)$, where $\varepsilon$ is the empty sequence of formulas.

(3) For each F-atomic formula A, $\vdash_T A_-(A;A)$.

(4) For each clause $\Sigma,\Pi \rightarrow \Delta$ in C, where $\Sigma$ is a sequence of $P\cup F$-atomic formulas and $\Pi$ is a sequence of N-atomic formulas or $P\cup N\cup F$-free formulas, if $\Delta$ is $P\cup N\cup F$-free or a P-atomic formula, then $\vdash_T A_+^{\$,k}(\Sigma),\Gamma \rightarrow \Delta$ for some number k; if $\Delta$ is a $N\cup F$-atomic formula, then $\vdash_T \Pi \rightarrow A_-^{\$,k}(\Sigma;\Delta)$ for some number k.

(5) For each pair $\Sigma$, $\Sigma'$ of finite sequences of $P\cup F$-atomic formulas such that each formula in $\Sigma$ occurs in $\Sigma'$, and for each $N\cup F$-atomic formula R,

$$\vdash_T A_+(\Sigma') \rightarrow A_+(\Sigma),$$

$$\vdash_T A_-(\Sigma;R) \rightarrow A_-(\Sigma';R),$$

and

$$\vdash_T \neg A_+(\Sigma) \rightarrow A_-(\Sigma;R).$$

(6) For each finite sequence $\Sigma$ of $P\cup F$-atomic formulas, each $N\cup F$-atomic formula R and each substitution $\theta$,

$$\vdash_T A_+(\Sigma)\theta \equiv A_+(\Sigma\theta),$$

and

$$\vdash_T A_-(\Sigma;R)\theta \equiv A_-(\Sigma\theta;R\theta).$$

Suppose that the language of T contains all the symbols of $LJ_=[\mathbb{C}\cup \$]$ other than symbols in F.

Then, for each finite sequent $\Sigma$ of $P\cup F$-atomic formulas and each $(P,N,F)_+$ sequent $\Gamma \to \Delta$, $\vdash_{LJ_=[\mathbb{C} \cup \$]} \Sigma, \Gamma \to \Delta$ implies $\vdash_T A_+^{\$,k}(\Sigma), \Gamma \to \Delta$ for some number k.

Colloary 3. Let $\mathbb{C}$, $\$$, $A_+$, $A_-$ and T be the same as in Theorem 1. Moreover suppose that

(7) for each finite sequence $\Sigma$ of $P \cup F$-atomic formulas and each $N \cup F$-atomic formula R,

$$\vdash_{LJ_=[\mathbb{C} \cup \$]} \Sigma \to A_+(\Sigma)$$

and

$$\vdash_{LJ_=[\mathbb{C} \cup \$]} \Sigma, A_-(\Sigma;R) \to R,$$

and

(8) $LJ_=[\mathbb{C} \cup \$]$ is an extension of T.

Then, for each $(P,N,F)_+$ sequent $\Gamma \to \Delta$,

$$\vdash_{T[\{A_+^{\$,k}(\varepsilon) \mid k \in \omega\}]} \Gamma \to \Delta \text{ if and only if } \vdash_{LJ_=[\mathbb{C} \cup \$]} \Gamma \to \Delta.$$

Let $\mathbb{C}$ be a set of $P \cup N \cup F$-clauses. Let $\Sigma$ be a finite sequence of $P \cup F$-atomic formulas, and A a $N \cup F$-atomic formula.

A formula is a positive (P,N,F)-section of $\Sigma$ for $\mathbb{C}$ if it has the form $\bigwedge \Gamma \supset \bigvee \Delta$, where

(i) $\Gamma$ is a finite sequence of formulas which are either N-atomic or $P \cup N \cup F$-free,

(ii) $\Delta$ consists of at most one P-atomic or $P \cup N \cup F$-free formula,

(iii) for some subsequence $\Sigma'$ of $\Sigma$ the sequent $\Sigma', \Gamma \to \Delta$ belongs to $\mathbb{C}$,

(iv) no formula occurs twice in $\Gamma$, and

(v) each variable occurring free in $\Gamma \to \Delta$ occurs in $\Sigma$.

A formula is negative (P,N,F)-section of $\Sigma$ and R for $\mathbb{C}$ if it has the form $\bigwedge\!\!\!\bigwedge \Gamma$, where

(i) $\Gamma$ is a finite sequence of formulas which are either N-atomic or $P \cup N \cup F$-free,

(ii) for some subsequence $\Sigma'$ of $\Sigma$ the sequent $\Sigma', \Gamma \to R$ belongs to $\mathbb{C}$,

(iii) no formula occurs twice in $\Gamma$,

(iv) each variable occurring free in $\Gamma$ occurs in $\Sigma$ or R.

Let $A_+(\Sigma)$ be the following formula:

$\bigwedge\!\!\!\bigwedge \{A | A$ is a positive (P,N,F)-section of $\Sigma$ for $\mathbb{C}\}$

$\wedge \bigwedge\!\!\!\bigwedge \{B | B$ is a P-atomic formula occurring in $\Sigma\}$.

Let $A_-(\Sigma;R)$ be the following formula:

$A_+(\Sigma) \supset \bigvee\!\!\!\bigvee \{A | A$ is a negative (P,N,F)-section of $\Sigma$ and R for $\mathbb{C}\} \vee R$

if R is N-atomic;

$A_+(\Sigma) \supset \bigvee\!\!\!\bigvee \{A | A$ is a negative (P,N,F)-section of $\Sigma$ and R for $\mathbb{C}\}$

$\vee \bigvee\!\!\!\bigvee \{\bar{s}=\bar{t} | P(\bar{t})$ occurs in $\Sigma\}$

if R is an F-atomic formula of the form $P(\bar{s})$.

The sequences $A_+$ and $A_-$ are called the canonical $(P,N,F)_+$-sequence for $\mathbb{C}$ and the canonical $(P,N,F)_-$-sequence for $\mathbb{C}$, respectively.

Proposition 4. Let $\mathbb{C}$ be a $P \cup N \cup F$-closed set of $P \cup N \cup F$-clauses. Let $A_+$ and $A_-$ be the canonical $(P,N,F)_+$-sequence for $\mathbb{C}$ and $(P,N,F)_-$-sequence for $\mathbb{C}$, respectively.

(1) If A is a P-atomic formula, then $\vdash_{LJ_=} A_+(A) \to A$.

(2) If A is an N-atomic formula, then $\vdash_{LJ_=} A \to A_-(\varepsilon;A)$.

(3) If A is an F-atomic formula, then $\vdash_{LJ_=} A_-(A;A)$.

(4) If $\Sigma,\Pi \to \Delta$ is a clause in $\mathbb{C}$, where $\Sigma$ is a finite sequence of P$\cup$F-atomic formulas and $\Pi \to \Delta$ is a $(P,N,F)_+$ sequent, and if $\Sigma^*$ is a finite sequence of P$\cup$F-atomic formulas such that each formula in $\Sigma$ occurs in $\Sigma^*$ and each variable free in $\Gamma \to \Delta$ occurs in $\Sigma^*$, then $\vdash_{LJ_=} A_+(\Sigma^*),\Pi \to \Delta$.

(5) If $\Sigma,\Pi \to R$ is a clause in $\mathbb{C}$, where $\Sigma$ is a finite sequenc of P$\cup$F-atomic formulas, $\Pi$ is a sequence of N-atomic or P$\cup$N$\cup$F-free formulas and R is a N$\cup$F-atomic formula, and if $\Sigma^*$ is a finite sequence of P$\cup$F-atomic formulas such that each formula in $\Sigma$ occurs in $\Sigma^*$ and each variable free in $\Gamma$ occurs $\Sigma^*$ or R, then $\vdash_{LJ_=} \Pi \to A_-(\Sigma^*;R)$.

(6) If $\Sigma$ and $\Sigma'$ is finite sequences of P$\cup$F-atomic formulas such that each formula in $\Sigma$ occurs in $\Sigma'$, and if R is a N$\cup$F-atomic formula, then

$$\vdash_{LJ} A_+(\Sigma') \to A_+(\Sigma),$$

$$\vdash_{LJ} A_-(\Sigma;R) \to A_-(\Sigma';R),$$

and

$$\vdash_{LJ} \neg A_+(\Sigma) \to A_-(\Sigma;R).$$

(7) If $\Sigma$ is a finite sequence of P$\cup$F-atomic formulas, R is a N$\cup$F-atomic formula and $\theta$ is a substitution, then

$$\vdash_{LJ_=} A_+(\Sigma)\theta \equiv A_+(\Sigma\theta),$$

and

$$\vdash_{LJ_=} A_-(\Sigma;R)\theta \equiv A_-(\Sigma\theta;R\theta).$$

(8) If $\Sigma$ is a finite sequence of P$\cup$F-atomic formulas and

R is a $N \cup F$-atomic formula, then

$$\vdash_{LJ_{=}[C]} \Sigma \to A_{+}(\Sigma),$$

and

$$\vdash_{LJ_{=}[C]} \Sigma, A_{-}(\Sigma; R) \to R.$$

Corollary 5. Let $\mathbb{C}$, $A_{+}$ and $A_{-}$ be the same as in Proposition 4. Let $\mathbb{S}$ be a finite set of (P,N,F)-basic sequents. Suppose that the sequent $\to \exists \bar{y} P(\bar{x}, \bar{y})$ belongs to $\mathbb{S}$, where P is a predicate symbol in $P \cup F$ and $\bar{x}$ is not empty. Then, for each $(P,N,F)_{+}$ sequent $\Gamma \to \Delta$, $\vdash_{LJ_{=}[\{A_{+}^{\mathbb{S},k}(i) \mid k \in \omega\}]} \Gamma \to \Delta$ if and only if $\vdash_{LJ_{=}[\mathbb{C} \cup \mathbb{S}]} \Gamma \to \Delta$.

Proposition 6. For each finite set $\mathbb{A}$ of sentences there exist a finite set G of new predicate symbols, a finite set $\mathbb{C}$ of $P \cup N \cup F \cup G$-clauses whose sequent formulas are quantifier free, and a finite set $\mathbb{S}$ of $(P,N,F \cup G)$-basic sequents such that $LJ_{=}[\mathbb{C} \cup \mathbb{S}]$ is conservative over $LJ_{=}[\mathbb{A}]$ and the sequent $\to \exists \bar{y} P(\bar{x}, \bar{y})$ belongs to $\mathbb{S}$ for some predicate symbol P in $P \cup F$, where $\bar{x}$ is not empty.

Axiomatization Theorem. Let L be a first order language without function symbols. Let $\mathbb{A}$ be a finite set of sentences of L. Let P, N, F and G be mutually disjoint sets of predicate symbols with $P \cup N \cup F \subseteq L$ and $L \cap G = \phi$. Let $\mathbb{C}$ and $\mathbb{S}$ be a finite set of $P \cup N \cup F \cup G$-clauses and a finite set of $(P,N,F \cup G)$-basic sequents, respectively, such that $LJ_{=}[\mathbb{C} \cup \mathbb{S}]$ is conservative over $LJ_{=}[\mathbb{A}]$, and for some predicate symbol P in $P \cup F \cup G$ the sequent $\rightarrow \exists \bar{y} P(\bar{x}, \bar{y})$ belongs to $\mathbb{S}$, where $\bar{x}$ is not empty. Let $A_+$ and $A_-$ be the canonical $(P,N,F \cup G)_+$-sequence for $\mathbb{C}$ and the canonical $(P,N,F \cup G)_-$-sequence for $\mathbb{C}$, respectively. Then the $(P,N,F)_+$ part of $LJ_=[\mathbb{A}]$ is axiomatized by the system of axioms $\{A_+^{S,k}(\varepsilon) \mid k \in \omega\}$.