

EXISTENCE OF DISTRIBUTION FUNCTIONS AND AVERAGE ORDER OF
ARITHMETICAL FUNCTIONS.

By

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1. Let $f(n)$ be a real sequence indexed by positive integers. f is limit-periodical B^λ , $\lambda \geq 1$, if given $\varepsilon > 0$, there exists a periodical function P_ε such that

$$\overline{\lim}_{x \rightarrow +\infty} \frac{1}{x} \sum_{n \leq x} |f(n) - P_\varepsilon(n)|^\lambda \leq \varepsilon.$$

It is known that there is a very close correspondence between B^λ and $\mathcal{L}^\lambda \left(\prod_p \mathbb{Z}_p, \prod_p dm_p \right)$, where \mathbb{Z}_p is the ring of the p -adic integers, dm_p the normalized Haar measure on \mathbb{Z}_p , p describing the whole set of the prime numbers (see, for instance, [1].)

2. If f belongs to B^1 , f real, then, the distribution function $\sigma_f(\cdot)$ of f exists ([2]) and moreover, if $\sigma_f(\cdot)$ is continuous in t , then, the function $I_{t,f}(\cdot)$ defined by

$$I_{t,f}(n) = \begin{cases} 1 & \text{if } f(n) < t \\ 0 & \text{if not} \end{cases}$$

is an element of B^1 ([3]).

From this, it is very easy to deduce the fact that given some real f in B^1 , if there exists an $r > 1$ such that

$$\overline{\lim}_{x \rightarrow +\infty} \frac{1}{x} \sum_{n \leq x} |f(n)|^r < +\infty,$$

then, for any λ satisfying $1 \leq \lambda < r$, we have $f \in B^\lambda$,
(for, if σ_f is continuous in t , then $I_{t,f}(n) f(n)$ belongs to B^1 . Now since

$$\sum_{n \leq x} |f(n) - I_{t,f}(n) f(n)|^\lambda = \sum_{n \leq x} |f(n)|^\lambda |1 - I_{t,f}(n)|,$$

we get the required result by use of Hölder inequality, taking the limit for t tending to infinity.)

3. The study of elements of B^1 with values 1 or 0 only, led me very recently to the following result ([4]).

THEOREM. Let $f(n)$ be an element of B^1 with values 1 or 0 only. Suppose that $\lim_{x \rightarrow +\infty} \frac{1}{x} \sum_{n \leq x} f(n)$ is not zero. Define an arithmetical function $\delta(n)$ by :

$$\delta(n) = 0 \quad \text{if} \quad f(n) = 0$$

$$\delta(n) = \min_m \{m - n | m > n\} \quad \text{if} \quad f(n) = 1.$$

Then, $\delta(n) \in B^1$.

The proof is not difficult (see [4]) ; it needs in an essential way the existence of the correspondence between B^1 and $\mathcal{L}^1 \left(\prod_p \mathbb{Z}_p, \prod_p dm_p \right)$. The other ingredients are the fact that the application $x \mapsto x+1$ from $\prod_p \mathbb{Z}_p$ to $\prod_p \mathbb{Z}_p$ is ergodic, and a (not so) well-known result of M. Kac ([5]) on Poincaré cycles. I would like to know if it is possible to get this result without the use of the correspondence between B^1 and $\mathcal{L}^1 \left(\prod_p \mathbb{Z}_p, \prod_p dm_p \right)$. If no other method exists, then, as far as I know, this theorem would be the first of this kind.

4. Application.

I shall restrict myself to a very special case, which is a good illustration of the usefulness of what has been explained before.

Consider the set of the square-free numbers. It was proved by Erdős ([6]) that, if $\delta(n)$ is defined by :

$$\delta(n) = 0 \quad \text{if } n \text{ is not square-free,}$$

$$\delta(n) = \min_{m \text{ square-free}} \{m-n \mid m > n, m \text{ square-free}\}$$

if n is square-free,

then : $\sum_{n \leq x} \delta(n)^\alpha \sim C(\alpha)x$, if $0 \leq \alpha \leq 2$, and Hooley ([7]) extended this result to the case $0 \leq \alpha \leq 3$. The existence of a distribution function for δ was proved by Mirsky ([8]).

A direct application of what has been explained before is possible, since the characteristic function of the square-free numbers is $|\mu(n)|$, where μ is the Möbius function, which is B^1 , with a not zero mean value : the theorem, in 3, gives that $\delta(n)$ is B^1 ; since δ takes only integral values, what has been stated in 2, gives that :

a. the set of the square-free numbers such that $\delta(n) = k$, $k \geq 0$ fixed, admits a density d_k and $\sum_{k \geq 0} d_k = 1$. Moreover, the characteristic function of such a set is in B^1 .

b. δ is an element of B^λ for any λ satisfying $1 \leq \lambda < 3$. (for $\delta(n)$ is in B^1 and $\lim_{n \rightarrow +\infty} \frac{1}{x} \sum_{n \leq x} \delta(n)^3$ exists.)

c. A necessary and sufficient condition for δ to be in B^∞ is that

for any $\alpha \geq 0$,

$$\overline{\lim}_{x \rightarrow +\infty} \frac{1}{x} \sum_{n \leq x} \delta(n)^\alpha < +\infty \quad (\#)$$

(This (#) is conjectured by Erdős ([6]).)

In the same way, it is immediate that a, b, c still hold if in place of the whole set of the square-free numbers, we restrict ourself to some subset A defined by

$$A = \left\{ n \in \mathbb{N}^* \mid \begin{array}{l} n \text{ square-free,} \\ n \equiv b \pmod{a} \end{array} \right\}$$

with $(a,b) = 1$, $a > b \geq 1$.

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