数理解析研究所講究録
第 518 巻 1984 年 12-20

NON-ITERATED TORUS KNOTS WHICH GIVE LENS SPACES AFTER DEHN SURGERY

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Let K be a knot in S^3 and $r \in \mathbb{Q} \cup \{\infty\}$. $\chi_{S^3}(K;r)$ denotes the closed orientable 3-manifold obtained by Dehn surgery of type r on S^3 along K.

We say that a knot K has Property $\tilde{P}$ if the fundamental group of $\chi_{S^3}(K;r)$ is not a finite cyclic group for a rational number r. For example, any non-torus two bridge knot has Property $\tilde{P}$ ([T]). By Gordon ([G. Theorem 1.11]), it is known that all non trivial knots "almost" have Property $\tilde{P}$. It is clear that if a knot has Property $\tilde{P}$ then it has Property P. Note that from a trivial knot, torus knots and twice iterated torus knots lens spaces are obtained by Dehn surgery ([M],[FS],[G]). Although these knots have Property P, they have not Property $\tilde{P}$ except the trivial knot. Therefore the converse does not hold. One may think that any knot has Property $\tilde{P}$ except trivial knot, torus knots and twice iterated torus knots. But we show that there exist non-iterated torus knots which give lens spaces after Dehn surgery in this note:

Main Theorem.

1. $L(k(2n+1)^2, 2k(2n+1)+1) \cong \chi_{S^3}(K(n+1,n;-k); -k(2n+1)^2)$,

2.1. $L(16k-2, 6k-1) \cong \chi_{S^3}(K(3,1;k); 16k-2)$,

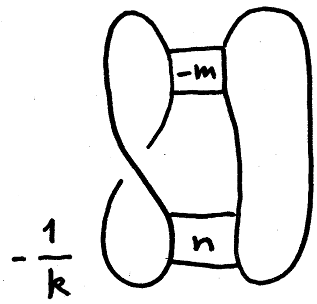
2.2. $L(19,11) \cong \chi_{S^3}(K(2,1;-2); -19)$,

2.3. $L(47,13) \cong \chi_{S^3}(K(4,1;2); 47)$.

2.4. $L(32,7) \cong \chi_{S^3}(K(5,1;1); 32)$,

Moreover all these knots $K(m,n;k)$ are neither the torus knots nor the iterated torus knots except $K(1,0;-k)$, $K(n+1,n;\pm 1)$ and $K(3,1;1)$.

The knot $K(m,n;k)$ ($m, n, k \in Z$, $mn \geq 0$) of the main theorem is obtained from a component of the two bridge link with the normal form $(2(2mn+m-n), 2n+1)$ by performing Dehn surgery of type $-1/k$ on S^3 along the another component of the link. That is $K(m,n;k)$ has the following surgery description:



, where  denotes a n full twists of two strands.

Outline of the proof.

By the link calculcs, it is proved that these lens spaces are obtained by Dehn surgery along these $K(m,n;k)$'s. But we omit the proof. To show the latter statement, we need several propositions but omit the proofs.

First we compute the Alexander polunomial of $K(m,n;k)$ by an usual way.

Proposition 1. The Alexander polynomial of $K(m,n;k)$ is given as follows: $A(t) =$

$$(t-1)\{(f^{m}-1)(g^{n+1}-1) - t^{k(n+m)}(f^{m-1}-1)(g^{n}-1)\}/(t^{m+n}-1)(f-1)(g-1),$$

where $f = t^{k(m+n)-1}$ and $g = t^{k(m+n)+1}$.

From Proposition 1, it follows easily,

Corollary 2.

$$\deg A(t) = \begin{cases} (m+n)\{k(m+n-1)-1\}+n-m+2 & (\ k>0\ ), \\ (m+n)\{(-k)(m+n-1)-1\}+m-n & (\ k<0\ ), \end{cases}$$

where $A(t)$ is normalized if $k < 0$.

$$A(-1) = \begin{cases} 2mn+m-n & (\ k\text{: odd, } m+n\text{: odd }), \\ k(n-m+2)+1 & (\ k\text{: any, } m,n\text{: odd }), \\ k(m-n)-1 & (\ k\text{: any, } m,n\text{: even }), \\ -1 & (\ k\text{: even, } m\text{: even, } n\text{: odd}), \\ 1 & (\ k\text{: even, } m\text{: odd, } n\text{: even}). \end{cases}$$

Remark. More presicely the expanded form of the (normalized) Alexander polynomial of $K(m,n;k)$ is as follows:

$$A(t) = (t-1)t^{k(m+n)(m+n-2)+n-m+1}\left(\sum_{s=0}^{k-1} t^{(m+n)s}\right)$$

$$+ (t-1)\sum_{j=0}^{n-2}\sum_{i=0}^{m-3} t^{k(m+n)(i+j+1)+j-i}\left(\sum_{s=0}^{k-1} t^{(m+n)s}\right)$$

$$+ (t-1)\sum_{j=1}^{n-1} t^{k(m+n)+j+1}\left(\sum_{s=0}^{kj-1} t^{(m+n)s}\right)$$

$$+ (t-1)\sum_{i=1}^{m-2} t^{k(m+n)-i-1}\left(\sum_{s=0}^{ki-1} t^{(m+n)s}\right) + t^{k(m+n)+1-m}$$

$$- (t-1)\sum_{s=0}^{k-1} t^{(m+n)s} \quad (k > 0).$$

$$A(t) = (t-1)t^{(-k)(m+n)(m+n-2)+m-n-1}\left(\sum_{s=0}^{-k-1} t^{(m+n)s}\right)$$

$$+ (t-1)\sum_{j=0}^{m-3}\sum_{i=0}^{n-2} t^{(-k)(m+n)(i+j+1)+j-i}\left(\sum_{s=0}^{-k-1} t^{(m+n)s}\right)$$

$$+ (t-1)\sum_{j=1}^{m-2} t^{(-k)(m+n)+j+1}\left(\sum_{s=0}^{-kj-1} t^{(m+n)s}\right)$$

$$+ (t-1)\sum_{i=1}^{n-1} t^{(-k)(m+n)-i-1}\left(\sum_{s=0}^{-ki-1} t^{(m+n)s}\right) + t^{(-k)(m+n)-n}$$

$$- (t-1)\sum_{s=0}^{-k-1} t^{(m+n)s} \quad (k < 0).$$

Let K_1 and K_2 be knots in S^3. By $K_1 \sim k_2$ we denote that K_1 and K_2 have the equivalent Alexander polynomials.

For $K(m,n;k)$, we have the following.

Corollary 3.

1. $K(m,0;k) \sim T(km-1,m-1)$,

2. $K(0,n;k) \sim T(kn+1,n)$,

3. $K(1,n;k) \sim T(k(n+1)+1,n+1)$,

4. $K(m,-1;k) \sim T(k(1-m)+1,1-m)$,

5. $K(n+2,n;1) \sim T(2n+1,2n+3)$,

6. $K(n,n;-1) \sim T(-2n+1,2n+1)$,

7. $K(n+3,n;-1) \sim [2n^2+6n+3,2;n+1,n+2]$,

8. $K(n+1,n;-1) \sim [-(2n^2+2n+1),2;-(n+1),n]$,

9. $K(n+1,n;1) \sim [2n^2+2n+1,2;n+1,n]$,

10. $K(n-1,n;-1) \sim [-(2n^2-2n-1),2;1-n,n]$,

11. Otherwise,

where $[p_1,q_1;p_2,q_2]$ denotes the (p_1,q_1)-cable of the torus knot of type (p_2,q_2).

Calculating the surgery descriptions of $K(m,n;k)$, we can show the following.

Proposition 4. The knots from 1 to 10 in Corollary 3 are genuine torus knots or twice iterated torus knots.

To identify a given knot as an non-torus knot or an non-iterated torus knot by the Alexander polynomial, the following is useful.

Proposition 5. Let K be a knot in S^3 and $A(t)$ be the

Alexaner polynomial of K.

1. If $A(-1) = \pm 1$ and $\deg A(t) \not\equiv 0 \ (4)$, then K is neither a torus knot nor an iterated torus knot.

2. Let $A(-1) = p \ (\neq \pm 1)$. Either if $p-1$ does not devide $\deg A(t)$ or if $p-1$ divides $\deg A(t)$, $q-1 = \deg A(t)/(p-1)$ and $(p,q) \neq 1$, then K is not the torus knot of type (p,q).

Applying Proposition 5, we have the following:

From Corollary 2., $A(-1) = \pm 1$ holds in the following cases.

$A(-1) = -1$ iff $K(2m,2n-1;2k)$, $K(2n,2n;k)$, $K(2n+3,2n-1;1)$ and $K(2n-1,2n-1;-1)$,

$A(-1) = +1$ iff $K(2m-1,2n;2k)$, $K(2n+1,2n-1;k)$, $K(2n,2n+2;-1)$ and $K(2n+2,2n;1)$.

Corollary 6. $K(2m,2n-1;2k)$, $K(2n,2n;k)$ ($k > 0$), $K(2n+3,2n-1;1)$, $K(2n+1,2n-1;k)$ ($k < 0$) and $K(2n,2n+2;-1)$ are neither the torus knots nor the iterated torus knots.

We use the following result.

Theorem. (Moser [M], Fintsushel and Stern [FS] and Gordon [G])

1. (Moser) $L(pq \pm 1, q^2) \cong \chi_{S^3}(T(p,q);\ pq \pm 1)$,

$$L(p,q)\#L(q,p) \cong \chi_{S^3}(T(p,q);\ pq),$$

2. (Fintsushel and Stern, and Gordon) A lens space is obtained by performing framed surgery on S^3 along a nontrivial iterated torus knot if and only if $L(4pq \pm 1, (2q)^2) \cong \chi_{S^3}([(2pq \pm 1, 2; p, q];\ 4pq \pm 1)$.

A connected sum of lens spaces is obtained by performing framed surgery on S^3 along a nontrivial iterated torus knot if and only if $L(q'pq \pm 1, q'q^2)\#L(q', \pm 1) \cong \chi_{S^3}([q'pq \pm 1, q'; p, q];\ (q'pq \pm 1)q')$.

By Corollary 6, $K(n+1,n;-k)$ (n:odd, k:even), $K(3,1;k)$ ($k<0$), $K(2,1;-2)$, $K(4,1;2)$ and $K(5,1;1)$ in our theorem are neither the torus knots nor the iterated torus knots. The rest to prove are $K(n+1,n;-k)$ (n:odd, k:odd or n:even, k:any) and $K(3,1;k)$ ($k>1$).

We prove here only the case $K = K(3,1;k)$ ($k>1$). By the remark after Corollary 2., $A(t) = (t-1)t^{8k-1}(\sum_{s=0}^{k-1} t^{4s})$

$$+ (t-1)t^{4k}(\sum_{s=0}^{k-1} t^{4s}) + (t-1)t^{4k-2}(\sum_{s=0}^{k-1} t^{4s}) + t^{4k-2}$$

$$- (t-1)\sum_{s=0}^{k-1} t^{4s}.$$

Moreover by Corollary 2, $\deg A(t) = 12k - 4$ and $A(-1) = 1$. Note that the degree of the higher terms decreases by 4.

Suppose K be the nontrivial torus knot T of the type (p,q) (p>q). Then the type of K is (p,4) by the above note. Thus $A_T(-1) = p$ ($\neq \pm 1$). This contradicts to $A_K(-1) = 1$.

Suppose K be an nontrivial iterated torus knot. Since a certain lens space is obtained by Dehn surgery along K by 2.1, K = J = [2pq 1,2;p,q] for some relatively prime integers p and q by 2 of the previous theorem. So $A_J(-1) \neq \pm 1$, this also contradicts to $A_K(-1) = 1$. Therefore K is not an iterated torus knot.

Similarly we can prove for the case K(n+1,n;1).

The exceptions can be seen by Proposition 4. ■

Remarks.

1. In [FS], Fintsushel and Stern shows that lens space L(9n,3n+1) is obtained by Dehn surgery along a certain knot K_n and that K_n (n:even) is not a torus knot by considering the double branched covering space of K_n.

We note that their K_n is our K(2,1;-n) and that the statement 1 of the main theorem includes their result.

2. K(3,1;-1) is the even prezel knot P(2,-3,-7). We do not know what extent lens spaces obtained by Dehn surgery along the (even) pretzel knots.

3. K(m,n;k) is obtained from Dehn surgery along a two bridge link and some families are obtained by Dehn surgery along some two bridge links.

To classify the results of Dehn surgery along the two bridge links and to describe the limiting phenomenon of breaking down hyperbolic Dehn surgery along the two bridge links seem to be interesting problems.

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