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Non Reversible Knots Exist

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$S^3 \supset K$, tame knot

$G = \pi_1(S^3 - K)$

Definition. K is reversible $\Leftrightarrow$ $\exists\, \rho : G \xrightarrow{\cong} G$ automorphism such that the induced $\xi : G/G' \xrightarrow{\cong} G/G'$ takes a generator to its inverse.

Examples:

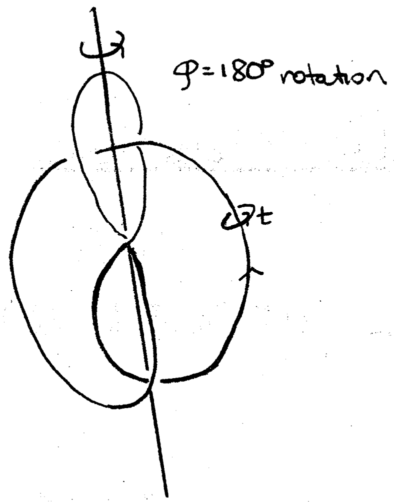
(i) K invertible $\Rightarrow$ K reversible

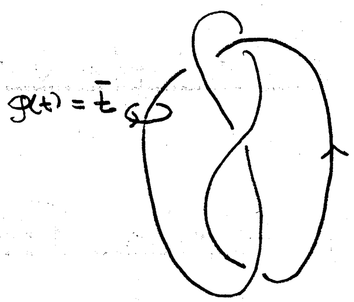
(ii) K positive amphicheiral $\Rightarrow$ K reversible.

Proof. (i) K invertible. So $\exists\, \varphi : S^3 \xrightarrow{\cong} S^3$ orientation preserving homeomorphism s.t. $\varphi(\vec{K}) = -\vec{K}$.

Take $\rho = \varphi_*$ (We may assume φ fixes the base point).

e.g.





(ii) K positive amphicheiral. So $\exists\ \psi : S^3 \xrightarrow{\cong} S^3$ orientation reversing homeomorphism s.t. $\psi(\vec{K}) = \vec{K}$.
Take $\rho = \psi_*$.

<u>Question</u>: Do non-reversible knots exist?
(<u>Answer</u>: Yes!)

<u>Notation</u>: $P = \langle x_1, \cdots, x_m \mid r_1, \cdots, r_n \rangle$, presentation of K.
$F = \langle x_1, \cdots, x_m \rangle$ free.

$$F \xrightarrow{\varphi} P \xrightarrow{\psi} P/P' \quad , \text{ canonical projections.}$$
$$\chi : F \to P/P'$$

$P/P' \cong \langle t \rangle$; choose $y \in P/P'$, a generator.

$$\left.\begin{array}{l} y \overset{\alpha_y}{\longmapsto} t \\ \bar{y} \overset{\alpha_{\bar{y}}}{\longmapsto} t \end{array}\right\} \text{ defines two isomorphisms.}$$

$$\left.\begin{aligned} A &= \left(\frac{\partial r_i}{\partial x_j}\right)^{\tilde{x}\tilde{\alpha}_y} \\ \bar{A} &= \left(\frac{\partial r_i}{\partial x_j}\right)^{\tilde{x}\tilde{\alpha}_{\bar{y}}} \end{aligned}\right\} \text{Laurent polynomial matrices.}$$

$\bar{A}$ is conjugate to A. (Here $\sim$ denotes extension to group rings) In future, we ignore $\tilde{\alpha}_y$, $\tilde{\alpha}_{\bar{y}}$.

<u>Definition</u>: $A, \bar{A}$ are the Alexander matrices of P.

<u>Definition</u>: (matrix equivalence).

(a) <u>Proper definition</u> (see [P1]) A, A_* matrices over $J\langle t\rangle$.

$A \sim A_* \equiv \exists\ (A, x), (A_*, x_*)$ satisfactory s.t. $(A,x) \sim (A_*, x_*)$.

(b) <u>Rough outline</u>:

The usual equivalence relation allows:

(0) Interchange of rows, Interchange of columns.

(1) $A \Longleftrightarrow \begin{pmatrix} A \\ * \end{pmatrix}$, $*$ = linear combination of the rows of A.

(II) $A \longleftrightarrow \begin{pmatrix} A & \begin{smallmatrix}0\\ \vdots\\ 0\end{smallmatrix} \\ * & 1 \end{pmatrix}$, $*$ arbitrary.

<u>We allow</u>:

(0) As above.

(I) As above.

(II) $A \longleftrightarrow \begin{pmatrix} A & \begin{smallmatrix}0\\ \vdots\\ 0\end{smallmatrix} \\ * & 1 \end{pmatrix}$, where if $* = (a_{(m+1)1}, a_{(m+1)2}, \dots, a_{(m+1)n})$,

$$\sum_{j=1}^{n} a_{(m+1)j}\,\tilde{x}(x_j - 1) = -t^{\delta} + 1, \quad \exists\ \delta \text{ integral}.$$

<u>Difference</u>: in (II), $*$ no longer arbitrary. (The reason is to ensure $\exists\ x'$ s.t. $\left(\begin{pmatrix} A & \begin{smallmatrix}0\\ \vdots\\ 0\end{smallmatrix} \\ * & 1 \end{pmatrix}, x'\right)$ is satisfactory)

<u>Theorem 1</u>. K reversible. Then $A \sim \bar{A}$ $\forall P$.

<u>Proof</u>. Consider first the special case $P = \langle x_1, \dots, x_m \mid r_1, \dots, r_{m-1} \rangle$ is Wirtinger. Then $r_i = x_a x_b \bar{x}_a \bar{x}_c$, $\exists\ a, b, c$, $\forall i$.

<u>Define</u> P's <u>reversed presentation</u> $P_R = \langle x_1, \dots, x_m \mid s_1, \dots, s_{m-1} \rangle$ by if $r_i = x_a x_b \bar{x}_a \bar{x}_c$ then $s_i \equiv \bar{x}_a \bar{x}_b x_a x_c$, $\forall i$.

$\exists$ an isomorphism $\lambda : P \overset{\cong}{\to} P_R$ induced by $x_j \mapsto \bar{x}_j$, $\forall j$.

λ induces $\zeta : \langle t \rangle \xrightarrow{\cong} P_R/P_R'$. Identify P_R/P_R' with $\langle t \rangle$ so that $\zeta(t) = \bar{t}$. Then

$$
\begin{aligned}
\chi(x_j) &= \zeta^{-1}(\psi_R \lambda \varphi(x_j)) \\
&= \zeta^{-1} \psi_R \varphi_R(\bar{x}_j) \\
&= \psi_R \varphi_R(x_j) \qquad (\because \zeta(t) = \bar{t})
\end{aligned}
$$

So

$$
\begin{tikzcd}
F \arrow[dr, "\varphi"] \arrow[ddr, "\chi"'] & & & F \arrow[dl, "\varphi_R"'] \arrow[ddl, "\chi"] \\
& P \arrow[r, "\lambda"] \arrow[d, "\psi"] & P_R \arrow[d, "\psi_R"] & \\
& \langle t \rangle \arrow[r, "\zeta"] & \langle t \rangle &
\end{tikzcd}
$$

K reversible $\Rightarrow \exists\, \rho$ s.t.

$$
\begin{tikzcd}
F \arrow[dr, "\varphi"] \arrow[ddr, "\chi"'] & & & & F \arrow[dl, "\varphi_R"'] \arrow[ddl, "\chi"] \\
& P \arrow[r, "\rho"] \arrow[d, "\psi"] & P \arrow[r, "\lambda"] \arrow[d] & P_R \arrow[d, "\psi_R"] & \\
& \langle t \rangle \arrow[r, "\xi"] & \langle t \rangle \arrow[r, "\zeta"] & \langle t \rangle &
\end{tikzcd}
\quad , \ \zeta \circ \xi = id..
$$

$$t \mapsto \bar{t} \mapsto t$$

Hence $A = \left(\partial r_i / \partial x_j\right)^{\tilde{\chi}} \sim \left(\partial s_i / \partial x_j\right)^{\tilde{\chi}}$.

($\lambda\rho$ gives sequence of Tietze transformations)

Suppose $r_i = x_a x_b \bar{x}_a \bar{x}_c$. Then $s_i = \bar{x}_a \bar{x}_b x_a x_c$.

So

$$
\begin{array}{lccc}
 & a & b & c \\
\partial r_i / \partial x_j = & 1 - x_c & x_a & -1 \\
 & & \downarrow \tilde{x} & \\
 & 1 - t & t & -1 \quad (\because P \text{ Wirtinger})
\end{array}
$$

$$
\begin{array}{lccc}
 & a & b & c \\
\text{and } \partial s_i / \partial x_j = & -\bar{x}_a + \bar{x}_a \bar{x}_b & -\bar{x}_a \bar{x}_b & \bar{x}_c \\
 & & \downarrow \tilde{x} & \\
 & -\bar{t} + \bar{t}^2 & -\bar{t}^2 & \bar{t} \\
 & & \wr \ (\times -t) & \\
 & 1 - \bar{t} & \bar{t} & -1 .
\end{array}
$$

Hence $(\partial s_i / \partial x_j)^{\tilde{x}} \sim \bar{A}$, and $A \sim \bar{A}$.

Generally: $Q = \langle y_1, \dots, y_n \mid s_1, \dots, s_0 \rangle$, arbitrary presentation of K. Alexander matrices B, $\bar{B}$.

$\exists\ \lambda : P \overset{\simeq}{\to} Q$ isomorphism s.t.

$$
\begin{tikzcd}
F \arrow[r] \arrow[dr, "x"'] & P \arrow[r, "\lambda"] \arrow[d] & Q \arrow[d] & \langle y_1, \dots, y_n \rangle \arrow[l] \arrow[dl, "x_*"] \\
 & \langle t \rangle \arrow[r, "\xi"] & \langle t \rangle &
\end{tikzcd}
$$

We may suppose $(\partial r_i/\partial x_j)^{\tilde{x}} = A$, $(\partial s_i/\partial x_j)^{\tilde{x}_*} = B$.

Then $\xi =$ identity $\Rightarrow A \sim B$; $\xi(t) = \bar{t} \Rightarrow A \sim \bar{B}$.

But $A \sim \bar{A}$, and $A \sim B \Leftrightarrow \bar{A} \sim \bar{B}$,

$\Rightarrow$ EITHER $B \sim A \sim \bar{A} \sim \bar{B}$

OR $B \sim \bar{A} \sim A \sim \bar{B}$

$\Rightarrow$ $B \sim \bar{B}$. Q.E.D.

<u>Theorem 2</u> (Partial converse to Thm. 1)

$A \sim \bar{A} \Leftrightarrow \exists\ \bar{\rho} : G/G'' \xrightarrow{\cong} G/G''$ s.t. the induced $\xi : G/G' \xrightarrow{\cong} G/G'$ takes a generator to its inverse.

<u>Proof.</u> See [P2], (1.5).

<u>Definition.</u> K <u>weakly reversible</u> $\Leftarrow\!\equiv \exists\ \bar{\rho} : G/G'' \xrightarrow{\cong} G/G''$ s.t. the induced $\xi : G/G' \xrightarrow{\cong} G/G'$ takes a generator to it inverse.

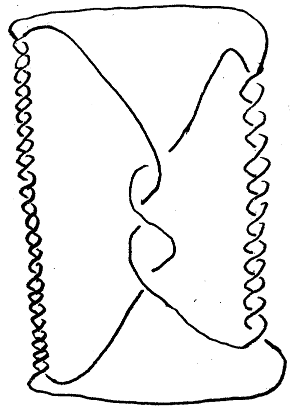
Theorem 1 + Theorem 2 $\Rightarrow$

<u>Summary</u> Reversible $\Rightarrow$ Weakly reversible

Weakly reversible $\Leftrightarrow A \sim \bar{A}$, $\forall\ p$.

<u>Proposition</u>. The pretzel knot (25, -3, 13) is not weakly reversible. Hence it is not reversible.

(25, -3, 13) =



<u>Proof</u>. $A = \begin{bmatrix} 5(1-t) & 2-t & 0 \\ 1-2t & 11(1-t) & 0 \end{bmatrix}$ is an Alexander matrix.

$A \sim \bar{A} \Rightarrow$ Row class $A \sim$ column class A (See [FS] for definitions etc.)

$R =$ ring of integers of $\mathbb{Q}(e)$, $e = \sqrt{-211}$.

$e^2 \equiv 1 \bmod 4 \Rightarrow R = \mathbb{Z} + \mathbb{Z}(\frac{1}{2}(1+e))$. ([ZS])

$R' = R[1/53]$.

Row class $A = ((52+e)/53, 11)_{R'}$

Column class $A = ((52+e)/53, 5)_{R'}$.

Equivalent $\overset{[P2]}{\Rightarrow}$ $I = (5\alpha, \alpha^2, 25)_R$ principle R ideal, $\alpha = (-3+e)/2$.

Write $I = R\beta$, $\beta = \frac{1}{2}(q_0 + q_1 e) \in R$.

Now $I \subset (25, \alpha)R$, and $|25| = 5^4$, $|\alpha| = \alpha\bar{\alpha} = 55$.

$\Rightarrow 5 \mid |r|$, $\forall r \in I$

$\Rightarrow 1 \notin I$.

Further $|\beta| \neq 5$, $\because |\beta| = \frac{1}{4}(q_0^2 + 211 q_1^2) \neq 5$ $\forall_{q_0, q_1 \in J}$.

But $25 \in I \Rightarrow \exists\, r \in R$ s.t. $25 = r\beta$.

Hence $|25| = 5^4 = |r|\,|\beta|$.

But $1 \notin I \Rightarrow |\beta| \neq 1$.

And $|\beta| \neq 5$, and similarly $|r| \neq 5$. Hence

EITHER $|\beta| = 25 \Rightarrow \beta = 5$,

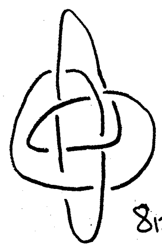
OR $|\beta| = (25)^2 \Rightarrow |r| = 1 \Rightarrow \beta = 25$.

But $I \ni \alpha^2 = \frac{1}{4}(-3+e)(-3+e) = (-101-3e)/2 \neq r \cdot 5$ or $r \cdot 25$ $\forall r \in R$

a contradiction to the principality of I.

Hence $A \not\sim \bar{A}$. Q.E.D.

Remark 1.



8_{17}

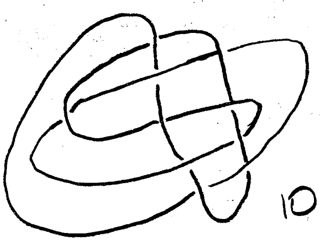
8_{17} is non positive amphicheiral, and non-invertible, but negative amphicheiral ([K]). Moreover A. Kawauchi has shown (see Appendix) 8_{17} is non-reversible. Further

8_{17} has a cyclic module; thus

$$A_{8_{17}} = [\Delta] = [\bar{\Delta}] = \bar{A} \quad (\Delta = \text{Alexander polynomial})$$

$\Rightarrow$ 8_{17} is weakly reversible.

<u>Remark 2.</u>



10_{115}

10_{115} is non +ve amphicheiral, and non-invertible.

$$A_{115} = \begin{bmatrix} t^2-t+1 & 2t & 0 \\ 2(t-1) & t^4-8t^3+17t^2-12t+1 & 0 \end{bmatrix}$$

Hence row class = Column class, and the method of proof used in the proposition is innapplicable.

<u>Conjecture.</u> 10_{115} is not weakly reveirsible.

Finally, we construct an infinity of non reversible knots.

Set $A = \begin{bmatrix} 5(1-t) & 2-t \\ 1-2t & 11(1-t) \end{bmatrix}$, and $A(p,q) = \begin{bmatrix} \underbrace{A \cdots A}_{p} & & & 0 \\ & \underbrace{\bar{A} \cdots \bar{A}}_{q} & & \vdots \\ & & & 0 \end{bmatrix}$ $\quad 2(p+q) \times 2(p+q)+1$ matrix.

<u>Theorem 3.</u> $A(p,q) \sim A(p',q') \Leftrightarrow p=p', q=q'$.

<u>Proof.</u> $\Leftarrow$ clear

$\Rightarrow$ $\Delta_{A(p,q)} = \Delta_{A(p',q')} \Rightarrow p+q = p'+q'$.

Column class $A(p,q) = [(\alpha,5)_{R'}]^{p}[(\alpha,11)_{R'}]^{q}$

Row class $A(p',q') = [(\alpha,5)_{R'}]^{p'}[(\alpha,11)_{R'}]^{q'}$.

Equivalent $\Rightarrow I^n$ principle, $I = (5\alpha, \alpha^2, 25)_R$, $p-p' = n = q'-q$.

<u>Lemma:</u> R Dedikind domain, I^n principle $\Rightarrow I$ principle.

<u>Proof.</u> (i) Without valuations; see [P2].

(ii) With valuations; see [O], p. 47.

By the proposition, I is not principle $\Rightarrow n=0$, or $p=p'$, $q=q'$.

Q.E.D.

Set $K_1 = (25,-3,13)$ pretzel knot.

Define $K(p,q) = \underset{p}{\#} K_1 \underset{q}{\#} (-K_1)$.

Then $A_{K(p,q)} = A(p,q)$. Theorem 3 $\Rightarrow$

<u>Theorem 4</u>. $\{K(p,q)\}_{p>q}$ is an infinite collection of non reversible knots with mutually inequivalent Alexander matrices.

<u>Appendix</u>.

<u>Claim</u> 8_{17} is non reversible.

<u>Proof</u>. (Kawauchi) Let $K = 8_{17}$, $E = S^3 - \mathring{N}(K)$, $G = \pi_1(E)$, and suppose $\exists\ \rho : \begin{matrix} G \xrightarrow{\cong} G \\ t \mapsto \bar{t} \end{matrix}$.

As K is hyperbolic, $\exists \rho \Rightarrow \exists\ h : E \circlearrowleft$ of finite order such that $h_*(m) = -m + al$, $h_*(l) = \pm l$ in $H_1(\partial E)$. (Mostow rigidity). There are two cases:

(1) h orientation preserving $\Rightarrow h_*(l) = -l \Rightarrow a = 0$ ($\because h$ has finite order) $\Rightarrow h$ can be extended to a homeomorphism of S^3, contradicting the non invertibility of K.

This in fact shows: K hyperbolic, non-invertible, signature $\sigma(K) \neq 0 \Rightarrow K$ non reversible, a result first pointed out to me by M. Sakuma.

(ii) h is orientation reversing. So $h_*(\ell) = \ell$.

a) $\mathrm{Fix}(h) \neq \phi$.

Proof. $H_*(E) = H_*(S^1)$. Lefschetz number $\lambda(h) = 1 - (-1) = 2 \neq 0$. Lefschetz fixed point theorem $\Rightarrow \mathrm{Fix}(h) \neq \phi$.

b) We can assume $h^{2^m} =$ identity $(m \geq 1)$.

Proof. If the order of $h = 2^m n$, n odd, replace h by h^n.

c) Proof that $m \ngtr 2$

$h' = h^{2^{(m-1)}} : E \circlearrowleft$ is an orientation preserving involution. Hence $\exists\, \bar{h}' : S^3 \circlearrowleft$ orientation preserving involution such that $\bar{h}'(K) = K$. $\bar{h}'$ preserves the orientation of K, and $\phi \neq \mathrm{Fix}(h) \subset \mathrm{Fix}(\bar{h}')$. Hence by Smith theory, (note $H_*(E) \cong H_*(S^1)$) $\mathrm{Fix}(\tilde{h}') = S^1$, but $S^1 \cap k = \phi$. It now follows from [M] that the Alexander polynomial of K must be of the form $A_{8_{17}} = F(t)F_0(t)$, where $F_0(t) \underset{2}{\equiv} F(t)P_\lambda(t)$. Here λ is some odd number, $p_\lambda(t) = (t^\lambda - 1)/(t-1)$, and $\underset{2}{\equiv}$ means equal considered as $J_2\langle t\rangle$ polynomials. The Alexander polynomial of 8_{17} is not of this form.

(d) Proof that $m \neq 2$.

$h^2 = $ identity, h orientation reversing, $\mathrm{Fix}(h) \neq \phi \overset{[KH]}{\Longrightarrow}$ $A_{8_{17}} = F(t)^2$, for some $F(t) \in J\langle t\rangle$. The Alexander polynomial of 8_{17} is not of this form. Q.E.D.

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