Approximate Queueing System Model via N-dimentional Elementary Return Process

豊橋技術科学大学工学部

高橋 治久 (Haruhisa Takahashi)

秋丸 春夫 (Haruo Akimaru)

あらまし

　本論文では，計算機ネットワークなどの大規模な待ち合せシステムのモデル（近似）として，N次元の基本復帰拡散過程提案しその解析を行っている。大規模システム（ネットワークが主）の拡散近似は，Reiman 等 [2] により，反射壁境界を用いた解析が行なわれており，極限重負荷定理（$\rho \to 1$ で近似が成立するという定理）が示されているが，ρ が小さければ，精度の点で問題がある。これを改良するため，本論文では Gelenbe [6] の結果を N次元に拡張し，N次元帰本復帰過程の前進方程式を導く。応用として対称コンピュータ通信システムの系内客数の公式を導く。

1. Introduction

We consider an n-dimensional diffusion process Y_t that arises in conjunction with large queueing systems such as networks of queues, preemptive-resume queueing systems, etc.. Its state space consists of the interior and boundaries of the parallelotope of R^n (i.e., $0 \leq x_i \leq L_i$, i=1,2,...n). On the interior of this state space, Y_t behaves like an ordinary n-dimentional diffusion process (Brownian motion with drift). Whenever Y_t reaches one of the (n-1)-dimensional hyperplanes (for example, $x_i=0$) it remains there for an exponentially distributed finite sojourn time and Y_t behaves this time as an (n-1)-dimentional diffusion process. After this finite sojourn time, a jump in the direction of x_i-axis occurs to a point whose x_i-coordinate is on the interior of $(0, L_i)$ according to some probability density. The process then starts from scratch. The behavior of Y_t on the lower dimentional hyperplane is defined in the same way. The process thus defined is a natural extension of one-dimensional elementary return process [4].

Several works have been devoted to the diffusion approximation to large queueing systems (mainly, to open queueing networks). Kobayashi [10] wrote out the forward equation for open queueing networks with the rough boundary condition that the approximating process must be restricted to the nonnegative orthant. Harrison [7] and Harrison and Reiman [8] showed the precise boundary conditions for a tandem queue which the reflected Brownian motion should obey. It was also shown that their reflected Brownian motion satisfies the heavy traffic limit

theorem [2].

Although the heavy traffic limit theorem was proved for the reflected Brownian motion, it has the disadvantage of failing to model explicitly the probability of an empty queue [2]. With this motivation Gelenbe [6] proposed a heuristic approach for modeling empty queue probabilities by applying one-dimensional elementary return process. Our primary goal here is to extend his work to multi-dimensional case and specifically, to derive the stationary forward equations of the n-dimentional elementary return process as a model of large queueing systems.

In Section 2 and 3 we define the multi-dimensional elementary return process and derive the corresponding stationary forward equations. Using the result derived in Section 3, we then derive in Section 4 the partial differential equations for open networks of queues and the preemptive-resume queueing systems. Furthermore the important remark for discretization of the process is proposed. In Section 5, we discuss the mean of the queue length of the pairs of communicating computers by analyzing the partial differential equations derived in section 4.

A general notation shall be used through the paper that the infinitesimal volume elements of R^n is denoted $dx = dx_1 dx_2 ... dx_n$.

2. Definition of the process

We shall define a multi-dimensional elementary return process Y_t or $Y(t)$, $t \geq 0$. The most general case shall be discussed where the state space S^n of the process is the parallelotope of R^n

$$0 \leq x_i \leq L_i, \quad L_i > 0 \ (i=1,2,...n).$$

Note that L_i may be infinite. S^n consists of its interior and the lower dimentional cells (boundaries). The number of the 0-cells (vertexes) of S^n is 2^n. Generally the number of the $(n-k)$-cells of S^n is $2^k \binom{n}{k}$, $k=1,2,\ldots n$ and sum to 3^n. We number these cells through so that the $(n-k)$-cells are denoted C_i, $(\sum_{j=1}^{k-1} 2^j \binom{n}{j} +1 \le i \le \sum_{j=1}^{k} 2^j \binom{n}{j})$ for $k>0$ and C_0 for $k=0$. Note that C_0 represents the interior of S^n. We write $C_j > (\not>) C_i$ when C_i is (not) a boundary of C_j. Let π^i be the natural projection map from R^n to R^{n-k} which is the sub-space of R^n parallel to the $(n-k)$-cell C_i. The index of the coordinate axis on $\pi^j(R^n)$ which is orthogonal to $\pi^i(R^n)$ is denoted $i*j$ if $C_j > C_i$.

On the interior of S^n, Y_t behaves as an n-dimensional Brownian motion which has the (constant coefficient) forward operator

$$L_0^* = \sum_{\phi\phi'=1}^{n} \frac{1}{2} v_{\phi\phi}^0 \frac{\partial^2}{\partial x_\phi \partial x_{\phi'}} - \sum_{\phi=1}^{n} m_\phi^0 \frac{\partial}{\partial x_\phi}$$

where the covariance matrix $(v_{\phi\phi'}^0)$ is non-negative definite. Whenever Y_t reaches an $(n-1)$-cell C_i, $(1\le i \le 2\binom{n}{1})$ at time t_1, it stays there for a finite sojourn time T_i and behaves as an $(n-1)$-dimensional Brownian motion with the (constant coefficient) forward operator

$$L_i^* = \sum_{\phi\phi'\neq i*0} \frac{1}{2} v_{\phi\phi}^i \frac{\partial^2}{\partial x_\phi \partial x_\phi} - \sum_{\phi\neq i*0} m_\phi^i \frac{\partial}{\partial x_\phi}$$

where the covariance matrix $(v_{\phi\phi'}^i)$ is non-negative definite. If Y_t reaches on an $(n-2)$-cell $C_j < C_i$ at t_1+T_i, it stays there for an finite sojourn time time T_j and behaves as an $(n-2)$-dimensional Brownian motion which has the (constant coefficient)

forward operator

$$L_j^* = \sum_{\phi\phi'\neq i*0,\ j*i} \frac{1}{2} v_{\phi\phi'}^j \frac{\partial^2}{\partial x_\phi \partial x_{\phi'}} - \sum_{\phi\neq i*0,\ j*i} m_\phi^j \frac{\partial}{\partial x_\phi}$$

with the nonnegative definite covariance matrix $(v_{\phi\phi'}^j)$. The behavior of Y_t and the forward operator on the lower dimensional cells are defined in the same way except for the 0-cells (vertexes). If one of the 0-cells is reached then Y_t stays there for an exponentially distributed finite sojourn time. The finite sojourn time T_k on C_k ($0 \leq k \leq 3^n-1$) is exponentially distributed if any boundary cell C_i of C_k (i.e., $C_i < C_k$) is not reached. As soon as the exponential sojourn time lapses before hitting the boundaries, a jump in the direction of x_{k*m}-axis occurs to the interior of C_m from C_k, ($C_m > C_k$) according to some probability denity function over $0 < x_{k*m} < L_{k*m}$.

Let λ_{km} be the positive constant which represents the rate at which jumps occur to a point on the interior of C_m from C_k, ($C_m > C_k$). Then

$$\text{Prob}\left\{ T_k > t \,\middle|\, \text{any } C_i < C_k \text{ is not reached} \right\}$$

$$= \exp\left(-t \sum_m \lambda_{km}\right)$$

where the summation in the exponential function is taken over all m such that $C_m > C_k$. For our purpose, let the density function for the jump be the Dirac delta function $\delta\{x_{k*m}-M(k,m)\}$ in which $M(k,m)$ is defined as follows: let

$$(1) \qquad \text{sgn}(k,m) = \begin{cases} +1 & \text{if } x_{k*m} = 0 \text{ for } x \in C_k \\ -1 & \text{if } x_{k*m} = L_{k*m} \text{ for } x \in C_k \end{cases}$$

then

$$M(k,m) = \frac{1}{2}[1+\text{sgn}(k,m)] + \frac{1}{2}[1-\text{sgn}(k,m)](L_{k*m}-1).$$

After the jump, the process starts from scratch. The process Y_t thus defined is the Markov process because there are no point with memory.

3. Stationary forward equation

Let the alternative representation of the process be

$$Y(t) = (k, x^k)_t$$

where k represents the index of the cell on which the process sojourns and $x^k = \pi^k(Y)$. We assume the process has the transition probability dnsity defined by

$$p_t^{ij}(x^i, y^j) \triangle y^j$$

$$\cong \text{Prob} \left\{ (k, x^k)_t = (j, y^j), \ y^j \in \triangle y^j \ | \ (k, x^k)_0 = (i, x^i) \right\}$$

in which $\triangle y^j$ represents the small volume element on C_j. We decide $p_t^{ij}(x^i, y^j)$ represents the transition probability if C_j is a 0-cell.

Let $x^k = \pi^k x$. Let $\mathcal{Y}_k$, $k = 0, \ldots, 3^n - 1$ be the set of continuous functions $f_k: \pi^k(C_k) \to R^1$ that are twice continuously differentiable in x^k except at z^k whose component x_i of R^n, if contained, is 1 or $L_i - 1$ for some i. Let $f_k(x^k) \in \mathcal{Y}_k$ be the density function on C_k that satisfies

$$\sum_{k=0}^{3^n - 1} \int_{\pi^k C_k} f_k(x^k) \, dx^k = 1 .$$

For the density function f_k, we define the transition operator by

$$(T_t^* f)_j (z^j) = \sum_k \int_{\pi^k C_k} f_k(x^k) \, p_t^{kj}(x^k, z^j) \, dx^k$$

where $f = (f_0, \ldots, f_{3^n-1})$ and $z^j \in \pi^j C_j$. Then

the forward operator of the process is

(2) $$A^* = \lim_{t \to 0^+} \frac{T_t^* - I}{t}$$

where I represents the unit operator on $\mathcal{Y}_0 \times \mathcal{Y}_1 \times \cdots \times \mathcal{Y}_{3^n-1}$. The

stationary forward equation of the process can be represented in

the form

(3) $$(A^* f)_i = 0, \quad i = 0, 1, \ldots, 3^n - 1 .$$

It will be convenient to define the differential operators

$$J_\phi^i [f_i(.)] = -\frac{1}{2} \sum_{\phi'} v_{\phi\phi'}^i \frac{\partial f_i(x^i)}{\partial x_\phi} + m_\phi^i f_i(x^i)$$

and

$$H^{ik} [f_i(.)] = \lim_{x_{k*i} \to N(k,i)} J_{k*i}^i [f_i(.)]$$

where $N(k,i) = (1/2)[1 - \text{sgn}(k,i)] L_{k*i}$, $i = 1, \ldots, 3^n - 1$, and ϕ, ϕ'

represent indexes of the components of x^i and the summation is

taken over all such indexes. The following theorem gives the

concrete form of the equation (3). We shall maintain all of the

notation established earlier.

Theorem 1. Assume that Y_t has a stationary density function

$f_k(\pi^k x)$, $k = 0, 1, \ldots, 3^n - 1$. Then it is the solution of the

equations

$$-L_0^* f_0(x) = \sum_{i=1}^{2n} \lambda_{i0} f_i(x^i) \delta(x_{i*0} - M(i,0))$$

....

$$-L_k^* f_k(x^k) = \sum_{m \text{ s.t. } C_m > C_k} \left\{ -\text{sgn}(k,m) H^{mk} [f_m(x^m)] \right.$$

$$\left. - {}_{km} f_k(x^k) \right\} + \sum_{j \text{ s.t. } C_k > C_j} \lambda_{jk} f_j(x^j) \delta(x_{j*k} - M(j,k))$$

$$(1 \leq k \leq 3^n - 2^n - 1)$$

....

$$0 = \sum_{m \text{ s.t. } C_m > C_k} -\text{sgn}(k,m) H^{mk} \left[f_m(x^m) \right] - \lambda_{km} f_k$$

$$(3^n - 2^n \leq k \leq 3^n - 1)$$

with boundary conditions

$$\lim_{x_\phi \to 0} f_k(x^k) = \lim_{x_\phi \to L_\phi} f_k(x^k) = 0$$

where x_ϕ is the component of x^k, $0 < k \leq 3^n - 1$.

Remark. If all L_i are finite and there are no absorbing states, Y_t is positive recurrent and has a stationary density.

Proof. Let define

$$(a_t^* f)_i = (T_t^* f)_i - f_i(x^i)$$

for any $f_i \in \mathcal{Y}_i$, $(i=0,\ldots,3^n-1)$. We need to show the concrete form of the equation

(4) $\quad \lim\limits_{t-0^+} \dfrac{1}{t}(a_t^* f)_i = 0.$

Since the transition probability densities $p_t^{kj}(x^k, z^j)$, $(k \neq j, C_j \not>,\not< C_k)$ vanish with higher order than $p_t^{ki}(x^k, z^i)$, $(C_j >,< C_k)$ as $t \to 0$, we have

(5) $\quad (a_t^* f)_i (z^i) = \displaystyle\sum_{k \text{ s.t. } C_i > C_k} \int_{\pi^k C_k} f_k(x^k) p_t^{ki}(x^k, z^i)\, dx^k$

$$+ \left[\int_{\pi^i C_i} f_i(x^i)\, p_t^{ii}(x^i, z^i)\, dx^i - f_i(z^i) \right]$$

$$+ \sum_{k \text{ s.t. } C_k > C_i} \int_{\pi^k C_k} f_k(x^k)\, p_t^{ki}(x^k, z^i)\, dx^k,$$

$$+ \mathcal{E}(t)\ ,\ \text{(for small } t > 0)$$

where $\mathcal{E}(t)/t = 0, (t \to 0)$. In the integrand of the first term of (5), the transition probability density is given by

$$p_t^{ki}(x^k, z^i)$$

$$= \int_{\pi^k C_k} p_t^{kk}(x^k, z^k)\{1 - \exp(-\lambda_{ki} t)\}\, \delta\{\pi^k z^i - z^k,\ z_{k*i} - M(k,i)\}\, dz^k$$

$$+ O(t)$$

Hence the first term in the right hand side of (5) multiplied by $1/t$ as $t-0$ becomes

(6) $\quad \displaystyle\sum_{k \text{ s.t. } C_i > C_k} \lambda_{ki}\, f_k(x^k)\, \delta(x_{k*i} - M(k,i)).$

Let $q_t^i(x^i, z^i)$ be the transition probability density of the Brownian motion on C_i (with the forward operator L^*_i). Then

$$p_t^{ii}(x^i, z^i) = \exp\left(- \sum_{k \text{ s.t. } C_k > C_i} \lambda_{ik}\, t\right) q_t^i(x^i, z^i)\ .$$

Thus the second term in the right hand side of (5) can be written as follows:

(7)

$$\left[\int_{\pi^i c_i} f_i(x^i) \, q_t^i(x^i, z^i) \, dx^i - f_i(z^i) \right] \exp\left(- \sum_{j \text{ s.t. } c_j > c_i} \lambda_{ij} \, t\right)$$

$$+ f_i(z^i) \left[\exp\left(- \sum_{j \text{ s.t. } c_j > c_i} \lambda_{ij} \, t\right) - 1 \right] .$$

Letting $t \to 0$ in (7) multiplied by $1/t$, we obtain

$$(8) \qquad L_i^* f_i(x^i) - \sum_{j \text{ s.t. } c_j > c_i} \lambda_{ij} \, f_i(x^i) .$$

Let

$$(9) \quad S_t = \sum_{i \text{ s.t. } c_k > c_i} \int_{\pi^i c_i} \int_{\pi^k c_k} f_k(x^k) \, p_t^{ki}(x^k, z^i) \, dx^k dz^i$$

$$= \int_{\pi^k c_k} f_k(x^k) \left[\sum_{i \text{ s.t. } c_k > c_i} \int_{\pi^i c_i} p_t^{ki}(x^k, z^i) \, dz^i \right] dx^k,$$

the last equality resulting by interchanging orders of integration. Since the transition density functions $p_t^{kj}(x^k, z^j)$, ($k = j$, $c_j > , < c_k$) vanish with higher order than $p_t^{ki}(x^k, z^i)$, ($c_k > , < c_i$) as $t \to 0$, we have

$$\sum_{i \text{ s.t. } c_k > c_i} \int_{\pi^i c_i} p_t^{ki}(x^k, z^i) \, dz^i$$

$$= 1 - \int_{\pi^k c_k} p_t^{kk}(x^k, z^k) \, dz^k - \sum_{j \text{ s.t. } c_j > c_k} \int_{\pi^j c_j} p_t^{kj}(x^k, z^j) \, dz^j$$

$$+ \, \varepsilon(t) \, , \quad (\text{for small } t > 0),$$

where $\lim\limits_{t\to 0} \mathcal{E}(t)/t = 0$. Thus

$$(10) \quad S_t = \int_{\pi^k C_k} f_k(z^k)\,dz^k - \int\int_{\pi^k C_k} f_k(x^k) p_t^{kk}(x^k, z^k)\,dx^k dz^k$$

$$- \int_{\pi^k C_k} \sum_{j \text{ s.t. } c_j > c_k}^{\prime} f_k(x^k) \int_{\pi^j C_j} p_t^{kj}(x^k, z^j)\,dx^j\,dx^k$$

$$+ \mathcal{E}(t), \qquad \text{(for small } t),$$

where $\mathcal{E}(t)/t = 0$, $(t \to 0)$. Using the same discussion as before, the first two terms and the last term in the right hand side of (10) multiplied by $1/t$ as $t \to 0$ become

$$\int_{\pi^k C_k} \left[-L_k^* f_k(x^k) + \sum_{j \text{ s.t. } c_j > c_k} \lambda_{kj} f_k(x^k) \right] dx^k$$

and

$$- \int_{\pi^k C_k} \sum_{j \text{ s.t. } c_j > c_k} \lambda_{kj} f_k(x^k)\,dx^k, \quad (t - 0)$$

respectively. Thus

$$\lim_{t \to 0} S_t/t = \int_{\pi^k C_k} -L_k^* f_k(x^k)\,dx^k$$

or in the alternative expression,

$$(11) \qquad\qquad = \int_{\pi^k C_k} \text{div } J^k[f_k(.)]\,dx^k.$$

where div $= (\partial/\partial x_\phi)$. Applying the well known divergence theorem to (11), we obtain

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(12) $\quad \lim_{t \to 0} S_t/t = \sum_{i \text{ s.t. } c_k > c_i} \int_{\pi^i c_i} -\text{sgn}(i,k) \, H^{ki}[f_k] \, dz^i$.

On the other hand from the first equation of (9) multiplied by 1/t as $t \to 0$, we have

(13) $\quad \lim_{t \to 0} S_t/t$

$= \sum_{i \text{ s.t. } c_k > c_i} \int_{\pi^i c_i} \lim_{t \to 0} \frac{1}{t} \int_{\pi^k c_k} f_k(x^k) \, p_t^{ki}(x^k, z^i) \, dx^k dz^i$

Comparing (12) with (13), we have

(14) $\quad - \text{sgn}(i,k) \, H^{ki}[f_k]$

$= \lim_{t - 0} (1/t) \int_{\pi^k c_k} f_k(x^k) \, p_t^{ki}(x^k, z^i) \, dx^k$,

for $c_k > c_i$. From (4), (6), (8), and (14) we obtain the forward equations in the theorem. Note that since the n-cell and the 0-cells have no upper cells and lower cells respectively, we can obtain the equations putting the related functions to be 0. Finally, since the boundaries act as absorbing boundaries for the diffusion process on c_k, $(k=0,\dots,3^{n-1})$ until the next jump occurs, we obtain the boundary condition in the theorem. This completes the proof.
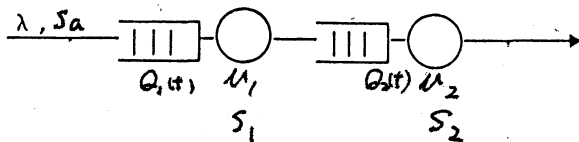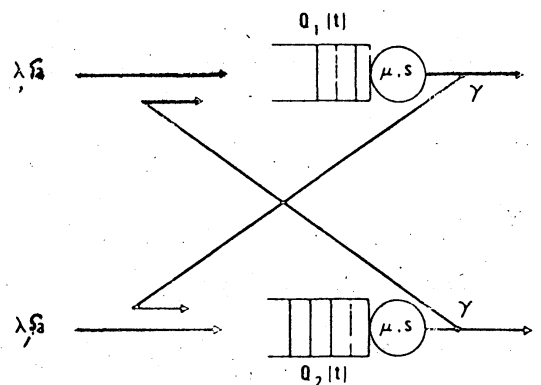

FIG. 1.

Fig. 2

4. Approximation for some queueing processes

We consider in this section some examples of queues which can be modeled with the two-dimensional elementary return process. Although we restrict our attention to the two-dimensional case for simplicity, most of techniques we use will be extended to higher dimensions.

From Theorem 1, the partial differential equations of the two-dimensional elementary return process on R_+^2 is written as follows;

$$\text{div } J^0(x) = \sum_{i=1}^{2} \lambda_{i0} f_i(x_{3-i}) \; \delta(x_i-1)$$

$$(15) \qquad (\partial/\partial x_{3-i}) J^i(x_{3-i}) = \lim_{x_i \to 0} -J_i^0(x) - \lambda_{i0} f_i(x_{3-i})$$

$$+ \lambda_{3i} f_3 \; \delta(x_{3-i}-1) , \qquad (i=1,2)$$

$$0 = \lim_{x_1 \to 0} -J^2(x_1) + \lim_{x_2 \to 0} -J^1(x_2) - (\lambda_{31} + \lambda_{32}) f_3 ,$$

$$\lim_{x_i \to 0} f_0(x_1, x_2) = 0 , \quad (i=1,2)$$

$$\lim_{x_i \to 0} f_i(x_{3-i}) = 0 , \quad (i=1,2)$$

where f_3 is the probability mass at the origin,

$$J_i^0(x) = m_i^0 f_0 - \frac{1}{2} \sum_{j=1}^{2} v_{ij}^0 \frac{\partial f_0}{\partial x_j}$$

$$J^0 = (J_i^0)$$

and

$$J^i(x_{3-i}) = m^i f_i - \frac{1}{2} v^i \frac{\partial f_i}{\partial x_{3-i}} , \qquad (i=1,2).$$

We determine the coefficients of (15) based on the fact that the diffusion process approximates the number in queue as soon as the number in queue moves from zero to one.

Example 1. First we consider the tandem queues of Figure 1 which is composed of two single server stations arranged in tandem. Custmers arrive at station 1 according to a renewal process whose interarrival times have mean $1/\lambda$ and variance S_a. Having completed service there, they proceed to station 2, and after completing service at station 2 they leave the system. Each station has a single server, and service times at station i, (i=1,2) are i.i.d. with mean $1/\mu_i$ and variance S_i. Let $Q_i(t)$ denote the number of customers in station i at time t, and let $Q(t)=(Q_1(t),Q_2(t))$. The coefficients of (15) are determined as follows.

(i) As soon as a busy period $(Q_1(t)>0,\ Q_2(t)>0)$ starts, the process $Q(t)$ approximately behaves as the ordinary two-dimensional diffusion process with the coefficients determined by "central-limt-theorem"-type argument [10]. Thus from [10]

$$m^0 = \begin{pmatrix} \lambda - \mu_1 \\ \mu_1 - \mu_2 \end{pmatrix}$$

$$v^0 = \begin{pmatrix} \lambda^3 S_a + \mu_1^3 S_1 & -\mu_1^3 S_1 \\ -\mu_1^3 S_1 & \mu_1^3 S_1 + \mu_2^3 S_2 \end{pmatrix} .$$

(ii) On the half-line $Q_1=0(>0),\ Q_2>0(=0)$, $Q_2(t)$ $(Q_1(t))$ behaves approximately as the one-dimensional diffusion process with infinitesimal mean $m^1=-\mu_2$ $(m^2=\lambda-\mu_1)$ and infinitesimal variance $v^1=\mu_2^3 S_2 (v^2=\lambda^3 S_a+\mu_1^3 S_1)$. The sojourn time on this half-line

behaves approximately as the Markovian arrival with the rate λ (μ_1). Thus we set $\lambda_{10}=\lambda$ and $\lambda_{20}=\mu_1$.

(iii) At the origin $Q_1=0$, $Q_2=0$, $Q_1(t)$ approximately behaves as the Markovian arrival with rate λ . Thus we can set $\lambda_{32}=\lambda$. Since there is no arrival at station 2 when $Q_1=0$, we set $\lambda_{31}=0$.

Example 2.   The second example is the pairs of identical computers in reciprocal communication (see Figure 2). This model was discussed by Foshini [5] under the heavy traffic condition. He solved the mean queue length applying the two-dimensional reflected diffusion process [8]. We will also discuss the mean queue length of this model in the next section.

The mean rate and the variance of exogenous interarrivals of jobs are denoted by λ and S_a respectively. Service times at each station are i.i.d. with mean $1/\mu$ and variance S . Having completed service at each station, jobs proceed to other computer for services with probability γ and with probability $1-\gamma$ jobs leaves the system. Using the same discussion in Example 1, the parameters of this model become as follows;

$$m_1^0 = m_2^0 = \lambda - \mu \ (1-\gamma)$$

$$v_{11}^0 = v_{22}^0 = \lambda^3 S_a + \mu^3 s + \mu \gamma \ (1-\gamma+\gamma \mu^2 s)$$

$$v_{12}^0 = v_{21}^0 = -2\mu^3 s \gamma$$

$$m^1 = m^2 = \lambda - \mu$$

$$v^1 = v^2 = \lambda^3 S_a + \mu^3 s$$

$$\lambda_{10} = \lambda_{20} = \lambda + \gamma \mu$$

and

$$\lambda_{31} = \lambda_{32} = \lambda \quad .$$

We have discussed thus far only the queueing network models. The multi-dimensional elementary return process is applicable to another kind of queueing systems as described in the next example.

Example 3. Consider the preemptive-resume queueing systems which handle two types of customers. In this system, one type of customers is given a preemptive priority over the other type and served by a single server. Each customers arrive at the queue according to a renewal process whose interarrival times have mean $1/\lambda_i$ and variance a_i, $i=1$ for high priority class and $i=2$ for low priority class. The service times for class i are i.i.d. with mean $1/\gamma_i$ and variance S_i. The same discussion as before yields

$$m^0 = \begin{pmatrix} \lambda_1 - \gamma_1 \\ \lambda_2 \end{pmatrix}$$

$$v^0 = \begin{pmatrix} \lambda_1^3 a_1 + \gamma_1^3 s_1 & 0 \\ 0 & \lambda_2^3 a_2 \end{pmatrix}$$

$$m^1 = \lambda_1 - \gamma_1$$

$$v^1 = \lambda_1^3 a_1 + {}_1^3 s_1$$

$$m^2 = \lambda_2 - {}_2$$

$$v^2 = \lambda_2^3 a_2 + {}_2^3 s_2$$

$$\lambda_{32} = \lambda_{10} = \lambda_2$$

and

$$\lambda_{31} = \lambda_{20} = \lambda_1 .$$

Notice that since the changes in the number of the class 1 customers is completely independent of the class 2 customers, the covariance between these two classes is zero.

Finally we remark the discretization of the density. A direct method (see [6]) is given by setting

$$P(0,0) = f_3$$

$$P(i,0) = \int_{i-1}^{i} f_1(x)\, dx \qquad , \ i>0$$

$$P(0,j) = \int_{j-1}^{j} f_2(x)\, dx \qquad , \ j>0$$

and

$$P(i,j) = \int_{i-1}^{i} \int_{j-1}^{j} f_0(x,y)\, dx\, dy \quad , \ i,j>0$$

where $P(i,j)$ is the approximate statistical-equalibrium probability that $Q_1=i$ and $Q_2=j$ customers simultaneously exist in the queue. This discretization, however has the disadvantage of failing to model the probability that one customer exists in the queue. This disadvantage is caused by the jump that reduces the occupation time of the section between $x_i=0$ and 1, $i=1,2$.

The alternative method of discretization which overcomes this disadvantage is given by

$$P(0,0) = f_3$$
$$P(i,0) = K_1 f_1(i) \quad , \quad i>0$$
$$P(0,j) = K_2 f_2(j) \quad , \quad j>0$$

and

$$P(i,j) = K_0 f_0(i,j) \quad , \quad i,j>0$$

where

$$K_1 = \int_0^\infty f_1(x)\, dx / \sum_{i=1}^\infty f_1(i)$$

$$K_2 = \int_0^\infty f_2(x)\, dx / \sum_{j=1}^\infty f_2(j)$$

and

$$K_0 = \int_0^\infty \int_0^\infty f_0(x,y) \, dx \, dy / \sum_{i,j=1}^\infty f_0(i,j) \quad .$$

This method is based on the concept of local occupation time. The jump does not have the effect any more to reduce the probability of one customer in each queue.

5 Formulas for mean

In this section we discuss the formulas for the equalibrium mean of the symmetric pairs of communicating computers. From Example 2 of section 4, the partial differential equations with the boundary conditions are rewritten as follows:

$$(16) \quad \tfrac{1}{2} A \frac{\partial^2 f_0}{\partial x_1^2} + B \frac{\partial^2 f_0}{\partial x_1 \partial x_2} + \tfrac{1}{2} A \frac{\partial^2 f_0}{\partial x_2^2} - C \frac{\partial f_0}{\partial x_1} - C \frac{\partial f_0}{\partial x_2}$$

$$= - \beta f_1(x_1) \, \delta(x_2 - 1) - \beta f_2(x_2) \, \delta(x_1 - 1)$$

$$(17) \quad -\tfrac{1}{2} D \frac{\partial^2 f_i}{\partial x_i^2} + E \frac{\partial f_i}{\partial x_i} = -H_{3-i}(x_i) - \gamma f_i(x_i) + \lambda f_3 \, \delta(x_i - 1)$$

$$(i = 1, 2)$$

$$(18) \quad K_1 + K_2 + 2 \lambda f_3 = 0$$

$$f_0(x_1, 0) = f_0(0, x_2) = 0$$

$$f_1(0) = f_2(0) = 0$$

where

$$H_{3-i}(x_i) = \lim_{x_{3-i} \to 0} \quad -\tfrac{1}{2} B \frac{\partial f_0}{\partial x_i} - \tfrac{1}{2} A \frac{\partial f_0}{\partial x_{3-i}} + C f_0$$

$$K_i = \lim_{x_i \to 0} -\tfrac{1}{2} D \frac{\partial f_i}{\partial x_i} + E f_i \quad , \qquad (i = 1, 2)$$

and where

$$A = \lambda^3 s_a + \mu^3 s + \mu \gamma (1 - \gamma + \gamma \mu^2 s)$$

$$B = -2 \mu^3 s \gamma$$

$$C = \lambda - \mu (1 - \gamma)$$

$$D = \lambda^3 s_a + \mu^3 s$$

$$E = \lambda - \mu$$

$$\beta = \lambda + \gamma \mu$$

and λ, s_a, μ, s, γ which are established in Example 2 are maintained.

Let define the Laplace transformation

$$L_0(s_1, s_2) = \int_0^\infty \int_0^\infty f_0(x_1, x_2) \exp(-s_1 x_1 - s_2 x_2) \, dx_1 \, dx_2$$

$$L_i(s_{3-i}) = \int_0^\infty f_i(x_{3-i}) \exp(-s_{3-i} x_{3-i}) \, dx_{3-i}$$
$$(i=1,2)$$

and

$$F_i(s_{3-i}) = \int_0^\infty H_i(x_{3-i}) \exp(-s_{3-i} x_{3-i}) \, dx_{3-i} \quad .$$
$$(i=1,2)$$

Taking the two-dimensional Laplace transformation of (16), we have

(19) $\quad [\frac{1}{2} A s_1^2 + B s_1 s_2 + \frac{1}{2} A s_2^2 - C s_1 - C s_2] L_0(s_1, s_2)$

$$+ F_1(s_2) + F_2(s_1)$$

$$= - \beta L_1(s_2) \exp(-s_1) - \beta L_2(s_1) \exp(-s_2) \quad .$$

Furthermore Laplace transformation of (17) and use of (18) yield

(20) $\quad F_i(s_{3-i}) = [\frac{1}{2} B s_{3-i}^2 - C s_{3-i} - \beta] L_i(s_{3-i})$

$$- \lambda f_3(1 - \exp(-s_{3-i})) \quad .$$

Let the power series expansion for L_0, L_i, $i=1,2$ be

$$L_0(s_1, s_2) = a_0 + a_1 s_1 + a_1 s_2 + a_2 s_1 s_2 + \cdots$$

and

$$L_2(s) = L_1(s) = b_0 + b_1 s + \ldots\ldots \quad .$$

Substitute (20) into (19) and the power series expansion for L_0, L_1, L_2 into the equation thus derived to get the coefficients of the s_1^2, $s_1 s_2$, and s_1. Then we obtain

$$\frac{1}{2} A a_0 - C a_1 + \frac{1}{2} D b_0 - E b_1 + \lambda f_3/2 = -\frac{\beta}{2} b_0$$

(21)

$$B a_0 - 2 C a_1 = 2 \beta b_1$$

and

(22) $\qquad -C a_0 - E b_0 - \lambda f_3 = \beta b_0 \quad .$

Notice that

$$a_0 = L_0(0,0) = \int_0^\infty \int_0^\infty f_0(x_1, x_2) \, dx_1 \, dx_2$$

$$b_0 = L_1(0) = L_2(0) = \int_0^\infty f_1(x_2) \, dx_2$$

and thus

(23) $\qquad a_0 + 2 b_0 + f_3 = 1 \quad .$

Let the stationary probability that the process sojourns on $x_i = 0, i = 1, 2$ be

(24) $\qquad 1 - \rho = b_0 + f_3.$

Then we have from (22), (23), and (24)

(25) $\quad \lambda (1 - \rho) + C \rho = b_0(C + \lambda - E - \beta)$.

The right hand side of (25) vanishes and we obtain

(26) $\qquad \rho = \lambda / \mu (1 - \gamma) \quad .$

This coincides the traffic intensity at each queue [2]. Although our approach is heuristic, we have had the precise modeling of

empty queue probability as well as the one-dimensional case [6].

Notice that the mean $E[x_1]=E[x_2]$ is given by

$$E[x_1] = \int_0^\infty x_1 \left[\int_0^\infty f_0(x_1,x_2)\, dx_2 + f_2(x_1) \right] dx_1$$

$$= -\frac{\partial L_0}{\partial s_1}(0,0) - \frac{\partial L_2}{\partial s_1}(0) = -a_1 - b_1 \quad .$$

We have thus proved the following result.

Theorem 2. Suppose that the equations (16), (17), and (18) have astationary solution. Then the mean $E[x_i]$ satisfies

$$(27) \qquad E[x_i] = \frac{-1}{2\beta C - 2CE} [(\beta - C)\{A(2\rho + m - 1) + (D + \beta)(1$$

$$-\rho - m) + \lambda m\} - (E\beta - BC)(2\rho + m - 1)]$$

where m represents the probability mass at the origin.

Theorem 2 states that if the steady state probability of empty queues is known, then the approximate mean queue length can be given by (27). Since the approximation is made assuming the Markovian arrival at the empty queues, it is reasonable selection to set

$$(28) \qquad m = (1 - \rho)^2 ,$$

the equality being due to Jackson network result.

Assuming (28), we have

$$(29) \qquad E[x_i] = [\gamma\rho^2(1-\gamma-\gamma\mu^2 s) + (\rho + \gamma)(1 - \rho)\rho$$

$$+ \rho(\lambda^2\rho s_a + \mu^2 s) + \rho(1 - \rho)^2$$

$$- \rho^2\{(\rho-1)(\rho + \gamma) + 2\mu s\gamma(\lambda - \mu(1 - \gamma)\}] /$$

$$2(\gamma + 1)(1-\rho - \gamma)$$

For uncoupled M/G/1 system, from the Pollaczek-Khinchin formula the mean queue length is

$$\rho \left\{ 1 + \frac{\rho(1 + \mu^2 s)}{2(1 - \rho)} \right\}$$

while (29) becomes

$$E[x_i] = \rho \left\{ \frac{1 + \rho^2}{2} + \frac{\rho + \mu^2 s}{2(1 - \rho)} \right\} \quad .$$

This appears to be a better approximation than

$$\frac{\rho + \mu^2 s}{2(1 - \rho)}$$

which is the result of Foschini [5] derived by means of the two-dimensonal reflected diffusion process approximation [7].

6 Concluding remarks

Computing the density f of our basic differential equation (15) appears to be difficult problem. The most simple form solution can be derived as a product form. It will be easy to show that if (and only if) the covariance coefficients v_{ij}^0, $i \neq j$ are zero and $v_{ii}^0 = v^{3-i}$, $m_i^0 = m^{3-i}$, (i=1,2), $\lambda_2 = \lambda_{10} = \lambda_{32}$, $\lambda_1 = \lambda_{20} = \lambda_{31}$, then there exists the separable exponential form solution:

$$f_{3-i}(x_i)$$
$$= f_3 \exp(G_{3-i} x_i) \frac{2\lambda_{3-i}}{v^{3-i}} \int_0^{x_i} \exp(-G_{3-i} x) \, F(x) \, dx$$

and

$$f_0(x) = \frac{4\lambda_1 \lambda_2}{v^1 v^2} f_3 \exp(G_1 x_1 + G_2 x_2) \prod_{i=1}^{2} \int_0^{x_i} \exp(-G_i x) \, F(x) \, dx$$

where $G_i = 2m^i / v^i$ and

$$F(x) = \begin{cases} -1 , & 0 \leq x \leq 1 \\ 0 , & x > 1 \end{cases}$$

This solution, however, does not give any additional useful

information on the stationary distribution. Although it is difficult to find out even the special solution of (15), it seems worth-while to attach this problem for the purpose of its practical utilization.

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