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OPTIMUM BLOCK PLAN FOR A FRACTIONAL 2^m FACTORIAL DESIGN

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1. Introduction. Consider a 2^m factorial experiment with m factors. An assembly (or treatment combination) is represented by an m-rowed vector $(j_1, j_2, \cdots, j_m)$, where j_t (the level of t-th factor) is equal to 0 or 1. As unknown effects, θ_0, θ_t, and in general, $\theta_{t_1 \cdots t_k}$ denote the general mean, main effects of t-th factor, and k-factor interactions of corresponding factors, respectively. For a fixed integer $\ell\,(1 \leqq \ell \leqq m)$, let $\underline{\theta}$ be the $\nu \times 1$ vector composed of effects up to ℓ-factor interactions, where $\nu = \sum_{i=0}^{\ell}\binom{m}{i}$, i.e.,

$$\underline{\theta}' = (\theta_0; \theta_1, \theta_2, \cdots, \theta_m; \theta_{12}, \cdots, \theta_{m-1\,m}; \cdots; \theta_{12\cdots\ell}, \cdots, \theta_{m-\ell+1\cdots m}).$$

Assume throughout this paper that $(\ell+1)$-factor and higher order interactions are negligible and that the m factors do not involve block factors. Let T be a fractional 2^m factorial (2^m-FF) design which is a suitable set of N assemblies. Note that the assemblies in T are not always distinct. We now consider the estimation of a $\nu_0 \times 1$ vector of linear parametric functions $\underline{\theta}_0 = C\,\underline{\theta}$ with the design T. For an $N \times 1$ observation vector $\underline{y}_T$ of T (whose observations are assumed to be independent random variables with common variance), consider the model

$$E(\underline{y}_T) = E\,\underline{\theta} \tag{1.1}$$

where $\mathcal{E}(\cdot)$ stands for an expected value and E is the $N \times \nu$ design matrix with elements ± 1 (see, e.g., Yamamoto, Shirakura and Kuwada (1975)). The best linear unbiased estimate of $\underline{\theta}_0$ can then be given by

$$(1.2) \qquad \hat{\underline{\theta}}_0 = K E' \underline{y}_T ,$$

where K is a $\nu_0 \times \nu$ matrix satisfying $KM = C$. Here $M = E'E$ is called the information matrix of T. Note that there does not always exist K satisfying $KM = C$ for a given C. However we consider a design T for which there exists such a matrix K, because of the estimability of $\underline{\theta}_0$. When $\underline{\theta}_0 = \underline{\theta}$, i.e., $\nu_0 = \nu$ and $C = I$ (identity matrix of appropriate order), T corresponds to a 2^m-FF design of resolution $2\ell+1$. On the other hand, when $\underline{\theta}_0 = (\theta_1, \cdots, \theta_m; \cdots ; \theta_{12\cdots\ell-1}, \cdots, \theta_{m-\ell+2\cdots m})'$, ($\nu_0 = \nu - 1 - \binom{m}{\ell}$), T corresponds to a 2^m-FF design of resolution 2ℓ (see Box and Hunter (1961)).

In view of (1.1) and (1.2), N homogenuous observations are necessary in order to estimate $\underline{\theta}_0$. After planning a design T, however, it may occur that the N observations can not be obtained simultaneously by physical, chemical and/or ecomomical reasons, etc. In this case, it is required to divide T into some blocks. Of course, the number of such blocks (say k) must be small compared to N. The problem is to constitute the k blocks such that they make an influence on the estimate $\hat{\underline{\theta}}_0$ as small as possible. As measures for constituting such blocks, we use well-known three norms of a confounding matrix of T. In Section 2, we introduce the three norms of a confounding matrix for r block effects and present a procedure for the constitution of at most $k = 2^r$ blocks for a design T. Furthermore, some properties of these norms are discussed for a 2^m-FF design of resolution 2ℓ or $2\ell+1$. In particular, it is shown that for any 2^m-FF design of even resolution, there exist at most 2^r blocks for which the three norms are zero. A design T whose information matrix M is of the form $M = n_1 I + n_2 G$, where $N = n_1 + n_2$ and $n_2 \geqq 0$, has various desirable properties (see Bhaskararao (1966), Raghavarao (1959, 1960, 1971), and Cheng (1980), etc.). In the above, G denotes a matrix of appropriate order whose elements are all one. For the constitution of 2 ($r = 1$) blocks for such a design T, Section 3 gives a lower bound on one norm which may be preferred over the other two norms. Some examples in which this bound is attained are also given. Section 4 deals with the constitution of 2 blocks for a balanced fractional 2^m factorial (2^m-BFF) design of resolution $2\ell+1$. It may be noted that the above type design is a special case of this design. A 2^m-BFF design of resolution V ($\ell = 2$) was first discussed by Srivastava (1970). Yamamoto. Shirakura and Kuwada (1975, 1976) have generalized to a design of resolution $2\ell+1$. Some properties of the preferable norm are presented for a 2^m-BFF design of resolution $2\ell+1$. Also, the norm are slightly modified. Indeed, the constitutions of 2 blocks minimizing the modified measure in some class are presented for A-optimal 2^m-BFF designs of resolution V for $m = 4$ and for the values of N satisfying $11 \leqq N \leqq 26$.

The results contained in this paper are also useful in the case where one does not know in advance whether block factors are exactly needed. Usually, the addition of new factors (nuisance factors) requires a larger number of assemblies and makes it difficult to plan a design in which the effects $\underline{\theta}_0$ is unconfounded with nuisance parameters.

2. Constitution of a confounding block plan. Let T be a 2^m-FF design with N assemblies whose α-th assembly is given by $(j_1^{(\alpha)}, j_2^{(\alpha)}, \cdots, j_m^{(\alpha)})$ for $\alpha = 1, \cdots, N$. For $2^r < N$, consider another 2^r design

D with r factors and N assemblies whose α-th assembly is given by $(d_1^{(\alpha)}, d_2^{(\alpha)}, \cdots, d_r^{(\alpha)})$ for $\alpha = 1, \cdots, N$. Suppose the number of distinct assemblies in D is k with $2^{r-1} < k \leq 2^r$. Further let [T:D] be a set of N assemblies obtained by juxtaposing T and D such that its α-th assembly is $(j_1^{(\alpha)}, \cdots, j_m^{(\alpha)}; d_1^{(\alpha)}, \cdots, d_r^{(\alpha)})$ for $\alpha = 1, \cdots, N$. The set [T ; D] can be considered as a 2^{m+r}-FF design with m+r factors. However, the r factors play here the role of block factors and D gives a block plan for T. That is, k blocks $B_{(d_1 \ldots d_r)}$ for T are constituted in a way such that the α-th assembly of [T : D] is $(j_1^{(\alpha)}, \cdots, j_m^{(\alpha)}; d_1^{(\alpha)}, \cdots, d_r^{(\alpha)})$ if and only if the α-th assembly of T belongs to $B_{(d_1^{(\alpha)} \ldots d_r^{(\alpha)})}$. Consider now the $N \times 1$ observation vector $\underline{y}_{[T:D]}$ of [T : D]. According to model (1.1), we then have the following model:

$$\mathcal{E}(\underline{y}_{[T:D]}) = E\underline{\theta} + X(D)\underline{\eta} , \tag{2.1}$$

where $\underline{\eta}' = (\eta_1, \cdots, \eta_r)$, ($\eta_\beta$ is the effect of β-th block factor), and X(D) is the $N \times r$ design matrix for $\underline{\eta}$ of D with elements ±1. Note that the level of j-th factor of the α-th assembly in D is 0 and 1 if and only if the (α, j) element of X(D) is -1 and 1, respectively. We still insist on using $\hat{\underline{\theta}}_0$ in (1.2) for the estimation of $\underline{\theta}_0$. Under model (2.1), the expected value of $\hat{\underline{\theta}}_0$, ($\underline{y}_T$ in $\hat{\underline{\theta}}_0$ is replaced with $y_{[T:D]}$), becomes

$$\mathcal{E}(\hat{\underline{\theta}}_0) = \underline{\theta}_0 + A(D)\underline{\eta}, \tag{2.2}$$

where $A(D) = KE'X(D)$ is said to be a confounding matrix of D. As measures how $\mathcal{E}(\hat{\underline{\theta}}_0)$ in (2.2) can be close to $\underline{\theta}_0$, the following three norms of A(D) may be considered:

$$\begin{aligned} &\text{(i)} \quad \|A(D)\|_1 = \max_{1\leq i\leq \nu_0} \{\textstyle\sum_{j=1}^{r} |a_{ij}|\} , \\ &\text{(ii)} \quad \|A(D)\|_2 = \max_{1\leq j\leq r} \{\textstyle\sum_{i=1}^{\nu_0} |a_{ij}|\} , \\ &\text{(iii)} \quad \|A(D)\|_3 = \{tr(A(D)'A(D))\}^{\frac{1}{2}} = \{\textstyle\sum_{i=1}^{\nu_0}\sum_{j=1}^{r} a_{ij}^2\}^{\frac{1}{2}}, \end{aligned} \tag{2.3}$$

where $a_{i,j}$ are the (i, j) elements of A(D). Let D_k^N denote the collection of all possible D with N assemblies, containing k distinct ones ($2^{r-1} < k \leqq 2^r$). For a given T, D in D_k^N is said to be a confounding block (CB) plan of order k when D is selected in a consideration of the norms $\|A(D)\|_i$, (i = 1,2,3). In particular, D in D_k^N is said to be an optimum confounding block (OCB) plan of order k for T with respect to the norm $\|A(D)\|_i$ if $\|A(D)\|_i$ is a minimum in the calss D_k^N for each i = 1,2,3. Also, the set $B_{T,k}$ of the corresponding k blocks $B_{(d_1 \ldots d_r)}$ is said to be an optimum block of order k for T. This idea is due to Hedayat, Raktoe and Federer (1974), in which they have given the norm $\|A\| = (\mathrm{tr}\, A'A)^{\frac{1}{2}}$ of alias matrix A as a measure in selecting a design.

2^m-FF designs of resolution 2ℓ or $2\ell+1$ are particularly important for our practical uses. Therefore we shall discuss some properties of the norms $\|A(D)\|_i$, (i = 1,2,3) for these designs.

Theorem 2.1. Let T be a 2^m-FF design of resolution 2ℓ. Suppose an $N \times r$ submatrix of E whose r columns correspond to r ℓ-factor interactions has k distinct rows ($2^{r-1} < k \leqq 2^r$). Then there exists an OCB plan D of order k for T such that $\|A(D)\|_i = 0$, (i = 1,2,3), i.e., A(D) is a zero matrix.

Proof. Denote the submatrix by Y. Assume that the r ℓ-factor interactions are at the s_1-th, s_2-th, $\cdots$, s_r-th positions of $\underset{\sim}{\theta}$ and let Z be the $\nu \times r$ matrix in which the i-th element of s_i-th row is 1 and 0 elsewhere. Then $Y = EZ$ holds. Put $X(D) = Y$. It is observed that $A(D) = KE'X(D) = KMZ = CZ = 0$. This completes the proof.

Theorem 2.2. Let T be a 2^m-FF design of resolution $2\ell+1$. Suppose that there exists the same submatrix as in Theorem 2.1. Then, there exists a CB plan D of order k for T such that $\|A(D)\|_i = 1$, $(i = 1,2)$, and $\|A(D)\|_3 = \sqrt{r}$. Moreover, in the plan D, every effect up to $(\ell-1)$-factor interactions is unconfounded with block effects $\underline{\eta}$.

Proof. Consider the same matrices Y and Z as in Proof of Theorem 2.1. Put $X(D) = Y$. Then, since $C = I$ and $K = M^{-1}$, we have $A(D) = Z$, which completes the proof.

Let Ω^m be the set of 2^m all distinct m-rowed vectors with elements 0 or 1. Consider the design $T_{\underline{\omega}}$ obtained from a design T by a level transformation $(j_1, \cdots, j_m) + \underline{\omega} = (j_1 + \omega_1, \cdots, j_m + \omega_m)$, (mod 2), where $(j_1, \cdots, j_m) \in T$ and $\underline{\omega} = (\omega_1, \cdots, \omega_m) \in \Omega^m$. (In particular, $T_{(0,\cdots,0)} = T$.) Then we have

Theorem 2.3. Let T be a 2^m-FF design of resolution 2ℓ or $2\ell+1$. Suppose D is a CB plan for T. Then for $\underline{\omega} \in \Omega^m$,

$$(2.4) \qquad \|A_T(D)\|_i = \|A_{T_{\underline{\omega}}}(D)\|_i \quad , \; i = 1,2,3$$

hold where $A_{T_{\underline{\omega}}}$ is the confounding matrix for $T_{\underline{\omega}}$.

Proof. Let $E_{T_{\underline{\omega}}}(D)$ and $M_{T_{\underline{\omega}}}(D)$ be respectively the design matrix and information matrix of $T_{\underline{\omega}}$. Then it can be shown that there exists the $\nu \times \nu$ diagonal matrix

$$P_{\underline{\omega}} = \mathrm{diag}(1;\ (-1)^{\omega_1}, \cdots, (-1)^{\omega_m};\ (-1)^{\omega_1+\omega_2}, \cdots, (-1)^{\omega_{m-1}+\omega_m}; \cdots ;\ (-1)^{\omega_1+\cdots+\omega_\ell}, \ldots, (-1)^{\omega_{m-\ell+1}+\cdots+\omega_m}),$$

which satisfies $E_{T_{\underline{\omega}}} = E_T P_{\underline{\omega}}$ (see Srivastava, Raktoe and Pesotan (1976)). Therefore we have $M_{T_{\underline{\omega}}} = P_{\underline{\omega}} M_T P_{\underline{\omega}}$. Also, in this case, there exists a $\nu_0 \times \nu_0$ matrix $Q_{\underline{\omega}}$ satisfying $Q_{\underline{\omega}} C P_{\underline{\omega}} = C$. In fact, put $Q_{\underline{\omega}} = C P_{\underline{\omega}} C'$. We thus have

$$(Q_{\underline{\omega}} K P_{\underline{\omega}}) M_{T_{\underline{\omega}}} = Q_{\underline{\omega}} K P_{\underline{\omega}} P_{\underline{\omega}} M_T P_{\underline{\omega}} = Q_{\underline{\omega}} K M_T P_{\underline{\omega}} = Q_{\underline{\omega}} C P_{\underline{\omega}} = C.$$

This means that if T is a 2^m-FF design of resolution 2ℓ (or $2\ell+1$), then T_ω is also of resoution 2ℓ (or $2\ell+1$). Therefore, $A_{T_\omega}(D) = (Q_\omega K P_\omega) E'_{T_\omega} X(D) = Q_\omega A_T(D)$ holds. Since Q_ω is a diagonal matrix such that the nonzero elements are ± 1, we have (2.4) from (2.3).

As the first problem, it is interesting that (optimum) CB plans of order 2 (i.e., $r = 1$) are constituted for designs. (In this case, note that $A(D)$ and $X(D)$ are column vectors.) In view of Theorem 2.1, we can always constitute an OCB plan of order 2 for a 2^m-FF design of even resolution. The following theorem implies that for a 2^m-FF design of odd resolution, we can obtain a CB plan better than one in Theorem 2.2 with respect to the norm $\| A(D) \|_1$:

Theorem 2.4. Let $\ell \leqq m-2$. Then there exists a CB plan D of order 2 for a 2^m-FF design of resoluton $2\ell+1$ such that $\| A(D) \|_1 = \frac{1}{2}$ holds. Moreover, in this plan D, any effects up to $(\ell-1)$-factor interactions are unconfounded with block effects $\underline{\eta}$.

Proof. Since $\ell \leqq m-2$, there exist two main effects θ_t and $\theta_{t'}$ such that $t, t' \notin \{t_1, \cdots, t_{\ell-1}, t'_1, \cdots, t'_{\ell-1}\}$ for two $(\ell-1)$-factor interactions $\theta_{t_1 \cdots t_{\ell-1}}$ and $\theta_{t'_1 \cdots t'_{\ell-1}}$. Denote a column vector of E corresponding to an effect θ_z in $\underline{\theta}$ by $\underline{e}_z$. Then consider the $N \times 1$ vector $\underline{x}$ as follows:

$$\underline{x} = \tfrac{1}{2}\{\underline{e}_t * (\underline{e}_{t_1 \cdots t_{\ell-1}} + \underline{e}_{t'_1 \cdots t'_{\ell-1}}) + \underline{e}_{t'} * (\underline{e}_{t_1 \cdots t_{\ell-1}} - \underline{e}_{t'_1 \cdots t'_{\ell-1}})\},$$

where $*$ denotes the product operation defined by $(a_1, a_2, \cdots, a_n)' * (b_1, b_2, \cdots, b_n)' = (a_1 b_1, a_2 b_2, \cdots, a_n b_n)'$. Then it is easy to verify that $\underline{x}$ is a vector with elements 1 or -1. From the property of design matrix E, $\underline{x}$ can also be written by

$$\underline{x} = \tfrac{1}{2}(\underline{e}_{t t_1 \cdots t_{\ell-1}} + \underline{e}_{t t'_1 \cdots t'_{\ell-1}} + \underline{e}_{t' t_1 \cdots t_{\ell-1}} - \underline{e}_{t' t'_1 \cdots t'_{\ell-1}}).$$

This means that $\underline{x}$ can be expressed as a linear combination of columns of E corresponding to the four ℓ-factor interactions $\theta t\, t_1 \cdots t_{\ell-1}$, $\theta t\, t'_1 \cdots t'_{\ell-1}$, $\theta t'\, t_1 \cdots t_{\ell-1}$ and $\theta t' t'_1 \cdots t'_{\ell-1}$. Suppose now $\underline{z}$ is a $\nu \times 1$ vector with elements 0, 1 or -1 obtained from $\underline{\theta}$ by replacing the above first three ℓ-factor interactions with 1, the last one with -1, and the other effects with 0. Further put $X(D) = \underline{x}$. Then we have $X(D) = \frac{1}{2} E \underline{z}$. Therefore, $A(D) = M^{-1} E' X(D) = \frac{1}{2} \underline{z}$. Hence $\|A(D)\|_1 = \frac{1}{2}$, which completes the proof.

Note that $\|A(D)\|_2 = 2$ and $\|A(D)\|_3 = 1$ for the plan D in Theorem 2.4. However, as compared to the norms (criteria) $\|A(D)\|_2$ and $\|A(D)\|_3$, the criterion $\|A(D)\|_1$ directly acts to reduce the bias for each estimate in $\hat{\underline{\theta}}_0$. In a sense, $\|A(D)\|_1$ may be preferred over $\|A(D)\|_2$ and $\|A(D)\|_3$. We henceforth investigate an OCB plan of order 2 for a 2^m-FF design of odd resoluiton with respect to $\|A(D)\|_1$. For simplicity, we write $\|A(D)\|$ for $\|A(D)\|_1$.

3. OCB plan of order 2 for some design. In this section, an optimum block $B_{T,2}$ with respect to $\|A(D)\|$ is characterized for a 2^m-FF design of resolution $2\ell+1$ whose information matrix is of the form $M = n_1 I + n_2 G$, where $N = n_1 + n_2$ and $n_2 \geqq 0$.

The design T is called an orthogonal 2^m-FF (OFF) design of resolution $2\ell+1$ if $M = n_1 I$, ($n_2 = 0$ in the above). Such a design has various desirable properties. It is well known that the design is optimal with respect to most established criteria (e.g., A-, D- and E-criteria due to Kiefer (1959)). It is cear that minimizing of $\|A(D)\|$ is equivalent to that of $\|E' X(D)\|$. We have $\|E' X(D)\| \geqq 2s$, (s is a nonnegative integer), in the design, since N and therefore, $\|E' X(D)\|$ are even.

Example 1. Consider

$$T = \{ \Omega(5,0),\ \Omega(5,2),\ \Omega(5,4) \} ,$$

where $\Omega(m, h)$ is the set of $\binom{m}{h}$ distinct m-rowed vectors in Ω^m with weight h (number of 1's). Then, it is easy to see that T is a 2^5-OFF design of resolution V with $N = n_1 = 16$ aseemblies. Let

$$B_{(0)} = \{\Omega(5,0),\ \Omega(5,4)\},\quad B_{(1)} = \{\Omega(5,2)\} .$$

be two blocks for T. For the corresponding plan D, we then have $\|E' X(D)\| = 4$ and therefore, $\|A(D)\| = \frac{1}{4}$. In fact, this design D turns out to be an OCB plan of order 2 for T. Thus $B_{T,2} = \{B_{(0)}, B_{(1)}\}$ is an optimum block.

Consider now a design T with $n_2 > 0$. This design can be obtained by adding n_2 assemblies $(1, 1, \cdots, 1)$ to the above orthogonal design. Recently, Cheng (1980) has shown that the design with $n_2 = 1$ is optimal with respect to a generalized type criterion (defined by the same author (1978)).

The following lemma can easily be proved:

Lemma 3.1. For $M = n_1 I + n_2 G$, $(n_2 > 0)$,

$$M^{-1} = v_1 I - v_2 (G - I) ,$$

where $v_1 = \{n_1 + (\nu-1)n_2\}/\{n_1(n_1 + \nu n_2)\}$ and $v_2 = n_2/\{n_1(n_1 + \nu n_2)\}$.

Let $\underline{b} = E' X(D)$ and assume there does not exist a plan D in D_2^N such that $\underline{b} = \underline{0}$ ($\underline{0}$ is a zero vector of appropriate order). Then we establish:

Theorem 3.2. For a design T such that $M = n_1 I + n_2 G$, $(n_2 > 0)$, and for a plan D in D_2^N,

$$\|A(D)\| \geq \frac{1}{n_1 + \nu n_2} \tag{3.1}$$

holds with equality if and only if $\underline{b}' = \pm(1, 1, \cdots, 1)$.

Proof. Let $\underline{b}' = (b_1, b_2, \cdots, b_\nu)$ and $b = \sum_{i=1}^{\nu} b_i$. Further suppose the b_i's are arrenged in ascending order of magnitude such that $b_{(1)} \leqq b_{(2)} \leqq \cdots \leqq b_{(\nu)}$. Then

$$(3.2) \qquad \|A(D)\| = \|M^{-1}\underline{b}\| = \max_{1\leqq i\leqq \nu} |(v_1+v_2)b_i - b v_2| .$$

where v_1 and v_2 are in Lemma 3.1. We divide two cases $b > 0$ and $b < 0$.

(i) The case of $b > 0$. In this case, $b_{(\nu)} \geqq 1$, since b_i's are integers. Take $p = \{v_1 - (\nu-1)v_2\}/v_2$ (> 0). Then it is easily seen that

$$(3.3) \qquad (p+\nu)b_{(\nu)} - b - p = p(b_{(\nu)} - 1) + \nu b_{(\nu)} - b \geqq 0.$$

Therefore,

$$(v_1+v_2)b_{(\nu)} - b v_2 \geqq v_2 p = \frac{1}{n_1 + \nu n_2} > 0$$

holds. Hence from (3.2) we have

$$(3.4) \qquad \|A(D)\| \geqq (v_1+v_2)b_{(\nu)} - b v_2 \geqq \frac{1}{n_1 + \nu n_2} .$$

The equality in (3.3) holds if and only if $b_{(\nu)} = 1$ and $b = \nu$. This means $\underline{b}' = (1, 1, \cdots, 1)$. It follows from (3.4) that the equality of (3.1) holds if and only if $\underline{b}' = (1, 1, \cdots, 1)$.

(ii) The case of $b < 0$. In this case, $b_{(1)} \leqq -1$. By an argument similar to the case (i), it can be shown that

$$\|A(D)\| \geqq -\{(v_1+v_2)b_{(1)} - b v_2\} \geqq v_2 p = \frac{1}{n_1 + \nu n_2} .$$

Also, it can be shown that the equality holds if and only if $\underline{b}' = -(1, \cdots, 1)$. This completes the proof.

Example 2. Consider

$$T = \{\Omega(5,0),\ \Omega(5,1),\ \Omega(5,2),\ \Omega(5,3),\ \Omega(5,4),\ \Omega(5,5),\ \Omega(5,5)\} .$$

Then, it is easy to see that T is a 2^5-FF design of resolution V with $N = 33$ assemblies such that $M = 32I + G$. That is, $\nu = 16$, $n_1 = 32$ and $n_2 = 1$. Let

$$B_{(0)} = \{\Omega(5,0),\ \Omega(5,2),\ \Omega(5,4),\ \Omega(5,5)\},$$
$$B_{(1)} = \{\Omega(5,1),\ \Omega(5,3),\ \Omega(5,5)\}$$

be two blocks for T. It is easily seen that $\underline{b} = E'X(D) = (1, \cdots, 1)'$ for the corresponding plan D in D_2^N. Since N is odd, $\underline{b}$ is not equal to $\underline{0}$ at all. Therefore, it follws from Theorem 3.2 that $\|A(D)\| = (n_1 + \nu n_2)^{-1} = 1/48$ is a minimum in D_2^N. Hence D is an OCB plan of order 2 for T and $B_{T,2} = \{B_{(0)}, B_{(1)}\}$ is an optimum block.

4. OCB plans of order 2 for balanced designs. CB plans in D_2^N are developed for 2^m-BFF designs of resolution $2\ell+1$. We first define a balanced design. A 2^m-FF design of resolution $2\ell+1$ is called balanced if the covariance matrix $\mathrm{Var}(\hat{\underline{\theta}}) = M^{-1}\sigma^2$ of $\hat{\underline{\theta}} = \hat{\underline{\theta}}_0$ in (1.2) is invariant under any permutation of m factors, i.e., if for two estimates $\hat{\theta}_{t_1 \cdots t_u}$ and $\hat{\theta}_{t'_1 \cdots t'_v}$ in $\hat{\underline{\theta}}$,

$$\mathrm{Var}(\hat{\theta}_{t_1 \cdots t_u}) = \mathrm{Var}(\hat{\theta}_{\tau(t_1 \cdots t_u)}),$$
(4.1)
$$\mathrm{Cov}(\hat{\theta}_{t_1 \cdots t_u}, \hat{\theta}_{t'_1 \cdots t'_v}) = \mathrm{Cov}(\hat{\theta}_{\tau(t_1 \cdots t_u)}, \hat{\theta}_{\tau(t'_1 \cdots t'_v)})$$

where $\mathrm{Var}(\cdot)$ and $\mathrm{Cov}(\cdot,\cdot)$ stand for variance and covariance of estimates, respectively, and τ is an element of the symmetric group $\mathfrak{G} = \{\tau\ ;\ \tau = \begin{pmatrix} 1 & 2 & \cdots & m \\ \tau(1) & \tau(2) & \cdots & \tau(m) \end{pmatrix}\}$ of order m. We here give the practical restriction $2\ell \leq m$. It is known from Srivastava (1970) and Yamamoto, Shirakura and Kuwada (1975, 1976) that a 2^m-BFF design of resolution $2\ell+1$ with N assemblies is equivalent to a balanced array of size N, m constraints, strength 2ℓ and index set $\mathfrak{U} = \{\mu_1, \mu_2, \cdots, \mu_{2\ell}\}$, (simply B-array[$N$, m, 2ℓ, $\mathfrak{U}$]), provided M is nonsingular. For the save of space, see the above papers for the definition of a blanced array. In particular, a B-array[N, m, 2ℓ, $\mathfrak{U}$] is called an orthogonal array of size N, m constraints, strength 2ℓ and index λ if $\lambda = \mu_0 = \mu_1 = \cdots = \mu_{2\ell}$. As a natural consequence, the above orthogonal array is equivalent to a 2^m-OFF design of resolution $2\ell+1$ with $M = 2^{2\ell}\lambda I$ (i.e., $N = n_1 = 2^{2\ell}\lambda$ in Section 3). A design with $M = n_1 I + n_2 G$, $(n_2 > 0)$, in Section 3, is also equivalent to a B-array[$N = n_1 + n_2$, m, 2ℓ, $\mathfrak{U}$] where $\mathfrak{U} = \{\mu_0 = \lambda, \cdots, \mu_{2\ell-1} = \lambda, \mu_{2\ell} = \lambda + n_2\}$ and $n_1 = 2^{2\ell}\lambda$.

Now, for a $\nu \times 1$ vector $\underline{c}$, attach the same subindices as the effects of $\underline{\theta}$ to the elements of $\underline{c}$, i.e.,

$$\underline{c}' = (c_0; c_1, \cdots, c_m; c_{12}, \cdots, c_{m-1m}; \cdots; c_{12\cdots\ell}, \cdots, c_{m-\ell+1\cdots m}).$$

Further, consider a $\nu \times 1$ vector $\tau(\underline{c})$ for $\tau \in \mathfrak{G}$ defined by

$$\tau(\underline{c})' = (c_0; c_{\tau(1)}, \cdots, c_{\tau(m)}; c_{\tau(12)}, \cdots, c_{\tau(m-1m)}; \cdots; c_{\tau(12\cdots\ell)}, \cdots, c_{\tau(m-\ell+1\cdots m)}).$$

Then, we have the following lemma which can easily be proved:

Lemma 4.1. For any $\tau \in \mathfrak{G}$,

$$\underline{c}_1' \tau(\underline{c}_2) = \tau^{-1}(\underline{c}_1)' \underline{c}_2$$

holds where $\underline{c}_1$ and $\underline{c}_2$ are $\nu \times 1$ vectors.

Theorem 4.2. Let T be a 2^m-BFF design of resolution $2\ell+1$ with N assemblies and D be a plan in D_2^N. Then, for any $\tau \in \mathfrak{G}$,

$$\|A(D)\| = \|A^{\tau}(D)\|$$

holds where $A^{\tau}(D) = M^{-1}\tau(E'X(D))$.

Proof. For $\tau_1, \tau_2 \in \mathfrak{G}$, suppose $V(\tau_1(\underline{\theta}), \tau_2(\underline{\theta}))\sigma^2$ is the covariance matrix of $\hat{\underline{\theta}}$ in which the row and column orders correspond to $\tau_1(\underline{\theta})$ and $\tau_2(\underline{\theta})$, respectively. In particular, note that $V(\underline{\theta}, \underline{\theta}) = M^{-1}$ since $\text{Var}(\hat{\underline{\theta}}) = M^{-1}\sigma^2$. It follows from Lemma 4.1 that

$$A^{\tau}(D) = M^{-1}\tau(E'X(D)) = V(\underline{\theta}, \underline{\theta})\,\tau(E'X(D))$$
$$= V(\underline{\theta}, \tau^{-1}(\underline{\theta}))(E'X(D)).$$

Therefore, we have

$$\tau^{-1}(A^{\tau}(D)) = V(\tau^{-1}(\underline{\theta}), \tau^{-1}(\underline{\theta}))(E'X(D)).$$

From (4.1), $V(\underline{\theta}, \underline{\theta}) = V(\tau^{-1}(\underline{\theta}), \tau^{-1}(\underline{\theta}))$ holds. Hence it can be shown that

$$\|A^{\tau}(D)\| = \|\tau^{-1}(A^{\tau}(D))\| = \|M^{-1}E'X(D)\| = \|A(D)\|.$$

A design T is called a simple array with parameters $\lambda_0, \lambda_1, \cdots, \lambda_m$, (simply, S-array$[m; \lambda_0, \lambda_1, \cdots, \lambda_m]$), if the assemblies in T can be obtained by λ_h repetitions of the set $\Omega(m, h)$ for each $h = 0, 1, \cdots, m$, where $\Omega(m, h)$ are given in Example 1 in Section 3. It is easily seen that an S-array$[m; \lambda_0, \cdots, \lambda_m]$ is a B-array$[N, m, 2\ell, \underline{\mu}]$, where

$$N = \sum_{h=0}^{m} \lambda_h \binom{m}{h},$$
$$\mu_i = \sum_{h=0}^{m} \lambda_h \binom{m-2\ell}{h-i}, \quad i = 0, 1, \cdots, 2\ell.$$

As will be seen from Chopra (1975), Chopra and Srivastava (1973), Shirakura (1976, 1977), Srivastava and Chopra (1971, 1974), etc., most of balanced arrays are of simple types for practical values of m and N. Therefore, it is desirable to study CB plans of order 2 for 2^m-BFF designs of resolution $2\ell+1$ which are obtained from simple arrays.

Theorem 4.3. Let T be an S-array$[m; \lambda_0, \cdots, \lambda_m]$ with $N = \sum_{h=0}^{m} \lambda_h \cdot \binom{m}{h}$. Then, for any $D \in \mathcal{D}_2^N$ and any $\tau \in \mathfrak{G}$, there exists D_0 in $\mathcal{D}_2^N$ such that

$$\tau(E' X(D)) = E' X(D_0).$$

Proof. The design matrix E is represented by

$$E = (\underline{e}_0;\ \underline{e}_1, \cdots, \underline{e}_m;\ \underline{e}_{12}, \cdots, \underline{e}_{m-1\,m};\ \cdots;\ \underline{e}_{12\cdots \ell}, \cdots, \underline{e}_{m-\ell+1\cdots m}),$$

where $\underline{e}_z$ is an $N \times 1$ vector corresponding to θ_z in $\underline{\theta}$. Suppose $\tau(E)$ is the $N \times \nu$ matrix obtained from E by exchanging $\underline{e}_z$ with $\underline{e}_{\tau(z)}$. (In particular, $\underline{e}_{\tau(0)} = \underline{e}_0$.) Further define the $N \times m$ matrices

$$E_1 = (\underline{e}_1, \cdots, \underline{e}_m) \quad \text{and} \quad \tau(E_1) = (\underline{e}_{\tau(1)}, \cdots, \underline{e}_{\tau(m)}).$$

Since T is a simple array, both E_1 and $\tau(E_1)$ then include just λ_h matrices of size $\binom{m}{h} \times m$ with elements ± 1 composed of $\binom{m}{h}$ distinct m-rowed vectors in which the numbers of 1's are h for each $h = 0, 1, \cdots, m$. Therefore, there exists a permutation matrix P of order N such that

$$\tau(E_1) = P E_1. \tag{4.2}$$

On the other hand, we have

$$\underline{e}_{\tau(t_1 t_2 \cdots t_u)} = \underline{e}_{\tau(t_1)} * \underline{e}_{\tau(t_2)} * \cdots * \underline{e}_{\tau(t_u)},$$

where $*$ denotes the product operator defined in Theorem 2.4. By (4.2), $\underline{e}_{\tau(t_i)} = P \underline{e}_{t_i}$ hold for $i = 1, \cdots, u$. Thus it can be shown that

$$\begin{aligned} \underline{e}_{\tau(t_1 t_2 \cdots t_u)} &= (P \underline{e}_{t_1}) * \cdots * (P \underline{e}_{t_u}) = P(\underline{e}_{t_1} * \cdots * \underline{e}_{t_u}) \\ &= P \underline{e}_{t_1 \cdots t_u}. \end{aligned}$$

Also, $\underline{e}_0 = P \underline{e}_0$, since the elements of $\underline{e}_0$ are all one. Therefore, the matrix P satisfies $\tau(E) = PE$. This means that

$$\tau(E' X(D)) = \tau(E)' X(D) = E' P' X(D).$$

From the one-to-one correspondence of $X(D)$ and D, there exists D_0

in D_2^N satisfying $X(D_0) = P'X(D)$. This completes the proof.

Theorems 4.2 and 4.3 are useful in determining an OCB plan D for a design T in which $\|A(D)\|$ has a minimum over D_2^N. Suppose $E'X(D) = \underset{\sim}{c} = (c_0; c_1, \cdots, c_m; \cdots, c_{m-\ell+1\cdots m})'$ and consider a subclass $D^{*N}_2 = \{D \in D_2^N;\ c_1 \leqq c_2 \leqq \cdots \leqq c_m\}$. Suppose now D^* is an OCB plan over D_2^N. Let $\tau \in \mathfrak{S}$ be a permutation such that $(c_1^*, \cdots, c_m^*)$ transforms $(c^*_{\tau(1)}, \cdots, c^*_{\tau(m)})$ with $c^*_{\tau(1)} \leqq \cdots \leqq c^*_{\tau(m)}$, where $\underset{\sim}{c}^* = E'X(D^*) = (c_0^*; c_1^*, \cdots, c_m^*; \cdots, c^*_{m-\ell+1\cdots m})'$. Then, by Theorems 4.2 and 4.3, there exists $D^*_0 \in D^{*N}_2$ satisfying $\|A(D^*)\| = \|A^{\tau}(D^*)\| = \|A(D^*_0)\|$. This means that D^*_0 is also an OCB plan over D_2^N. That is, an OCB plan D (w.r.t. $\|A(D)\|$) over D_2^N can be selected in D^{*N}_2.

However, D_2^N includes in general too many plans, which is not practical. We thus consider the reasonable subclass of D_2^N,

$$S_2^N = \{D \in D_2^N;\ w(D) = [\tfrac{N}{2}]\}\ ,$$

which gives a balanced block plan for a design in some sense. In the above, w(D) denotes the number of 1's in D and [x] denotes the greatest integer less than or equal to x. Also, we take no interest in the confouding of the general mean and block effect. Therefore, the following slightly modified norm may be considered:

$$\|A^*(D)\| = \max_{1\leqq i\leqq \nu-1} |a_i^*|\ ,$$

where $A^*(D)$ is the $(\nu-1) \times 1$ vector obtained by removing the first element of $A(D)$ and a_i^*'s are the elements of $A^*(D)$. Our interest now lies in a CB plan such that $\|A^*(D)\|$ is a minimum over S_2^N.

The vector $A^*(D)$ can be rewritten by $A^*(D) = M^*E'X(D)$, where M^* is the $(\nu-1) \times \nu$ matrix obtained by removing the first row of M^{-1}.

By arguments similar to Proofs of Theorems 4.2 and 4.3, we then have the following two theorems:

Theorem 4.4. Consider the design T and plan D of Theorem 4.2. Then, for any $\tau \in \mathfrak{G}$,

$$\|A^*(D)\| = \|A^{*\tau}(D)\| \quad ,$$

where $A^{*\tau}(D) = M^*\tau(E'X(D))$.

Theorem 4.5. Consider the simple array T of Theorem 4.3. Then for any $D \in S_2^N$ and $\tau \in \mathfrak{G}$, there exists $D_0 \in S_2^N$ such that

$$\tau(E'X(D)) = E'X(D_0).$$

From the above theorems, we observe similarly that an OCB plan (w.r.t. $\|A^*(D)\|$) over S_2^N can also be selected in the restricted class

$$S_2^{*N} = \{D \in S_2^N \;;\; D \in D_2^{*N}\} \;.$$

On the other hand, the calculation of $\|A(D)\|$ or $\|A^*(D)\|$ requires the inverse of $\nu \times \nu$ matrix M. However, Yamamoto, Shirakura and Kuwada (1976) have shown that for a 2^m-BFF design of resolution $2\ell+1$, M^{-1} has at most $\binom{\ell+3}{3}$ distinct elements. Furthermore, Shirakura and Kuwada (1976) have given an explicit expression of those elements.

In Tables 1 and 2, optimum blocks $B_{T,2}$ corresponding to OCB plans D (w.r.t. $\|A^*(D)\|$) over S_2^N are listed for A-optimal 2^4-BFF designs T of resolution V (minimizing $\mathrm{tr}\,M^{-1}$) for the values of N with $11 \leqq N \leqq 26$. Such A-optimal balanced designs have been already given by Srivastava and Chopra (1971). Note that a B-array[N, m = 4, 4, $\mathfrak{U} = \{\mu_0, \mu_1, \mu_2, \mu_3, \mu_4\}$] is equivalent to an S-array[m = 4; $\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4$], where $N = \lambda_0 + 4\lambda_1 + 6\lambda_2 + 4\lambda_3 + \lambda_4$ and $\mu_i = \lambda_i$ $(i = 0,1,2,3,4)$. Therefore, 2^4-BFF designs of resolution V can be represented by the sets $\Omega(m = 4, h)$ of assemblies

Table 1

Optimum blocks for 2^4-BFF designs of resolution V

N	$\mu_0\ \mu_1\ \mu_2\ \mu_3\ \mu_4$	T	$B_{(0)}$	$\|A^*(D)\|$
12	1 0 1 1 1	{Ω(0), Ω(2), Ω(3), Ω(4)}	{Ω(2)}	0.25
13	2 0 1 1 1	{Ω(0), Ω(0), Ω(2), Ω(3), Ω(4)}	{Ω(2), Ω(4)}	0.1429
15	1 1 1 1 0	{Ω(0), Ω(1), Ω(2), Ω(3)}	{Ω(1), Ω(3)}	0.2
16	1 1 1 1 1	{Ω(0), Ω(1), Ω(2), Ω(3), Ω(4)}	{Ω(1), Ω(3)}	0
17	2 1 1 1 1	{Ω(0), Ω(0), Ω(1), Ω(2), Ω(3), Ω(4)}	{Ω(0), Ω(1), Ω(3)}	0.0370
18	2 1 1 1 2	{Ω(0), Ω(0), Ω(1), Ω(2), Ω(3), Ω(4), Ω(4)}	{Ω(1), Ω(3), Ω(4)}	0.0833
19	3 1 1 1 2	{Ω(0), Ω(0), Ω(0), Ω(1), Ω(2), Ω(3), Ω(4), Ω(4)}	{Ω(0), Ω(1), Ω(3), Ω(4)}	0.0437
21	1 2 1 1 2	{Ω(0), Ω(1), Ω(1), Ω(2), Ω(3), Ω(4), Ω(4)}	{Ω(1), Ω(2), Ω(4)}	0.1398
22	1 1 2 1 1	{Ω(0), Ω(1), Ω(2), Ω(2), Ω(3), Ω(4)}	{Ω(1), Ω(2), Ω(4)}	0.125
23	2 1 2 1 1	{Ω(0), Ω(0), Ω(1), Ω(2), Ω(2), Ω(3), Ω(4)}	{Ω(0), Ω(0), Ω(2), Ω(3)}	0.0690
24	2 1 2 1 2	{Ω(0), Ω(0), Ω(1), Ω(2), Ω(2), Ω(3), Ω(4), Ω(4)}	{Ω(0), Ω(0), Ω(2), Ω(3)}	0
25	1 2 1 2 2	{Ω(0), Ω(1), Ω(1), Ω(2), Ω(3), Ω(3), Ω(4), Ω(4)}	{Ω(0), Ω(1), Ω(3), Ω(3)}	0.0395
26	2 2 1 2 2	{Ω(0), Ω(0), Ω(1), Ω(1), Ω(2), Ω(3), Ω(3), Ω(4), Ω(4)}	{Ω(1), Ω(1), Ω(3), Ω(4)}	0.0625

Table 2

Optimum blocks for 2^4-BFF designs of resolution V

N	$\mu_0\ \mu_1\ \mu_2\ \mu_3\ \mu_4$	T	$B_{(0)}$	$\|A^*(D)\|$
11	1 0 1 1 0	{Ω(0), Ω(2), Ω(3)}	{Ω(0), (1,0,1,0), (0,1,0,1), (0,0,1,1), (1,0,1,1), (0,1,1,1)}	0.3331
14	0 1 1 1 0	{Ω(1), Ω(2), Ω(3)}	{(1,0,0,0), (0,0,0,1), (0,1,1,0), (0,1,0,1), (0,0,1,1), (1,1,1,0), (1,0,1,1)}	0.25
20	1 2 1 1 1	{Ω(0), Ω(1), Ω(1), Ω(2), Ω(3), Ω(4)}	see Example 3, (ii)	0.1429

(h = 0,1,2,3,4). Briefly, we write $\Omega(h) = \Omega(4, h)$ for $h = 0, \cdots, 4$. Fortunately, the optimum blocks $B_{T,2}$ turn out to be simply expressed by $\Omega(h)$, $(h = 0, \cdots, 4)$, except N = 11, 14 and 20. Tabe 1 lists these blocks. Also, Table 2 lists those blocks for N = 11, 14 and 20. In the tables, note that the other set $B_{(1)}$ can be given by $B_{(1)} = T - B_{(0)}$ for each T. For reference, the values of $\|A^*(D)\|$ are given in the tables. Again from the above theorems, it is observed that an OCB plan is not unique for a given design. Furthermore, from Theorem 2.3, if D is an OCB plan for a design T, then it is also so for $\bar{T}$, where $\bar{T}$ is the design obtained by an interchange of 0 and 1 in T. Note that (2.4) holds for the norm $\|A^*(D)\|$ and that if T is an A-optimal design, then $\bar{T}$ is also so. The following example is helpful in referring to Tables 1 and 2:

Example 3.

(i) Consider a B-array[N = 12, m = 4, 4, μ = {1, 0, 1, 1, 1}] given by

$$T = \{\Omega(0), \Omega(2), \Omega(3), \Omega(4)\},$$

which is a 2^4-BFF design of resolution V with 12 assemblies. Then, T can be rewritten by the 4×12 matrix

$$T = \left[\begin{array}{c|cccccc|cccc|c} 0 & 1&1&1&0&0&0 & 1&1&1&0 & 1 \\ 0 & 1&0&0&1&1&0 & 1&1&0&1 & 1 \\ 0 & 0&1&0&1&0&1 & 1&0&1&1 & 1 \\ 0 & 0&0&1&0&1&1 & 0&1&1&1 & 1 \end{array}\right],$$

whose α-th column denotes the α-th assembly in T. Let D_1 = (1; 0,0,0, 0,0,0; 1,1,1,1; 1) and D_2 = (1; 1,1,0,0,0,0; 1,1,0,0; 1) be CB plans in S_2^N, whose α-th elements denote the α-th assemblies in D_1 and D_2, respectively. Then $\|A^*(D_1)\| = 0.25$ and $\|A^*(D_2)\| = 0.5$. This means that D_1 is a CB plan better than D_2 for T. In fact, $\|A^*(D_1)\|$ is a minimum over S_2^N. Hence D_1 is an OCB plan and $B_{T,2} = \{B_{(0)} = \{\Omega(2)\}$, $B_{(1)} = T - B_{(0)} = \{\Omega(0), \Omega(3), \Omega(4)\}\}$ is an optimum block (w.r.t. $\|A^*(D)\|$) over S_2^N for the design T. This is the indication of Table 1 for N = 12.

(ii) Consider a B-array[20, 4, 4, {1, 2, 1, 1, 1}] given by

$$T = \{\Omega(0),\ \Omega(1),\ \Omega(1),\ \Omega(2),\ \Omega(3),\ \Omega(4)\}\ ,$$

which is a 2^4-BFF design of resolution V with 20 assemblies. Similarly, T can be rewritten by the following

$$T = \left[\begin{array}{c|c|c|c|c|c} 0 & 1000 & 1000 & 111000 & 1110 & 1 \\ 0 & 0100 & 0100 & 100110 & 1101 & 1 \\ 0 & 0010 & 0010 & 010101 & 1011 & 1 \\ 0 & 0001 & 0001 & 001011 & 0111 & 1 \end{array}\right] .$$

Let $D = (1; 0,1,0,0; 1,0,0,0; 1,1,1,1,1,1; 0,0,0,0; 1) \in S_2^N$. Then $\|A^*(D)\|$ = 0.1429, which is a minimum over S_2^N. Hence D is an OCB plan and $B_{T,2} = \{B_{(0)}, B_{(1)}\}$ is an optimum block (w.r.t. $\|A^*(D)\|$) over S_2^N for T, where

$$B_{(0)} = \{(1,0,0,0),\ (0,0,1,0),\ (0,0,0,1),\ (0,1,0,0),$$
$$(0,0,1,0),\ (0,0,0,1),\ \Omega(3)\}\ ,$$

$$B_{(1)} = T - B_{(0)} = \{\Omega(0),\ (0,1,0,0),\ (1,0,0,0),\ \Omega(2),\ \Omega(4)\}.$$

Remark. This paper deals with the constitution of block plan for a fractional design of 2^m type. However, the procedure is available for a general asymmetrical factorial design. Indeed, we may consider an $s_1 \times s_2 \times \cdots \times s_m \times 2^m$-FF design for the set [T : D] in Section 2.

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