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IDENTIFIABILITY OF LINEAR RETARDED SYSTEMS

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1. Introduction

In the field of control engineering the problem of identification or system parameter estimation has attracted much interest and has been investigated in many references. The identifiability, as a necessary step in the modelling process, becomes an important research subject. This problem is a kind of *inverse problem* and is nonlinear in parameters whenever the system is linear, so that it is difficult and interesting. When the system is described by a 1-dimensional heat equation with an unknown potential, this problem is connected with the Gel'fand-Levitan theory and is studied to a great extent. In the aspect we refer to a work by Suzuki [1,2] who has solved the problem in a satisfactory manner (see also [3],[4],[5]).

In this paper we study the identifiability problem in the case where the system is described by a linear functional differential equation of retarded type. In the retarded system the unknown parameters are coefficient matrices, delays and a kernel function of retardation. To identify the parameters we use a method which depends on semigroup approach on the product space $Z_2 = R^n \times L_2(-h, 0; R^n)$. Using an abstract identifiability result in general Banach spaces and the spectral decomposition on Z_2, we establish some identifiability criteria for such parameters. The results are stated in terms of the generalized eigenfunctions of the transposed system and the structural operator F (see [7]) assuming that the system of generalized eigenfunctions is complete in Z_2.

2. Spectral Analysis of Linear Retarded Systems

We consider the system described by the linear autonomous functional differential equation in R^n :

$$(RS)\quad \begin{cases} \dot{y}(t) = L(y_t) + f(t) \quad \text{a.e. for } t > 0, & (2.1) \\ y(0) = y_0^0, \quad y(s) = y_0^1(s) \quad s \in [-h, 0), & (2.2) \end{cases}$$

where $y_0^0 \in R^n$, $y_0^1(\cdot) \in L_2(-h, 0; R^n)$, $f(\cdot) \in L_2^{loc}(R^+; R^n)$ and $L : C([-h, 0] ; R^n) \to R^n$ is a linear operator given by

$$L(y_t) = \int_{-h}^{0} d\eta(s) y(t+s). \tag{2.3}$$

Here η is an $n \times n$ matrix function of bounded variation on $[-h, 0]$ and the integral in (2.3) means the Stieltzes integral in R^n. The notation y_t denotes the t-translated state (or t-segment) of (RS), i.e.,

$$y_t(s) = y(t+s) \qquad s \in [-h, 0] \qquad \text{(cf. Hale [11]).}$$

Throughout this paper we suppose that η in (2.3) is given by

$$\eta(s) = - A_0\eta_0(s) - \sum_{r=1}^{k} A_r\eta_r(s) - \int_{s}^{0} B(u)du, \tag{2.4}$$

where η_r is the characteristic function of the interval $(-\infty, -h_r]$, $(r = 0,1, \cdots, k)$, $0 = h_0 < h_1 < h_2 < \cdots < h_k \le h$ are non-negative constants, A_r $(r = 0,1,\cdots, k)$ are $n \times n$ matrices and $B(\cdot)$ is an $n \times n$ matrix function in $L_2(-h, 0; R^{n\times n})$. Then (2.1) can be written as

$$\dot{y}(t) = \sum_{r=0}^{k} A_r y(t-h_r) + \int_{-h}^{0} B(s)y(t+s)ds + f(t) \quad \text{a.e.} \quad t > 0. \tag{2.5}$$

We study the system (RS) in the space $Z_2 = R^n \times L_2(-h, 0; R^n)$. The element $y \in Z_2$ will be denoted by (y^0, y^1), $y^0 \in R^n$, $y^1 \in L_2(-h, 0; R^n)$ and the space Z_2 is a Hilbert space with inner product

$$\langle y,z\rangle_{Z_2} = \langle y^0, z^0\rangle_{R^n} + \int_{-h}^{0} \langle y^1(s), z^1(s)\rangle_{R^n} ds$$

$$\text{for any } y = (y^0, y^1),\ z = (z^0, z^1) \in Z_2, \tag{2.6}$$

where $\langle\ ,\ \rangle_{R^n}$ denotes the inner product in R^n. So that the norm of Z_2 is given by

$$\| y \|_{Z_2} = (|y^0|^2_{R^n} + \| y^1(\cdot) \|^2_{L_2(-h,\ 0;\ R^n)})^{1/2}$$

$$\text{for } y \in Z_2. \tag{2.7}$$

If $y^0 = (y^0_0, y^1_0) \in Z_2$ and $f(\cdot) \in L_2^{loc}(R^+; R^n)$, then there exists a unique solution $y(t) = y(t; y_0, f)$, $t \geq 0$ of (RS) in the space $W^{(1)}_{2,loc}(R^+; R^n)$ (see Delfour and Mitter [6, Thm. 3.1]). Moreover the following continuous dependence holds ;

For any fixed $t > 0$, there exists a contant $K_t > 0$ such that

$$\| y(\cdot;\ y_0, f) \|_{W_2^{(1)}(0,\ t;\ R^n)} \leq K_t(\| y^0 \|_{Z_2} + \| f(\cdot) \|_{L_2(0,\ t;\ R^n)}). \tag{2.8}$$

Here $W_2^{(1)}(a, b; R^n)$ denotes the Sobolev space of y in $L_2(a, b; R^n)$ with distributive derivative $\dot{y}$ in $L_2(a, b; R^n)$ and $W^{(1)}_{2,loc}(R^+; R^n)$ denotes the set of all y which belong to $W_2^{(1)}(0, t; R^n)$ for all $t > 0$.

We now define the operator $T_S(t) : Z_2 \to Z_2$ for $t \geq 0$ by

$$T_S(t)y_0 = (y(t;\ y_0, 0),\ y_t(\cdot;\ y_0,\ 0)) \quad \text{for each } y_0 \in Z_2. \tag{2.9}$$

The following results are well-known. For proofs and details, see [8],[9],[10],

[11] and [7].

(I) The family of operators $\{ T_S(t);\ t \geq 0 \}$ is a C_0-semigroup on Z_2 and

$T_S(t)$ is compact for all $t \geq h$;

(II) The infinitesimal generator A_S of $T_S(t)$ is given by (2.10)

$$D(A_S) = \{ x = (x^0, x^1) : x^1(\cdot) \in W_2^{(1)}(-h, 0; R^n) \text{ and } x^1(0) = x^0 \} \quad (2.11)$$

and

$$A_S x = \left(\int_{-h}^{0} d\eta(s) x^1(s),\ \frac{dx^1}{ds} \right) \quad \text{for } x = (x^0, x^1) \in D(A_S). \quad (2.12)$$

By (2.4) the first coordinate of $A_S x$ is given by

$$A_0 x^0 + \sum_{r=1}^{k} A_r x^1(-h_r) + \int_{-h}^{0} B(s) x^1(s) ds \ ; \quad (2.13)$$

(III) Define the characteristic matrix $\Delta(\lambda)$ by

$$\Delta(\lambda) = \lambda I - \int_{-h}^{0} d\eta(s) e^{\lambda s}. \quad (2.14)$$

Then $\sigma(A_S)$ is the point spectrum and is given by $\sigma(A_S) = \{ \lambda: \det \Delta(\lambda) = 0 \}$. Since the characteristic function $\det \Delta(\lambda)$ is an entire function of λ, all zeros of $\det \Delta(\lambda) = 0$ are of finite order. For each $\lambda \in \sigma(A_S)$, we denote by M_λ the generalized eigenspace of A_S corresponding to λ. It is well known that M_λ is finite dimensional and there exists a natural number k_λ such that $M_\lambda = \text{Ker}\,(\lambda - A_S)^{k_\lambda}$ and

$$Z_2 = \text{Ker}\,(\lambda - A_S)^{k_\lambda} \oplus \text{Im}\,(\lambda - A_S)^{k_\lambda} \ \text{(direct sum)}. \quad (2.15)$$

The subspace M_λ is also represented by the range of projection operator

$$P_\lambda \phi = \frac{1}{2\pi i} \int_{\Gamma_\lambda} R(\mu; A_S) \phi d\mu \ , \qquad \phi \in Z_2, \quad (2.16)$$

where Γ_λ is any closed rectifiable curve containing λ inside and all other points of $\sigma(A_S)$ outside. To give a basis of M_λ we use the following notation introduced by Delfour and Manitius [7]. Define the operator $E_\lambda : R^n \to Z_2$ by

$$E_\lambda \xi = (\xi, e^{\lambda s}\xi) \in Z_2 \qquad \text{for} \quad \xi \in R^n. \tag{2.17}$$

Put $\dim M_\lambda = m_\lambda$. Then a basis $\{\phi_1, \cdots, \phi_{m_\lambda}\}$ of M_λ is given by

$$\phi_i = \sum_{j=0}^{m_\lambda - 1} \frac{1}{j!} \frac{d^j}{d\lambda^j} E_\lambda \xi^i_{j+1}, \quad i = 1, \cdots, m_\lambda, \tag{2.18}$$

where $\xi^i = (\xi^i_1, \xi^i_2, \cdots, \xi^i_m) \in R^n \times \cdots \times R^n$, $i = 1, \cdots, m_\lambda$, satisfy the linear equations

$$\sum_{j=l}^{m_\lambda} \left[\frac{1}{(j-l)!} \frac{d^{j-l}}{d\lambda^{j-l}} \Delta(\lambda)\right] \xi^i_j = 0, \quad l = 1, \cdots, m_\lambda, \tag{2.19}$$

and $\{\xi^1, \xi^2, \cdots, \xi^m\}$ is a set of linear independent solutions of (2.19). Since $A_S M_\lambda \subset M_\lambda$, there exists a $m_\lambda \times m_\lambda$ matrix Λ_λ such that

$$A_S \Phi_\lambda = \Phi_\lambda \Lambda_\lambda \quad \text{and} \quad T_S(t)\Phi_\lambda = \Phi_\lambda e^{\Lambda_\lambda t} \quad \text{for} \quad t \geq 0, \tag{2.20}$$

where $\Phi_\lambda = (\phi_1, \cdots, \phi_{m_\lambda})$.

(IV) For each $\lambda \in \rho(A_S) = C - \sigma(A_S)$, the resolvent $R(\lambda; A_S)$ can be represented by

$$R(\lambda; A_S)(x^0, x^1(s)) = \left(\Delta(\lambda)^{-1} a^0,\ e^{\lambda s}\Delta(\lambda)^{-1} a^0 - \int_0^s e^{\lambda(s-u)} x^1(u)\,du\right),$$

where

$$a^0 = x^0 - \int_{-h}^0 \int_0^s e^{\lambda(s-u)} d\eta(s) x^1(u)\,du. \tag{2.21}$$

Since the integral operators in (2.21) are compact (R^n is finite dimensional) we see easily that $R(\lambda; A_S)$ is compact for any $\lambda \in \sigma(A_S)$. It follows from (2.19) that the order of a pole λ of the resolvent $R(\lambda; A_S)$ equals to that of a zero of $\det \Delta(\lambda) = 0$.

(V) In this paragraph we state the converse statement of (II). Let A_S be the closed linear operator defined by (2.11) and (2.12). Then A_S generates a C_0-semigroup $T_S(t)$ on Z_2. If $f(\cdot) \in L_2^{loc}(R^+; R^n)$ and $y_0 \in Z_2$, then the first component of

$$T_S(t)y_0 + \int_0^t T_S(t-u)(f(u),0)du, \quad t \geq 0 \tag{2.22}$$

satisfies the equation (2.1). The second component of (2.22) is the t-segment of the solution of (2.1).

(VI) We give here some conditions for the completeness of the generalized eigenfunctions of A_S due to Manitius [9]. Let $\tilde{\eta}(s) = \eta(s) + A_0\chi_0(s)$, where $\eta(s)$ is given in (2.4). Define the linear bounded operator $H : L_2(-h, 0; R^n) \to L_2(-h, 0; R^n)$ by

$$(Hz)(s) = \int_{-h}^{s} d\tilde{\eta}(u)z(u-s), \quad s \in [-h, 0] \quad \text{for} \quad z \in L_2(-h, 0; R^n). \tag{2.23}$$

The system of generalized eigenfunctions of A_S is said to be complete in Z_2 if $\overline{\text{span}}\ \{ M_\lambda : \lambda \in \sigma(A_S) \} = Z_2$. The completeness holds if and only if

$$\text{Ker } H^* = \{0\}. \tag{2.24}$$

The condition (2.24) is equivalent to that the equation

$$\int_{-h}^{s} d\tilde{\eta}^T(u)z(u-s) = 0 \quad \text{for a.e.} \quad s \in [-h, 0] \tag{2.25}$$

admits the only one trivial solution $z(s) = 0$ a.e. in $[-h, 0]$, where the symbol ^{T} denotes the transpose operation. If $B(s) = 0$ a.e. $s \in [-h, 0]$ and $h_k = h$, then a necessary and sufficient condition for the completeness in Z_2 is that

$$\det A_k \neq 0, \tag{2.26}$$

which is a consequence from (2.24). For other useful criteria for completeness, see Delfour and Manitius [7,Sec. 5].

According to [7] define the structural operator $F : Z_2 \to Z_2$ by $F = \begin{pmatrix} I & O \\ O & H \end{pmatrix}$.
It is evident that F is bounded. We shall give a useful expression of the projection operator P_λ in (2.16). Let $\eta^T(s)$ be the transposed matrix of $\eta(s)$. We denote by $T_S^+(t)$ the C_0-semigroup in Z_2 corresponding to (RS) with $\eta = \eta^T$. Let A_S^+ be its infinitesimal generator. It is shown in [7] that if $\lambda \in \sigma(A_S)$, then $\bar{\lambda} \in \sigma(A_S^+)$ and $\dim M_\lambda = \dim \operatorname{Ker} (\bar{\lambda} - A_S^+)^{k_\lambda} = m_\lambda$. Let $\{ \psi_1, \cdots, \psi_{m_\lambda} \}$ be a basis of $\operatorname{Ker} (\bar{\lambda} - A_S^+)^{k_\lambda}$. Then the projection P_λ is given by the formula

$$P_\lambda \phi = \sum_{i=1}^{m_\lambda} \langle \psi_i, F\phi \rangle_{Z_2} \phi_i, \qquad \phi \in Z_2, \tag{2.27}$$

where $\{ \phi_1, \cdots, \phi_{m_\lambda} \}$ is the basis in (2.18). If $\phi = (\phi^0, \phi^1)$ satisfies $\phi^1 \in C([-h, 0]; R^n)$ and $\phi^1(0) = \phi^0$, then the coefficient $\langle \psi_i, F\phi \rangle_{Z_2}$ in (2.27) is written by

$$\langle \psi_i, F\phi \rangle_{Z_2} = \langle \psi_i(0), \phi(0) \rangle_{R^n} + \int_{-h}^{0} \int_{u}^{0} \langle \psi_i(u-s) d\eta(u), \phi(s) \rangle_{R^n} ds, \tag{2.28}$$

which corresponds to the bilinear form described by Hale [11, Sec. 7.3].

3. Abstract results on Controllability and Identifiability

In this section we give two abstract results needed later.

Let X and U be separable Banach spaces. Consider the control system on X

$$(A, B) : \dot{x}(t) = Ax(t) + Bu(t), \qquad t > 0, \tag{3.1}$$

where $x(t) \in X$, A generates a C_0-semigroup $T(t)$ on X, B is a bounded linear operator from U into X and $u(t)$ is a control function defined on U with values in X. The system (A, B) is called approximately controllable if

$$\overline{\bigcup_{t \ge 0} T(t)BU} = X, \tag{3.2}$$

where the upper bar denotes the closure in X.

In this section we assume that A satisfies the following assumption

H_1: A is a closed linear operator with compact resolvent.

Differently from Triggiani [12] the selfadjointness or normality of A is not assumed. By a familiar theorem [13, Thm 6.29], the assumption H_1 implies that the spectrum $\sigma(A)$ of A is a countable set and consists entirely of discrete eigenvalues with finite multiplicities. We put $\sigma(A) = \{\lambda_n : n=1,2,\cdots\}$.

The following decomposition results hold:

(VII) For each $\lambda_n \in \sigma(A)$ define the operator

$$P_n = \frac{1}{2\pi i}\int_{\Gamma_n} R(\lambda; A)d\lambda, \tag{3.3}$$

where Γ_n is a rectifiable, closed curve containing λ_n only, and put $M_n = P_n X$. Then P_n is the cannonical eigenprojector on X, i.e., $P_n^2 = P_n$ and $P_n P_j = 0$ if $n \neq j$, A commutes with P_n and M_n is A-invariant, i.e., $AM_n \subset M_n$, $\dim M_n < \infty$ and the operator A restricted to M_n, denoted by A_n, is bounded on M_n and $\sigma(A_n) = \{\lambda_n\}$. The closed subspace M_n is called the generalized eigenspace corresponding to the eigenvalue λ_n. Any element in M_n is called the generalized eigenfunction of A corresponding to λ_n [13, p.181].

(VIII) $T(t)$, $t \geq 0$ commutes with P_n and M_n is $T(t)$-invariant, i.e.,

$$T(t)M_n \subset M_n \qquad \text{for each } n = 1,2,\cdots . \tag{3.4}$$

(IX) If λ_n is a pole of order d_n of $R(\lambda; A)$, then

$$M_n = \text{Ker}\,(\lambda_n - A)^{d_n}, \quad \dim M_n = m_n < \infty \tag{3.5}$$

and

$$X = \text{Ker}\,(\lambda_n - A)^{d_n} \oplus \text{Im}\,(\lambda_n - A)^{d_n}. \tag{3.6}$$

Let $\{\phi_{n1}, \cdots, \phi_{nm_n}\}$ be a basis in M_n. Then by (VII) and (VIII), there exists curve Γ_n^m containing λ_N^m but not containing any of $\sigma(A) \cup (\sigma(A^m) - \{\lambda_n^m\})$ inside. Then by (3.3) and (3.17), we have

$$P_N^m x_i = \frac{1}{2\pi i}\int_{\Gamma_N^m} R(\lambda;\ A^m)x_i d\lambda$$

$$= \frac{1}{2\pi i}\int_{\Gamma_N^m} R(\lambda;\ A)x_i d\lambda = 0, \quad i = 1,\cdots, p. \tag{3.23}$$

Note that Γ_N^m does not contain any pole of $R(\lambda;\ A)$ inside. The conditions (3.23) and (3.18) imply that $M_N^m = \{0\}$, a contradiction. Therefore $\lambda_N^m \in \sigma(A)$. Then there exists a natural number N^* (for fixed N) such that

$$\lambda_N^m = \lambda_{N^*} \qquad \text{and} \tag{3.24}$$

$$P_N^m x_i = \frac{1}{2\pi i}\int_{\Gamma_N^m} R(\lambda;\ A)x_i d\lambda = P_{N^*}x_i, \quad i = 1,\cdots, p. \tag{3.25}$$

It follows from (3.25) and (3.18) that

$$M_N^m \subset M_{N^*}. \tag{3.26}$$

Hence we have from (3.24) and (3.26) that $\{\lambda_n^m\}_{n=1}^\infty \subset \{\lambda_n\}_{n=1}^\infty$ and the inclusions $M_n^m \subset M_{n^*}$ hold for all n. Since A^m satisfies H_2, we see that

$$X = \overline{\text{span}}\ \{ M_{n^*}:\ n = 1,2,\cdots\} \subset \overline{\text{span}}\ \{ M_n:\ n = 1,2,\cdots \} = X, \tag{3.27}$$

that is, A also satisfies H_2. It follows from (3.27) and (3.5) that $\{n\} = \{n^*\}$, so that $\lambda_n^m = \lambda_n$ and $M_n^m = M_n$ for all n. In fact if $M_n^m \not\subseteq M_n$, the completeness for A^m does not hold. Since $M_n^m = M_n = \text{span}\ \{ \phi_{n1},\cdots, \phi_{nK_n} \}$, we have by (3.4) and (3.10) that

$$(\phi_{n1},\cdots, \phi_{nK_n})e^{\tilde{A}_n^m t}\begin{pmatrix} \xi_{1,i} \\ \vdots \\ \xi_{K_n,i} \end{pmatrix} = (\phi_{n1},\cdots, \phi_{nK_n})e^{\tilde{A}_n t}\begin{pmatrix} \xi_{1,i} \\ \vdots \\ \xi_{K_n,i} \end{pmatrix},$$

an $m_n \times m_n$ constant matrix $\tilde{A}_n$ such that

$$(A\phi_{n1}, \cdots, A\phi_{nm_n}) = (\phi_{n1}, \cdots, \phi_{nm_n})\tilde{A}_n. \tag{3.7}$$

Furthermore we have that for each $x \in M_n$, there exists a column vector $\xi = (\xi_1, \cdots, \xi_{nm_n})^T$ such that

$$x = \sum_{j=1}^{m_n} \phi_{nj}\xi_j, \tag{3.8}$$

$$Ax = (\phi_{n1}, \cdots, \phi_{nm_n})\tilde{A}_n \tag{3.9}$$

and

$$T(t)x = (\phi_{n1}, \cdots, \phi_{nm_n})e^{\tilde{A}_n t} . \tag{3.10}$$

So that $T(t)$ can be extended for all $t \in R$ on M_n.

Let X^* be the dual space of X. Since the adjoint operator A^* satisfies H_1, similar decomposition results as above hold for A^*. Moreover the following fact holds:

(X) If λ_n is a pole of order d_n of $R(\lambda; A)$, then $\overline{\lambda_n}$ is a pole of order d_n of $R(\lambda; A^*)$ and $M_n^* = \text{Ker}\ (\overline{\lambda_n} - A^*)^{d_n}$, dim M_n = dim $M_n^* = m_n$. In addition

$$X^* = \text{Ker}\ (\overline{\lambda_n} - A^*)^{d_n} \oplus \text{Im}\ (\overline{\lambda_n} - A^*)^{d_n} \tag{3.11}$$

and

$$\text{Im}\ (\lambda_n - A)^{d_n} = X \ominus M_n = M_n^{*\perp} = (\text{Ker}\ (\overline{\lambda_n} - A^*)^{d_n})^{\perp}. \tag{3.12}$$

where $\perp$ denotes the orthogonal complement.

The system (A, B) is called spectrally controllable if all systems $(A_n, P_n B)$ are approximately controllable on M_n, $n = 1,2,\cdots$. It can be verified that the spectral controllability of (A, B) is equivalent to that

$$P_n BU = M_n, \quad n = 1,2,\cdots . \tag{3.13}$$

We next suppose that A satisfies the assumption

H_2: The system of generalized eigenfunctions of A is complete, i.e.,

$$\overline{\text{span}} \{ M_n : n = 1,2,\cdots \} = X. \tag{3.14}$$

PROPOSITION 3.1. Let A satisfy H_1 and H_2. Then (A, B) is approximately controllable if and only if (A, B) is spectrally controllable. Moreover if $U = C^p$ and $B : U \to X$ is given by

$$B(\eta_1,\cdots, \eta_p) = \sum_{i=1}^{p} x_i \eta_i, \quad x_i \in X, \quad \eta_i \in C^1, \tag{3.15}$$

then the (spectral) controllability condition (3.13) is equivalent to that

$$\text{span} \{ P_n x_1,\cdots, P_n x_p \} = M_n, \quad n = 1,2,\cdots . \tag{3.16}$$

Proof. If A is normal, A satisfies H_2. Hence this proposition extends Theorem 3.8 of Triggiani [12]. The detailed proof is omitted, since it can be carried as in Fattorini [15, Cor. 3.2].

For the given operator A, we consider the *model* operator A^m which generates a C_0-semigroup $T^m(t)$ and satisfies H_1. For A^m we denote the spectrum $\sigma(A^m)$ by $\{ \lambda_n^m : n = 1,2,\cdots \}$. Let M_n^m be the generalized eigenspace corresponding to the eigenvalue λ_n^m and P_n^m be the associated eigenprojector, i.e.,

$$P_n^m = \frac{1}{2\pi i} \int_{\Gamma_n^m} R(\lambda; A^m) d\lambda, \tag{3.17}$$

where Γ_n^m is a rectifiable closed curve containing λ_n^m only. Put $M_n^m = P_n^m X$, $\dim M_n^m = K_n$ and let $\{ \Phi_{n1},\cdots, \Phi_{nK_n} \}$ be a basis of M_n^m. We denote a basis of $M_n^m{}^*$ by $\{ \Phi^*_{n1},\cdots, \Phi^*_{nK_n} \}$, where $M_n^m{}^*$ is the generalized eigenspace corresponding to $\overline{\lambda_n^m}$ for A^{m*}.

The following Proposition is fundamental and is an extension of Nakagiri [4, Thm. 5.1].

PROPOSITION 3.2. Let A and A^m satisfy H_1 and let $(x_1, \cdots, x_p) \in X^p = X \times \cdots \times X$. If A^m satisfies H_2 and

$$\text{span}\ \{ P_n^m x_1, \cdots, P_n^m x_p \} = M_n^m, \quad n = 1,2,\cdots , \tag{3.18}$$

then A is identifiable, i.e.,

$$T(t)(x_1, \cdots, x_p) = T^m(t)(x_1, \cdots, x_p) \quad \text{in } X^p \quad \text{for } t > 0 \tag{3.19}$$

implies

$$A = A^m. \tag{3.20}$$

Remark 3.1. Let B be given by (3.15). Proposition 3.1 implies that the condition (3.18) is equivalent to the approximate controllability of the system (A^m, B). The condition (3.18) is also equivalent to the rank condition

$$\text{rank}\ \left(\langle x_i, \Phi^*_{nj} \rangle_{X,X^*} : \begin{matrix} i \to 1,\cdots, p \\ j \downarrow 1,\cdots, K_n \end{matrix} \right) = K_n \quad \text{for all} \quad n = 1,2,\cdots , \tag{3.21}$$

where $\langle\ ,\ \rangle_{X,X^*}$ is the duality pairing between X and X^*. A similar rank condition to (3.21) for abstract stabilizability is given by Suzuki and Yamamoto [14].

Proof of Proposition 3.2. Let (3.19) and (3.19) be satisfied. Then by taking Laplace transforms of (3.19), we have

$$R(\lambda; A)(x_1, \cdots, x_p) = R(\lambda; A^m)(x_1, \cdots, x_p) \tag{3.22}$$

for $\text{Re}\ \lambda$ sufficiently large. Since the resolvent operator is analytic on its domain, we see by analytic continuation that (3.22) holds for all $\lambda \in C - (\sigma(A) \cup \sigma(A^m))$. Let N be fixed. If $\lambda_N^m \notin \sigma(A)$, we can choose a rectifiable closed

$$t > 0, \quad i = 1, \cdots, p, \tag{3.28}$$

where $P_n^m x_i = P_n x_i = \sum_{j=1}^{K_n} \Phi_{nj} \xi_{j,i}$, $i = 1, \cdots, p$ and $\tilde{A}_n^m$, $\tilde{A}_n$ are the $K_n \times K_n$ matrices given in (IX). Differentiating (3.28) and letting $t \downarrow 0$, we obtain that

$$(\Phi_{n1}, \cdots, \Phi_{nK_n})(\tilde{A}_n^m - \tilde{A}_n)\,\Xi = (0, \cdots, 0), \tag{3.29}$$

where $\Xi = (\xi_{j,i} : \begin{smallmatrix} i \to 1, \cdots, p \\ j \downarrow 1, \cdots, K_n \end{smallmatrix})$. Since rank $\Xi = K_n$ by (3.18), (3.29) implies $(\Phi_{n1}, \cdots, \Phi_{nK_n})(\tilde{A}_n^m - \tilde{A}_n) = (0, \cdots, 0)$, and hence $\tilde{A}_n^m = \tilde{A}_n$. That is

$$A^m\big|_{M_n^m} = A_n^m = A\big|_{M_n} = A_n \quad \text{for all} \quad n = 1, 2, \cdots . \tag{3.30}$$

By the completeness of generalized eigenfunctions of A^m and (3.30), we conclude that $A = A^m$, which proves (3.20).

4. Identifiability of Linear Retarded Systems

Consider the following linear retarded system with p-numbers of initial values and initial functions and forcing functions :

$$(RS)_p \begin{cases} \dot{y}(t) = \sum_{r=0}^{k} A_r y(t-h_r) + \int_{-h}^{0} B(s)y(t+s)ds + f_i(t) \quad \text{a.e. for} \quad t > 0, & (4.1) \\ y(0) = y_{0,i}^0, \quad y(s) = y_{0,i}^1(s) \quad s \in [-h, 0) \quad (i = 1, \cdots, p). & (4.2) \end{cases}$$

Here A_r, h_r, $r = 0, 1, \cdots, k$, $B(\cdot)$, $y_{0,i} = (y_{0,i}^0, y_{0,i}^1)$, $i = 1, \cdots, p$, satisfy the assumptions in Section 2.

Throughout this section we suppose the following conditions :

(XI) The $n \times n$ matrices A_r, $r = 0, 1, \cdots, k$ (k is known), are unknown except that $A_r \neq 0$ for each $r = 0, 1, \cdots, k$ and the matrix function $B(\cdot)$ is unknown except that $B(\cdot) \in L_2(-h, 0; R^{n\times n})$;

XII) The initial conditions $y_{0,i} = (y_{0,i}^0, y_{0,i}^1) \in Z_2$, $i = 1, \cdots, p$, are known;

(XIII) The delay times h_r, $r = 1,\cdots, k$, are unknown but known that

$$0 = h_0 < h_1 < h_2 < \cdots\cdot < h_k \le h$$

and $h > 0$ is known;

(XIV) The forcing functions $f_i(\cdot) \in L_2^{loc}(R^+; R^n)$, $i = 1,\cdots, p$, are known.

A_r $(r = 0,1,\cdots, k)$, h_r $(r= 1,\cdots, k)$, $B(\cdot)$ are parameters in $(RS)_p$ to be identified. Under the above conditions there exist unique solutions $y_i(t; y_{0,i}, f_i)$, $i = 1,\cdots, p$, of $(RS)_p$.

By the model system $(RS)_p^m$ we understand the system (4.1),(4.2) in which A_r, $B(\cdot)$ and h_r are replaced by A_r^m, $B^m(\cdot)$ and h_r^m, respectively. The corresponding model states are denoted by $y_i^m(t; y_{0,i}, f_i)$, $i = 1,\cdots, p$. The corresponding kernel function in (2.4) is denoted by $\eta^m(s)$. All quantities subscripted by m are assumed to be known.

We shall say that A_r, $r = 0,1,\cdots, k$, $B(s)$ and/or h_r, $r = 1,\cdots, k$ are identifiable if

$$A_0 = A_0^m,\quad A_1 = A_1^m,\quad \cdots\cdots,\quad A_k = A_k^m, \tag{4.3}$$

$$B(s) = B^m(s) \quad \text{for a.e.} \quad s \in [-h, 0], \tag{4.4}$$

and/or $$h_1 = h_1^m,\quad \cdots\cdots,\quad h_k = h_k^m \tag{4.5}$$

follow from the relations

$$e_i(t) = y_i(t; y_{0,i}, f_i) - y_i^m(t; y_{0,i}, f_i) = 0 \quad \text{in } R^n, \quad t > 0,\ i = 1,\cdots,p. \tag{4.6}$$

Let $T_S^m(t)$ be the semigroup relating to the model system $(RS)_p^m$ which is given in Section 2. The infinitesimal generator of $T_S^m(t)$ is denoted by A_S^m. We denote $\sigma(A_S^m) = \{ \lambda_n^m : n = 1,2,\cdots\cdot \}$. In this case $\sigma(A_S^m) \cap \{\lambda : \mathrm{Re}\ \lambda > \delta \}$ is finite (may be empty) for any $\delta \in R$. Let $M_n^m = \mathrm{Ker}\ (\lambda_n^m - A_S^m)^{D_n}$ be the generalized eigenspaces corresponding to λ_n^m. We denote the basis of M_n^m by $\{\Phi_{n1}, \cdots, \Phi_{nK_n}\}$ (which can be choosen as in (2.18)). Let $\{\Psi_{n1}, \cdots, \Psi_{nK_n}\}$ be a basis of $\mathrm{Ker}\ (\bar{\lambda}_n^m - (A_S^m)^+)^{D_n}$ and let H^m be the operator given in (2.23) corresponding to $\eta^m(s)$. The cannonical eigenprojector of A_S^m corresponding to λ_n^m is denoted by P_n^m. Then we have the following theorem by using Proposition 3.2.

<u>THEOREM 4.1.</u> Let $f_i(\cdot) = 0$ in $L_2^{loc}(R^+; R^n)$, $i = 1, \cdots, p$. If

i) $$\mathrm{Ker}\ (H^m)^* = \{0\} \quad \text{and} \tag{4.7}$$

ii) the set of initial conditions $\{y_{0,1}, \cdots, y_{0,p}\}$ satisfies

$$\mathrm{rank}\ \left(\langle \Psi_{nj}, F y_{0,i} \rangle_{Z_2} : \begin{matrix} i \to 1, \cdots, p \\ j \downarrow 1, \cdots, K_n \end{matrix} \right) = K_n \quad \text{for all} \quad n = 1, 2, \cdots, \tag{4.8}$$

then all A_r, $B(s)$ and h_r are identifiable.

<u>Proof.</u> Let the assumption in this theorem and (4.6) be satisfied. Then by the definition of translation semigroups, we have

$$T_S(t)(y_{0,1}, \cdots, y_{0,i}) = T_S^m(t)(y_{0,1}, \cdots, y_{0,p}) \quad \text{in} \quad X^p \quad \text{for} \quad t > 0. \tag{4.9}$$

Observe that the equation (4.9) for $0 < t < h$ requires the observability of $y_{0,1}, \cdots, y_{0,p}$ in Z_2. Put $X = Z_2$ in Proposition 3.2. It is easy to verify that (4.8) is equivalent to the span condition (3.18). Then by Proposition 3.2 it follows that the conditions (4.7), (4.8) and (4.9) imply that

$$A_S x = A_S^m x \quad \text{for any} \quad x \in D(A_S^m). \tag{4.10}$$

The equation (4.10) implies that (by considering the first component of Z_2)

$$A_0 x^1(0) + \sum_{r=1}^{k} A_r x^1(-h_r) + \int_{-h}^{0} B(s) x^1(s) ds$$

$$= A_0^m x^1(0) + \sum_{r=1}^{k} A_r^m x^1(-h_r^m) + \int_{-h}^{0} B^m(s) x^1(s) ds$$

for any $x^1(\cdot) \in W_2^{(1)}(-h, 0; R^n)$. (4.11)

Let $\xi \in R^n$ and $\varepsilon > 0$ be fixed. Let $v_\varepsilon(s)$ be a function in $W_2^{(1)}(-h, 0; R^n)$ such that

$$v_\varepsilon(0) = \xi, \quad v_\varepsilon(s) = 0 \quad \text{if} \quad -h \le s \le -\varepsilon$$

and

$$\int_{-h}^{0} |v_\varepsilon(s)|^2 ds \le \varepsilon^2. \tag{4.12}$$

Then for each $\varepsilon \in (0, \min(h_1, h_1^m))$, we can apply (4.11) to $v_\varepsilon(s)$ to obtain

$$A_0\xi - A_0^m\xi = \int_{-h}^{0} (B^m(s) - B(s))v_\varepsilon(s)ds. \tag{4.13}$$

By (4.12), (XI) and the Schwartz inequality, we have

$$| A_0\xi - A_0^m\xi | \le \left(\int_{-h}^{0} \| B^m(s) - B(s) \|^2 ds\right)^{1/2} \left(\int_{-h}^{0} |v_\varepsilon(s)|^2 ds\right)^{1/2}$$

$$\le \varepsilon \| B^m(\cdot) - B(\cdot) \|_{L_2(-h, 0; R^{n\times n})}. \tag{4.14}$$

Letting $\varepsilon \to 0$, we obtain from (4.14) that

$$A_0\xi = A_0^m\xi \qquad \text{for any} \quad \xi \in R^n. \tag{4.15}$$

This shows $A_0 = A_0^m$. We consider the second step. We shall show $h_1 = h_1^m$ by contradiction. Assume contrary that $h_1 \ne h_1^m$, say $-h_1 < -h_1^m$. we now consider a function $W_\varepsilon(s)$ satisfying $W_\varepsilon(-h_1^m) = \xi$, $W_\varepsilon(s)$ vanishes outside the ε-neighborhood of $-h_1^m$ and satisfying the condition (4.12), where ξ is an any vector in R^n. Then applying similar method as above we see that $A_1^m = 0$, which contradicts the assumption in (XI). Hence $h_1 = h_1^m$ is proved. So that $A_1 = A_1^m$. Continuing this process, we can verify (4.3) and (4.5). Finally we shall prove (4.4). From the above arguments, it follows by (4.11) that

$$\int_{-h}^{0} (B(s) - B^m(s))x^1(s)ds = 0 \quad \text{for any} \quad x^1 \in W_2^{(1)}(-h, 0; R^n). \tag{4.16}$$

Since $W_2^{(1)}(-h, 0; R^n)$ is dense in $L_2(-h, 0; R^n)$, (4.16) holds for any x^1 in $L_2(-h, 0; R^n)$. This means $B(s) = B^m(s)$ for a.e. $s \in [-h, 0]$. Thus the proof is completed.

<u>Remark 4.1.</u> If the basis $\{ \Phi_{n1}, \cdots, \Phi_{nK_n} \}$ is choosen to be orthonormal system in Z_2 (that is possible by Schmidt's orthogonalization), the condition (4.8) can be replaced by

$$\text{rank } (\langle P_n^m y_{0,i}, \Phi_{nj} \rangle_{Z_2} : \begin{matrix} i \rightarrow 1,\cdots, p \\ j \downarrow 1,\cdots; K_n \end{matrix}) = K_n \quad \text{for all} \quad n = 1,2,\cdots. \tag{4.17}$$

<u>THEOREM 4.2.</u> Let $y_{0,i} = 0$ in Z_2, $i = 1,\cdots, p$ and let $f_i(t)$, $i = 1,\cdots, p$, be of the forms $g_i(t)\xi_i$, $\xi_i \in R^n$ and the coefficient time functions $g_i(t)$ are members of $L_2^{loc}(R^+; R)$. If the condition (4.7) is satisfied and if

i) $g_i(\cdot) \neq 0$ in $L_2^{loc}(R^+; R)$, $i = 1,\cdots, p$ and (4.18)

ii) the set $\{ \xi_1, \cdots, \xi_p \}$ in R^n satisfies

$$\text{rank } (\langle \Psi_{nj}(0), \xi_i \rangle_{R^n} : \begin{matrix} i \rightarrow 1,\cdots, p \\ j \downarrow 1,\cdots, K_n \end{matrix}) = K_n \quad \text{for all} \quad n = 1,2,\cdots, \tag{4.19}$$

then all A_r, $B(s)$ and h_r are identifiable.

<u>Proof.</u> Put

$$v_i(t) = \int_0^t T_S(t-s)(f_i(s),0)ds, \quad i = 1,\cdots, p. \tag{4.20}$$

Then by Webb [16,p.73] we see that the second component of $v_i(t)$, say $v_i^1(t)$, satisfies

$$(v_i^1(t))(s) = \begin{cases} 0 & \text{if} \quad t+s \le 0 \\ v_i^0(t+s) = y(t+s; 0, f_i) & \text{if} \quad t+s \ge 0. \end{cases} \tag{4.21}$$

$v_i^0(t)$ denotes the first component of $v_i(t)$. $v_i^m(t)$ can be defined similarly.

Let the assumptions in this theorem and (4.18),(4.19) be satisfied. Then by (2.22) and (4.21) the zero output errors $e_i(t) = y_i(t; 0,f_i) - y_i^m(t; 0,f_i) = 0$ in R^n, $t > 0$, $i = 1,\cdots, p$, implies that $v_i(t) = v_i^m(t)$ in Z_2, $t > 0$, $i = 1,\cdots, p$, i.e.,

$$\int_0^t g_i(t-u)(T_S(u)(\xi_i,0) - T_S^m(u)(\xi_i,0))du = 0 \quad \text{in} \quad Z_2, \quad t > 0, \quad i = 1,\cdots, p. \tag{4.22}$$

Hence for any x in Z_2 it follows from (4.18) that

$$\int_0^t g_i(t-u)R_i(u)du = 0 \quad \text{in} \quad R, \quad t > 0, \quad i = 1,\cdots, p, \tag{4.23}$$

where

$$R_i(t) = \langle T_S(t)(\xi_i,0) - T_S^m(t)(\xi_i,0), x\rangle_{Z_2}. \tag{4.24}$$

Since (4.18) is satisfied, we have by (4.23) and Titchmarsh's theorem on convolution equations [17,Thm. 15] that

$$R_i(t) = 0 \quad \text{for a.e.} \quad t > 0, \ i = 1,\cdots, p. \tag{4.25}$$

Since $x \in Z_2$ can be choosen arbitrary, (4.25) implies that

$$T_S(t)((\xi_1,0),\cdots, (\xi_p,0)) = T_S^m(t)((\xi_1,0),\cdots, (\xi_p,0)) \quad \text{in} \quad X^p \quad \text{for} \quad t > 0. \tag{4.26}$$

In this case the corresponding rank condition to (4.8) is given by (4.19). Thus as in the proof of Theorem 4.1, the conclusion of this theorem follows.

Remark 4.2. Put $B_0 = (\xi_1,\cdots,\xi_p)$, an $n \times p$ matrix, and define a bounded linear operator $B : R^p \to Z_2$ by $Bu = (B_0u,0)$. We can verify easily that the system (A_S^m, B) is spectrally controllable if and only if (4.19) is satisfied. Hence by Pandlfi [18], the condition (4.19) can be replaced by

$$\text{rank } [\Delta^m(\lambda), B_0] = n \quad \text{for all} \quad \lambda \in C, \tag{4.27}$$

where $\Delta^m(\lambda) = \lambda I - \int_{-h}^0 d\eta^m(s)e^{\lambda s}$ and $B_0 = (\xi_1,\cdots,\xi_p)$.

5. Examples

In this section we give some examples which illustrate the status of this paper. To apply Theorems 4.1, 4.2 in Section 4, we need exact informations on eigenvalues and eigenfunctions of the model system. We give such quantities of different nature in the following examples.

Example 5.1. Consider the following delay-differential equation on R^3:

$$\dot{y}(t) = A_{0,\varepsilon}y(t) + A_1y(t-1), \tag{5.1}$$

where

$$A_{0,\varepsilon} = \begin{pmatrix} 1 & 0 & 3 \\ 0 & -2 & 0 \\ 0 & 0 & 1+\varepsilon \end{pmatrix}, \quad A_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{5.2}$$

Then we have

$$\Delta(\lambda) = \begin{pmatrix} \lambda-1-e^{-\lambda} & 0 & \lambda-3 \\ 0 & \lambda+2+3+e^{-\lambda} & 0 \\ 0 & -2e^{-\lambda} & \lambda-(1+\varepsilon)-e^{-\lambda} \end{pmatrix} \tag{5.3}$$

and $$\det \Delta(\lambda) = (\lambda-1-e^{-\lambda})(\lambda+2+3e^{-\lambda})(\lambda-(1+\varepsilon)-e^{-\lambda}) = P_1(\lambda)P_3(\lambda)P_{1+\varepsilon}(\lambda). \tag{5.4}$$

It can be easily verified that if $0 < |\varepsilon| \le 1$, then all roots of $\det \Delta(\lambda) = 0$ are simple and are the union of the roots of $P_1(\lambda) = 0$, $P_3(\lambda) = 0$ and $P_{1+\varepsilon}(\lambda) = 0$. If $\varepsilon = 0$, then the roots of $P_1(\lambda) = 0$ are double roots of $\det \Delta(\lambda) = 0$ and the roots of $P_2(\lambda) = 0$ are simple roots.

First we consider the case where $\varepsilon = 1$. Put $\{\lambda_n^{(1)}\}_{n=1}^{\infty} = \{\lambda : P_1(\lambda) = 0\}$ $\{\lambda_n^{(2)}\}_{n=1}^{\infty} = \{\lambda : P_2(\lambda) = 0\}$ and $\{\lambda_n^{(3)}\}_{n=1}^{\infty} = \{\lambda : P_3(\lambda) = 0\}$. Approximate root values $\lambda_n^{(1)}$, $\lambda_n^{(2)}$, $\lambda_n^{(3)}$ are:

$$\lambda_1^{(1)} = 1.27846$$
$$\lambda_2^{(1)}, \lambda_3^{(1)} = -1.58832 \pm 4.15531\, i$$
$$\lambda_4^{(1)}, \lambda_5^{(1)} = -2.41763 \pm 10.686\, i$$

$$\lambda_6^{(1)}, \lambda_7^{(1)} = -2.8615 \pm 17.0561\ i$$
$$\lambda_8^{(1)}, \lambda_9^{(1)} = -3.16775 \pm 23.3856\ i$$
$$\lambda_{10}^{(1)}, \lambda_{11}^{(1)} = -3.40194 \pm 29.698\ i$$

etc.;

$$\lambda_1^{(2)} = 2.12003$$
$$\lambda_2^{(2)}, \lambda_3^{(2)} = -1.689 \pm 3.96275\ i$$
$$\lambda_4^{(2)}, \lambda_5^{(2)} = -2.44163 \pm 10.5987\ i$$
$$\lambda_6^{(2)}, \lambda_7^{(2)} = -2.87267 \pm 16.9996\ i$$
$$\lambda_8^{(2)}, \lambda_9^{(2)} = -3.17431 \pm 23.3438\ i$$
$$\lambda_{10}^{(2)}, \lambda_{11}^{(2)} = -3.4063 \pm 29.6649\ i$$

etc.;

$$\lambda_1^{(3)}, \lambda_2^{(3)} = -0.01035 \pm 2.28682\ i$$
$$\lambda_3^{(3)}, \lambda_4^{(3)} = -0.986369 \pm 7.98032\ i$$
$$\lambda_5^{(3)}, \lambda_6^{(3)} = -1.55292 \pm 14.1687\ i$$
$$\lambda_7^{(3)}, \lambda_8^{(3)} = -1.91812 \pm 20.4244\ i$$
$$\lambda_9^{(3)}, \lambda_{10}^{(3)} = -2.18595 \pm 26.6966\ i$$
$$\lambda_{11}^{(3)}, \lambda_{12}^{(3)} = -2.3972 \pm 32.9747\ i$$

etc.

The corresponding generalized eigenspaces are all one dimensional and are given by:

$$M_{\lambda_n}(1) = \text{span}\left\{ e^{\lambda_n^{(1)} s} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}, \qquad n = 1,2,3,\cdots\cdots \ ;$$

$$M_{\lambda_n}(2) = \text{span}\left\{ e^{\lambda_n^{(2)} s} \begin{pmatrix} \lambda_n^{(2)} - 3 \\ 0 \\ -1 \end{pmatrix} \right\}, \qquad n = 1,2,3,\cdots\cdots \ ;$$

$$M_{\lambda_n^{(3)}} = \text{span}\left\{ e^{\lambda_n^{(3)} s} \begin{pmatrix} 3(\lambda_n^{(3)}-3) \\ 2(\lambda_n^{(3)}-1) \\ -(\lambda_n^{(3)}+2)(4\lambda_n^{(3)}-1) \end{pmatrix} \right\} \quad , \quad n = 1,2,3,\cdots .$$

Observe that the above generalized eigenspaces consist of a complete system in $R^3 \times L_2(-1, 0; R^3)$ (det $A_1 \neq 0$). Let $p = 1$ and $y_{0,1} = ((1, 0, 0)^T, 0)$, then the condition (4.8) in Theorem 4.1 is satisfied, which is verified easily by using (4.27).

Next consider the case where $\varepsilon = 0$. In this case the generalized eigenspaces corresponding to $\lambda_n^{(1)}$ are two dimensional and are given by

$$M_{\lambda_n^{(1)}} = \text{span}\left\{ e^{\lambda_n^{(1)} s} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \; e^{\lambda_n^{(1)} s} \begin{pmatrix} 0 \\ 0 \\ -\lambda_n^{(1)} \end{pmatrix} + s e^{\lambda_n^{(1)} s} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\},$$

$$n = 1,2,3,\cdots .$$

The rest generalized eigenspaces corresponding to $\lambda_n^{(3)}$ are one dimensional and are given by

$$M_{\lambda_n^{(3)}} = \text{span}\left\{ e^{\lambda_n^{(3)} s} \begin{pmatrix} 3(\lambda_n^{(3)}-3) \\ 4\lambda_n^{(3)}-1 \\ -2(\lambda_n^{(3)}+2)(4\lambda_n^{(3)}-1) \end{pmatrix} \right\}, \quad n = 1,2,3,\cdots .$$

Let $p = 2$ and $y_{0,1} = ((1, 0, 0)^T, 0)$, $y_{0,2} = ((0, 0, 1)^T, 0)$, then the condition (4.8) in Theorem 4.1 is satisfied. Hence if the equation (5.1) is adopted as the model system equation, it requires at least two zero output errors when $\varepsilon = 0$ and requires at least one zero output error when $0 < |\varepsilon| \leq 1$ to solve the identifiability problem. But in both cases the multiplicity of the model system is not larger than the size 3 of the matrices $A_{0,\varepsilon}$ and A_1. In general, however, there is no relation between the multiplicity and the size of matrices. This example 5.1 also shows that the multiplicity is a discontinuous function of the data $\{A_{0,\varepsilon}, A_1\}$. That is, $A_{0,\varepsilon} \to A_{0,0}$ as $\varepsilon \to 0$ in the matrix norm, but the multiplicity jumps at $\varepsilon = 0$.

Example 5.2. Here we give an example which is a scalar equation with the multiplicity 2. Consider the following scalar equation

$$\dot{y}(t) = y(t) - y(t-1). \tag{5.5}$$

Then $\Delta(\lambda) = \lambda - 1 + e^{-\lambda}$ and the one real root of $\det \Delta(\lambda) = 0$ is 0 and this root is a double root. Other roots are all simple and have nonzero imaginary parts. These approximate root values are:

$$\lambda_1 = 0$$
$$\lambda_2, \lambda_3 = -2.0884 \pm 7.46149\, i$$
$$\lambda_4, \lambda_5 = -2.66407 \pm 13.8791\, i$$
$$\lambda_6, \lambda_7 = -3.0263 \pm 20.2238\, i$$
$$\lambda_8, \lambda_9 = -3.29168 \pm 26.5432\, i$$
$$\lambda_{10}, \lambda_{11} = -3.50127 \pm 32.8507\, i$$

etc.

The corresponding eigenspaces are

$$M_{\lambda_1} = \text{span}\{1, s\}, \quad M_{\lambda_n} = \text{span}\{e^{\lambda_n s}\}, \quad n = 2,3,\cdots.$$

Note that $\dim M_{\lambda_1} = 2$. If $p = 2$, $y_{0,1} = (1, 0)$, $y_{0,2} = (0, s)$, then the condition (4.8) in Theorem 4.1 is satisfied.

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