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Construction of Multivariable Schwarz-form Realizations via Orthogonal Polynomial Matrices

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I. Introduction

It is well-known that a continuous-time, time-invariant linear scalar system $H(s)=b(s)/a(s)$, where $a(s)$ and $b(s)$ are polynomials with $\deg a(s) = n$ and $\deg b(s) \leq n-1$, has the Schwarz-form realization (A,B,C) as follows.

$$A = \begin{bmatrix} 0 & 1 & & & & \\ -f_1 & 0 & 1 & & 0 & \\ & \ddots & \ddots & \ddots & & \\ & 0 & -f_{n-2} & 0 & 1 \\ & & & -f_{n-1} & -f_n \end{bmatrix} : n \times n \qquad B = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} : n \times 1$$

$$C : 1 \times n$$

A is determined by a(s) and called the Schwarz matrix of a(s) after [1]. Deriving A from a(s) is equivalent to applying the stability test of Routh-Hurwitz to a(s), and $\{f_i\}$ are related to entries of the Routh table of a(s) and to the Hurwitz determinants of a(s) [2]. This relation leads to an outstanding feature of A as follows: A is stable, (i.e., all eigenvalues of A have negative real parts,) if and only if all $\{f_i\}$ are positive. Another important property of A is that the Lyapunov equation $AX+XA'+BB'=0$ has a diagonal solution $X=\mathrm{diag}\{\delta_0,\cdots\cdots,\delta_{n-1}\}$, where $\{\delta_i\}$ are defined by

$$\delta_{n-1}=1/2f_n, \quad \delta_{i-1}=\delta_i/f_i \quad \text{for } i=1,2,\cdots,n-1.$$

This property also yields the same stability criterion of A as above, and thus the Routh-Hurwitz test is linked to the Lyapunov equation via A [2]. Moreover, the Schwarz-form realization is used for some lower order approximation methods including the Routh approximation [3].

The above situations, however, cannot be extended to multivariable systems. To construct the Schwarz-form realization of a prescribed multivariable system written in a matrix fraction description $H(s)=N(s)D(s)^{-1}$, where N(s) and D(s) are polynomial matrices, we need a polynomial matrix version of the Routh-Hurwitz test. But the results on the stability of polynomial matrices which have been obtained so far, such as [4], cannot be applied to our problem.

In this paper, we introduce the notion of orthogonal polynomial matrices to study the stability of D(s) and to derive the Schwarz-form realization of H(s).

We proceed as follows. In section II, we define a basis of polynomial matrices suitable to treat D(s), and derive the block-companion-form realization of H(s). A solution of a Lyapunov equation derived from this realization is used to define a matrix-valued inner product of polynomial matrices in section III. In section IV, we introduce an orthogonal system of polynomial matrices, which is related to the stability of D(s). An efficient algorithm for constructing the orthogonal polynomial matrices is presented. In the scalar case, this algorithm amounts to the reversed procedure of the Routh-Hurwitz test. By means of this algorithm, we derive the Schwarz-form realization of H(s) in section V.

We refer to [6] for fundamental arguments on multivariable systems.

II. Mathematical Preliminaries

Let H(s) be a strictly proper rational matrix of size q×p, which expresses the tranfer function matrix of a p-input-q-output, continuous-time, and time-invariant linear system. Suppose that H(s) is written in a right *matrix fraction description* (MFD) such as

$$H(s) = N(s)\ D(s)^{-1} \tag{2.1}$$

where N(s) and D(s) are polynomial matrices of sizes q×p and p×p, respectively. We further assume that D(s) is column-reduced (or column-proper), which can always be attained by obtaining, if necessary, another right MFD by multiplying both N(s) and D(s) from the right by an appropriate unimodular polynomial matrix [6].

Let m_i (i=1,2,····,p) denote the ith column-degree, i.e., the highest degree of all the polynomials in the ith column, of D(s). We can assume without loss of generality that

$$m \triangleq m_1 \geq m_2 \geq \cdots\cdots \geq m_p \geq 1. \tag{2.2}$$

The column-reducedness of D(s) is expressed as

$$n \triangleq \deg \det D(s) = \sum_{i=1}^{p} m_i. \tag{2.3}$$

Note that n amounts to the McMillan degree of H(s) if and only if the MFD (2.1) is irreducible.

For each j=0,1,····,m, we define an integer r(j) by

$$r(j) \triangleq \mathrm{Max}\ \{r \mid 1 \leq r \leq p \text{ and } m_r \geq m-j\ \}.$$

For example, if

$$p=4, \quad (m=)m_1=4, \quad m_2=m_3=2, \quad m_4=1,$$

then

$$r(0)=r(1)=1, \quad r(2)=3, \quad r(3)=r(4)=4.$$

In general, it is clear that

$$1 \leqq r(0) \leqq r(1) \leqq \cdots\cdot \leqq r(m-1) = r(m) = p$$

holds. It also follows from (2.2) that

$$m_i \geqq m-j \quad \text{if and only if} \quad i \leqq r(j)$$

for $1\leqq i\leqq p$ and $0\leqq j\leqq m$, from which we have

$$\sum_{j=0}^{m-1} r(j) = n \tag{2.4}$$

and

$$m_i = m - \mathrm{Min}\,\{j \mid 0\leqq j\leqq m-1 \text{ and } r(j)\geqq i\}. \tag{2.5}$$

Note that

$$r(j) = p \quad \text{for } \forall j \quad \text{if and only if} \quad m_i = m \quad \text{for } \forall i.$$

Let, for $j=0,1,\cdots,m$,

$$T_j(s) \triangleq \left[\begin{array}{cccc|c} s^j & & & & \\ & s^{j-m+m_2} & & & \\ & & \ddots & & O \\ & & & s^{j-m+m_{r(j)}} & \end{array}\right] \quad : r(j)\times p.$$

Note that

$$sT_j(s) = \Lambda_j \; T_{j+1}(s) \tag{2.6}$$

where

$$\Lambda_j \triangleq \left(I_{r(j)} \;\middle|\; 0 \right) \quad : r(j)\times r(j+1).$$

Suppose that a k×p polynomial matrix P(s) is written as

$$P(s) = \sum_{j=0}^{d} P_j \; T_j(s) \tag{2.7}$$

where d is an integer in $0 \le d \le m$, and P_j is a $k\times r(j)$ constant matrix. Then, the ith column-degree of P(s) is less than or equal to $d-m+m_i$, where a negative column-degree means that all the elements in the corresponding column of P(s) are 0. Conversely, any k×p polynomial matrix P(s), with the ith column-degree less than or equal to $d-m+m_i$ for $i=1,2,\cdots,p$, can always be written as (2.7). For convinience, we call d in (2.7) the *degree* of P(s) and write d=deg P(s), when $P_d \ne 0$. Especially if $P_d = I_{r(d)}$, P(s), consequently being of size r(d)×p, is said to be *monic*.

For example, since the ith column-degree of D(s) is m_i, D(s) is represented as

$$D(s) = \sum_{j=0}^{m} D_j \; T_j(s) \tag{2.8}$$

where D_j is a $p\times r(j)$ matrix. Moreover, the column-reducedness of D(s), (2.3), is equivalent to the nonsingularity of D_m. Hence, we can define a monic polynomial matrix $\bar{D}(s)$ by

$$\bar{D}(s) = D_m^{-1} \; D(s) = T_m(s) + \sum_{j=0}^{m-1} \bar{D}_j \; T_j(s)$$

where

$$\bar{D}_j \triangleq D_m^{-1} \; D_j .$$

On the other hand, from the strict properness of $H(s)=N(s)D(s)^{-1}$, the ith column-degree of $N(s)$ is less than m_i. Therefore, $N(s)$ is represented as

$$N(s) = \sum_{j=0}^{m-1} N_j \; T_j(s) \tag{2.9}$$

where N_j is a $q\times r(j)$ matrix.

Now, let

$$A \triangleq \left[\begin{array}{cc|c|ccc} O & \Lambda_0 & & & & \\ \hline & & \Lambda_1 & & & \\ & & & \ddots & & \\ & & & & & \Lambda_{m-2} \\ \hline -\bar{D}_0 & -\bar{D}_1 & \cdots & \cdots & \cdots & -\bar{D}_{m-1} \end{array}\right] \quad : n\times n$$

$$B \triangleq \left[\; O \;\middle|\; D_m'^{-1} \;\right]' \quad : n\times p$$

$$C \triangleq \left[\; N_0 \;\middle|\; N_1 \;\middle|\; \cdots \;\middle|\; N_{m-1} \;\right] \quad : q\times n$$

$$T(s) \triangleq \left[\; T_0'(s) \;\middle|\; T_1'(s) \;\middle|\; \cdots \;\middle|\; T_{m-1}'(s) \;\right]' \quad : n\times p.$$

Then, it follows from (2.6),(2.8) and (2.9) that

$$(\, sI_n - A \,)^{-1} \; B = T(s) \; D(s)^{-1} \tag{2.10}$$

and that

$$C \; T(s) = N(s). \tag{2.11}$$

Hence, we have

$$H(s) = C \; (sI_n - A)^{-1} \; B,$$

i.e., (A,B,C) is a realization of H(s). We call (A,B,C) the *controller block-companion-form realization* of H(s) or, more precisely, of the MFD (2.1). It is clear that (A,B) is controllable and has controllability indices $\{m_i\}$. In fact, (A,B,C) can be obtained immediately from the well-known controller-form realization [6] of the MFD (2.1), by a permutation on the ordering of state variables.

Next, we consider the following Lyapunov equation:

$$A X + X A' + B \Pi B' = 0 \tag{2.12}$$

where Π is a given p×p positive definite matrix and X is an unknown n×n matrix. From now on, we assume that a symmetric solution X of (2.12) has been obtained. It should be noted that the following three statements are equivalent.

(i) X is positive definite.

(ii) A is a stable matrix; i.e., all eigenvalues of A have negative real parts.

(iii) D(s) is a stable polynomial matrix; i.e., all zeros of det D(s) have negative real parts.

In the above case, X can be represented as

$$X = \int_0^\infty e^{tA} B \Pi B' e^{tA'} dt. \tag{2.13}$$

Let X be partitioned into blocks as

$$X = \begin{pmatrix} X_{0,0} & X_{0,1} & \cdots & X_{0,m-1} \\ X_{1,0} & \cdots & & \vdots \\ \vdots & & & \vdots \\ X_{m-1,0} & \cdots & & X_{m-1,m-1} \end{pmatrix}$$

where the size of $X_{i,j}$ is $r(i)\times r(j)$. Note that $X_{i,j}=X'_{j,i}$ holds for every i,j, from the symmetricity of X. Then, the equation (2.12) is represented as follows:

$$\Lambda_i\, X_{i+1,j} + X_{i,j+1}\, \Lambda'_j = 0 \qquad (0\le i,j\le m-2), \tag{2.14}$$

$$\sum_{i=0}^{m-1} \bar{D}_i\, X_{i,j} = X_{m-1,j+1}\, \Lambda'_j \qquad (0\le j\le m-2), \tag{2.15}$$

$$\sum_{i=0}^{m-1} \bar{D}_i\, X_{i,m-1} + \sum_{i=0}^{m-1} X_{m-1,i}\, \bar{D}'_i = \bar{\Pi}, \tag{2.16}$$

where $\bar{\Pi} = D_m^{-1}\,\Pi\, D_m'^{-1}$. For the later arguments, we further assume that, for $k=0,1,\cdots,m-1$,

$$X_k = \begin{pmatrix} X_{0,0} & \cdots & X_{0,k} \\ \vdots & & \vdots \\ X_{k,0} & \cdots & X_{k,k} \end{pmatrix} : \{\sum_{j=0}^{k} r(j)\}\times\{\sum_{j=0}^{k} r(j)\}$$

is nonsingular. (2.17)

Clearly, this assumption is satisfied in the case (i)-(iii) described above.

III. Inner Product of Polynomial Matrices

Given two polynomial matrices P(s) and Q(s), with their degrees less than or equal to m-1, such as

$$\begin{aligned} P(s) &= \sum_{j=0}^{m-1} P_j\, T_j(s) = \left(P_0 \middle| P_1 \middle| \cdots \middle| P_{m-1}\right) T(s) \quad : k\times p \\ Q(s) &= \sum_{j=0}^{m-1} Q_j\, T_j(s) = \left(Q_0 \middle| Q_1 \middle| \cdots \middle| Q_{m-1}\right) T(s) \quad : \ell\times p, \end{aligned} \tag{3.1}$$

we define the *inner product* of them by

$$\begin{aligned} \langle P(s),Q(s)\rangle &= \left(P_0 \middle| P_1 \middle| \cdots \middle| P_{m-1}\right) X \left(Q_0 \middle| Q_1 \middle| \cdots \middle| Q_{m-1}\right)' \\ &= \sum_{i=0}^{m-1}\sum_{j=0}^{m-1} P_i\, X_{i,j}\, Q_j' \qquad : k\times\ell. \end{aligned}$$

Especially, we have

$$\langle T_i(s),T_j(s)\rangle = X_{i,j}.$$

It is obvious that the following properties hold:

- $\langle P(s)+Q(s)\,,\,R(s)\rangle = \langle P(s),R(s)\rangle + \langle Q(s),R(s)\rangle,$ (3.2)

- $\langle K\,P(s)\,,\,L\,Q(s)\rangle = K\,\langle P(s),Q(s)\rangle\,L',$ (3.3)

- $\langle P(s),Q(s)\rangle = \langle Q(s),P(s)\rangle',$ (3.4)

where polynomial matrices P(s),Q(s),R(s) and constant matrices K,L are assumed to have suitable sizes. Furthermore, from (2.14),

- $\langle s\,P(s)\,,\,Q(s)\rangle + \langle P(s)\,,\,s\,Q(s)\rangle = 0$ (3.5)

holds if $\deg P(s) \leq m-2$ and $\deg Q(s) \leq m-2$. This property will play an important role in the later arguments.

Consider the case that $D(s)$ is a stable polynomial matrix. Let $h_D(t) : p\times p$ be the impulse response matrix of $D(s)^{-1}$. Then, for $P(s)$ and $Q(s)$ written as (3.1), we have from (2.10)

$$
\begin{aligned}
P(\tfrac{d}{dt})\ h_D(t) &= \left[P_0 \middle| P_1 \middle| \cdots \middle| P_{m-1}\right] e^{tA}\ B \\
Q(\tfrac{d}{dt})\ h_D(t) &= \left[Q_0 \middle| Q_1 \middle| \cdots \middle| Q_{m-1}\right] e^{tA}\ B
\end{aligned}
\qquad (t > 0) \tag{3.6}
$$

Since X is represented as (2.13) under the stability assumption, we obtain from (3.6) the following *time-domain expression* of the inner product:

$$
<P(s),Q(s)> = \int_0^{\infty} \{P(\tfrac{d}{dt})h_D(t)\}\ \Pi\ \{Q(\tfrac{d}{dt})h_D(t)\}'dt. \tag{3.7}
$$

The corresponding *frequency-domain expression* is as follows:

$$
<P(s),Q(s)> = \frac{1}{2\pi}\int_{-\infty}^{\infty} P(j\omega)\ D^{-1}(j\omega)\ \Pi\ D'^{-1}(-j\omega)\ Q'(-j\omega)\ d\omega. \qquad (j^2 = -1) \tag{3.8}
$$

It should be remarked that the meaning of the property (3.5) becomes clear in (3.7) and in (3.8).

So far, the inner product has been defined only for polynomial matrices having degrees less than or equal to $m-1$. Next we consider extending the inner product so that $<P(s),Q(s)>$ is defined even if one of $\{P(s),Q(s)\}$ has degree m. Of course, such an extension is not unique. However, if $D(s)$ is stable, a natural extension can be obtained by means of (3.7) as follows. Suppose that $\deg P(s) = m$ and that $\deg Q(s) \leq m-1$. Then $P(\frac{d}{dt})h_D(t)$ has a δ-function-like singularity at $t=0$, while $Q(\frac{d}{dt})h_D(t)$ does not have such a singularity, although possibly being discontinuous at $t=0$. Hence, replacing $\int_0^\infty$ by $\int_{0-}^\infty$ in (3.7), and using the formula

$$\int_{0-}^{\infty} \delta(t)\, f(t)\, dt = \frac{1}{2}\{\, f(0+) + f(0-)\,\},$$

we can obtain a finite value of (3.7). For instance, consider the case $P(s)=\bar{D}(s)$. From the definition of $h_D(t)$, we have

$$\bar{D}(\tfrac{d}{dt})h_D(t) = D_m^{-1}\,\delta(t),$$

which yields

$$\begin{aligned} \langle \bar{D}(s), Q(s)\rangle &= \int_{0-}^{\infty} D_m^{-1}\,\delta(t)\,\Pi\,\{Q(\tfrac{d}{dt})h_D(t)\}' dt \\ &= \frac{1}{2}\, D_m^{-1}\,\Pi\,\{Q(\tfrac{d}{dt})h_D(0+)\}'. \end{aligned}$$

Note that $Q(\frac{d}{dt})h_D(t)=0$ for $\forall t<0$. Assuming that $Q(s)$ is written as (3.1), we obtain from (3.6)

$$\begin{aligned} Q(\tfrac{d}{dt})h_D(0+) &= \left(Q_0 \middle| Q_1 \middle| \cdots \middle| Q_{m-1}\right) B \\ &= Q_{m-1}\, D_m^{-1}. \end{aligned}$$

Therefore we have

$$\langle \bar{D}(s), Q(s)\rangle = \frac{1}{2}\,\bar{\Pi}\, Q'_{m-1}.$$

Thus, the following important equation is obtained:

$$\langle \bar{D}(s), T_j(s) \rangle = \begin{cases} 0, & \text{for } j=0,1,\cdots,m-2, \\ \frac{1}{2}\,\bar{\Pi}, & \text{for } j=m-1. \end{cases} \tag{3.9}$$

It follows immediately from (3.9) that

$$\langle T_m(s), T_j(s) \rangle = \begin{cases} -\sum_{i=0}^{m-1} \bar{D}_i X_{i,j} = -X_{m-1,j+1} \Lambda_j', \quad \text{(see (2.15} \\ \qquad \text{for } j=0,1,\cdots,m-2, \\ \frac{1}{2}\bar{\Pi} - \sum_{i=0}^{m-1} \bar{D}_i X_{i,m-1}, \quad \text{for } j=m-1. \end{cases} \tag{3.10}$$

Clearly, (3.2)-(3.4) still hold for the extended inner product. (3.5) also holds if $\deg P(s) \leqq m-1$ and $\deg Q(s) \leqq m-1$, which can be proved by applying an integration by parts to (3.7). It should be noted that the same extension as shown above can be obtained also by means of (3.8).

Remark: Replacing $\int_0^\infty$ by $\int_{0+}^\infty$ in (3.7), we obtain another extension of the inner product. On this extension, $\bar{D}(s)$ is orthogonal to all the polynomial matrices with degrees less than or equal to m-1 (cf. (3.9)), and (3.5) does not hold when $\deg P(s) = \deg Q(s) = m-1$.

In the general case without the stability assumption on D(s), we choose to extend the inner product by (3.10), preserving the properties (3.2)-(3.4). Then, it is obvious that (3.9) holds. Moreover, it follows from (2.15) and (2.16) that (3.5) holds if $\deg P(s) \leqq m-1$ and $\deg Q(s) \leqq m-1$.

IV. Orthogonal Polynomial Matrices

In this section, we consider polynomial matrices $\{R_j(s) ; 0 \leq j \leq m\}$ such that

(i) $R_j(s)$ is of degree j and monic; i.e., it can be written as

$$R_j(s) = T_j(s) + \sum_{i=0}^{j-1} R_{j,i} T_i(s) \tag{4.1}$$

where $R_{j,i}$ is a $r(j)\times r(i)$ matrix;

(ii) $\langle R_j(s), T_i(s)\rangle = 0, \quad$ for $i=0,1,\cdots,j-1$.

It follows immediately that

$$\langle R_i(s), R_j(s)\rangle = 0, \quad \text{if } i \neq j; \tag{4.2}$$

i.e., $\{R_j(s)\}$ constitute an orthogonal system.

In terms of the coefficient matrices of $R_j(s)$ in (4.1), (ii) is represented by the following matrix equation:

$$\left(R_{j,0}\middle|\cdots\cdots\middle|R_{j,j-1}\right) X_{j-1} + \left(X_{j,0}\middle|\cdots\cdots\middle|X_{j,j-1}\right) = 0 \tag{4.3}$$

where $X_{m,i}$ $(0 \leq i \leq m-1)$ is defined additionally, according to the extension of the inner product, by

$$X_{m,i} \triangleq \langle T_m(s), T_i(s)\rangle.$$

We can see from (4.3) that, on the assumption (2.17), $\{R_j(s)\}$ satisfying (i) and (ii) always exist and are uniquely determined.

For $j=0,1,\cdots,m-1$, let

$$\Delta_j \triangleq \langle R_j(s),R_j(s)\rangle \quad : r(j)\times r(j) \text{ symmetric},$$

(Note that $\Delta_m=\langle R_m(s),R_m(s)\rangle$ cannot be defined.)

and let

$$\tilde{R}_j \triangleq \begin{pmatrix} I_{r(0)} & & & \\ R_{1,0} & I_{r(1)} & & \\ \vdots & \vdots & \ddots & \\ R_{j,0} & R_{j,1} & \cdots & I_{r(j)} \end{pmatrix}$$

$$: \{\sum_{i=0}^{j} r(i)\}\times\{\sum_{i=0}^{j} r(i)\}.$$

Then, from (4.2) we have

$$\tilde{R}_j \; X_j \; \tilde{R}_j' = \text{block diag } \{\Delta_0,\Delta_1,\cdots,\Delta_j\}. \tag{4.4}$$

It can be seen from (2.17) and (4.4) that all $\{\Delta_j\}$ are nonsingular. Moreover, since the stability of D(s) is equivalent to the positive definiteness of $X=X_{m-1}$, we obtain from (4.4) the following proposition.

Proposition 1

D(s) is stable, if and only if all $\{\Delta_j\}$ are positive definite.

Because of the condition (i), we can adopt $\{R_j(s)\}$ as a basis of polynomial matrices instead of $\{T_j(s)\}$. For instance, N(s) can be written as

$$N(s) = \sum_{j=0}^{m-1} \hat{N}_j R_j(s) \tag{4.5}$$

where $\hat{N}_j$ is a $q\times r(j)$ matrix (cf. (2.9)). Similarly, D(s) and $\bar{D}(s)$ ought to be represented by linear conbinations of $\{R_j(s)\ ;\ 0 \leq j \leq m\}$. In fact, the following equation holds.

Proposition 2

$$\bar{D}(s) = R_m(s) + \frac{1}{2} \bar{\Pi} \Delta_{m-1}^{-1} R_{m-1}(s) \tag{4.6}$$

Proof: $\bar{D}(s)$ is monic polynomial matrix of degree m satisfying (3.9). But such a polynomial matrix is unique, because of the nonsingularity of X assumed in (2.17). Thus, the above equation is readily proved by verifying that the right-hand side satisfies the same equation as (3.9).

In order to obtain $\{R_j(s)\}$, we may solve the linear equations (4.3) by a standard method, separately for $j=0,1,\cdots,m$. However, owing to the special property (3.5) of the inner product, there exists a recursive and more efficient algorithm to obtain $\{R_j(s)\}$, as presented below. In general, $\{R_j(s)\}$ are not sufficient to form a recursion. Hence, in the algorithm, we shall introduce new polynomial matrices $\{U_j(s)\ ;\ 0 \leq j \leq m\}$ with $U_j(s) : \{p-r(j)\}\times p$ and shall accomplish a recursion on

$$\left[\frac{R_j(s)}{U_j(s)}\right] \quad : p \times p, \quad j = 0, 1, \cdots\cdot, m.$$

Theorem 1

$\{R_j(s)\}$ are obtained by the following recursive algorithm.

Initialization: Set

$$\left[\frac{R_0(s)}{U_0(s)}\right]\begin{matrix}\}\ r(0) \\ \}\ p-r(0)\end{matrix} = I_p .$$

Recursion: Compute the followings, successively for $j=0,1,\cdots\cdot,m-1$.

$$\Delta_j = \langle R_j(s), R_j(s)\rangle \quad : r(j)\times r(j) \tag{4.7}$$

$$\Gamma_j = \langle s\,R_j(s), R_j(s)\rangle \quad : r(j)\times r(j) \tag{4.8}$$

$$\Theta_j = \langle U_j(s), R_j(s)\rangle \quad : \{p-r(j)\}\times r(j) \tag{4.9}$$

$$E_j = \Gamma_j\ \Delta_j^{-1} \quad : r(j)\times r(j) \tag{4.10}$$

$$F_j = \Delta_j\ \Lambda'_{j-1}\ \Delta_{j-1}^{-1} \quad : r(j)\times r(j-1) \tag{4.11}$$

$$G_j = \Theta_j\ \Delta_j^{-1} \quad : \{p-r(j)\}\times r(j) \tag{4.12}$$

$$\left[\frac{R_{j+1}(s)}{U_{j+1}(s)}\right]\begin{matrix}\}\ r(j+1) \\ \}\ p-r(j+1)\end{matrix} = \left[\frac{(sI_{r(j)}-E_j)R_j(s) + F_jR_{j-1}(s)}{U_j(s) - G_jR_j(s)}\right], \tag{4.13}$$

with the exception that (4.11) and the term '$+F_jR_{j-1}(s)$' in (4.13) are omitted when $j=0$.

Remark 1: On the assumption (2.17), the algorithm can always be carried out because of the nonsingularity of every Δ_j.

Remark 2: When $r(j)=p$ holds, we can do without $U_j(s) : 0\times p$ and omit (4.9),(4.12) and the term '$U_j(s)-G_jR_j(s)$' in (4.13).

Remark 3: Because of (3.5), we have

$$\Gamma_j = -\langle R_j(s), s\,R_j(s)\rangle = -\Gamma_j' ;$$

i.e., Γ_j is anti-symmetric.

Proof of Theorem 1: Together with (i),(ii), we consider the following statements.

(iii) $U_j(s)$ can be written as

$$U_j(s) = \Xi_j + \sum_{i=0}^{j-1} U_{j,i}\, T_i(s),$$

where $U_{j,i}$ is a $\{p-r(j)\}\times r(i)$ matrix and

$$\Xi_j \triangleq \left[\, 0 \;\middle|\; I_{p-r(j)} \,\right] \quad : \{p-r(j)\}\times p.$$

(iv) $\langle U_j(s), T_i(s)\rangle = 0$, for $i=0,1,\cdots,j-1$.

We shall prove by induction on j that $R_j(s)$ and $U_j(s)$ obtained by the algorithm satisfy (i)-(iv) for $j=0,1,\cdots,m$. The case $j=0$ is trivial. Assume that (i)-(iv) hold for $j=0,1,\cdots,k$ $(0\leq k\leq m-1)$. Then, (i) and (iii) for $j=k+1$ can be easily verified from (4.13) by noting that

$$\left(\frac{s\,T_k(s)}{\Xi_k}\right) = \left(\frac{T_{k+1}(s)}{\Xi_{k+1}}\right). \qquad \text{(cf. (2.6))}$$

Let

$$Q_k(s) \triangleq \begin{cases} (s\,I_{r(0)} - E_0)\,R_0(s) & \text{if } k=0 \\ (s\,I_{r(k)} - E_k)\,R_k(s) + F_k\,R_{k-1}(s) & \text{if } k \geq 1, \end{cases}$$

$$V_k(s) \triangleq U_k(s) - G_k\,R_k(s).$$

Then, the verification of (ii) and (iv) for j=k+1 is reduced to that of

$$\langle Q_k(s), T_i(s)\rangle = 0 \tag{4.14}$$

and

$$\langle V_k(s), T_i(s)\rangle = 0. \tag{4.15}$$

for $i=0,1,\cdots,k$. From the property (3.5), we have

$$\langle Q_k(s), P(s)\rangle$$

$$= \begin{cases} -\langle R_0(s), s\,P(s)\rangle - E_0\,\langle R_0(s), P(s)\rangle & \text{if } k=0 \\ -\langle R_k(s), s\,P(s)\rangle - E_k\,\langle R_k(s), P(s)\rangle + F_k\,\langle R_k(s), P(s)\rangle & \text{if } k\geq 1 \end{cases} \tag{4.16}$$

for any polynomial matrix $P(s)$ with $\deg P(s) \leq m-1$. Since both $R_k(s)$ and $R_{k-1}(s)$ satisfy (ii) because of the induction hypothesis, it follows immediately from (4.16) that (4.14) holds for $i=0,1,\cdots,k-2$. (4.14) for $i=k-1$ is verified as

$$\langle Q_k(s), T_{k-1}(s)\rangle = -\langle R_k(s), s\,T_{k-1}(s)\rangle + F_k\langle R_{k-1}(s), T_{k-1}(s)\rangle$$

$$= -\Delta_k \Lambda'_{k-1} + F_k \Delta_{k-1}$$

$$= 0 \quad \text{(from (4.11)).}$$

It also follows from (4.16) that

$$\langle Q_k(s), R_k(s)\rangle = -\langle R_k(s), s\,R_k(s)\rangle - E_k\langle R_k(s), R_k(s)\rangle$$

$$= \Gamma_k - E_k\,\Delta_k$$

$$= 0 \quad \text{(from (4.10)),}$$

which clearly implies (4.14) for $i=k$. (4.15) for $i=0,1,\cdots,k-1$ is obvious from (ii) and (iv) for $j=k$ in the induction hypothesis, and (4.15) for $i=k$ comes immediately from (4.12). Thus, (ii) and (iv) for $j=k+1$ have been verified. (Q.E.D.)

Corollary

$$\begin{cases} \Lambda_0\,R_1(s) = (s\,I_{r(0)} - E_0)\,R_0(s) \\ \Lambda_j\,R_{j+1}(s) = (s\,I_{r(j)} - E_j)\,R_j(s) + F_j\,R_{j-1}(s) \quad (1 \leqq j \leqq m-1) \end{cases} \tag{4.17}$$

Proof: Obvious from (4.13).

The stability test of D(s) via the algorithm and Proposition 1 is nothing but an efficient test of the positive definiteness of X taking advantage of the special structure (2.14)-(2.16). This procedure requires only $O(pn^2)$ operations, while a standard method such as the Cholesky factorization requires $O(n^3)$ operations. We note that, in view of the computational efficiancy, it is favorable to replace (4.7)-(4.9) by the followings:

$$\begin{cases} \Delta_j = \langle R_j(s), T_j(s) \rangle & (4.7)' \\ \Gamma_j = \langle s\, T_j(s), R_j(s) \rangle + R_{j,j-1} \Lambda_{j-1} \Delta_j & (4.8)' \\ \Theta_j = \langle \Xi_j, R_j(s) \rangle, & (4.9)' \end{cases}$$

which are justified by (i)-(iv).

When D(s) is a regular polynomial matrix in the sense that it can be written as

$$D(s) = \sum_{j=0}^{m} D_j s^j$$

with $\det D_m \neq 0$, it follows that $m=m_1=m_2=\cdots=m_p$ and that $r(0)=r(1)=\cdots=r(m)=p$. In this case, $\{U_j(s)\}, \{\Theta_j\}$ and $\{G_j\}$ do not appear substatially in the algorithm, and the recursion becomes much simpler as follows:

$$\begin{cases} R_0(s) = I_p, \quad R_1(s) = s\, I_p - E_0, \\ R_{j+1}(s) = (s\, I_p - E_j)\, R_j(s) + F_j\, R_{j-1}(s). \end{cases}$$

$$(1 \leqq j \leqq m-1)$$

Moreover, when p=1, i.e., when D(s) is a scalar polynomial of degree n (=m), every Γ_j becomes 0 because of its anti-symmetricity, and we have

$$\begin{cases} R_0(s) = 1, \quad R_1(s) = s, \\ R_{j+1}(s) = s\, R_j(s) + (\Delta_j / \Delta_{j-1})\, R_{j-1}(s). \end{cases} \qquad (4.18)$$

$$(1 \leqq j \leqq n-1)$$

Combining (4.18) with (4.6), we can see that (4.18) is nothing but the procedure to construct the Routh array of D(s) in reverse order, and that Proposition 1 is equivalent to the Routh-Hurwitz criterion.

Thus, the results of this section might be regarded as a matrix version of the Routh-Hurwitz stability test. However, it should be noted that we cannot find $R_m(s)$ and $R_{m-1}(s)$ directly from D(s) in the matrix case, whereas we can do it in the scalar case by decomposing D(s) into the even power part and the odd power part. Therefore, the method for the scalar case which produces $\{R_j(s)\}$ by lowering degrees from $R_m(s)$ and $R_{m-1}(s)$ cannot be extended to the matrix case, and we must produce $\{R_j(s)\}$ in reverse order by means of the algorithm in Theorem 1, which calls for a solution of the Lyapunov equation (2.12).

We note that a similar situation appears in the stability theory of discrete-time systems concerning the Schur-Cohn stability test and the Levinson algorithm, as mentioned, for instance, in [5].

V. Schwarz-form Realization

Now, we are ready to derive the Schwarz-form realization of $H(s) = N(s)D(s)^{-1}$.

Theorem 2

Let

$$\hat{A} \triangleq \begin{bmatrix} E_0 & \Lambda_0 & & & & \\ -F_1 & E_1 & \Lambda_1 & & & \\ & \ddots & \ddots & \ddots & & \\ & & & -F_{m-2} & E_{m-2} & \Lambda_{m-2} \\ & & & & -F_{m-1} & -F_m + E_{m-1} \end{bmatrix} \quad : n\times n \tag{5.1}$$

where F_m is additionally defined by

$$F_m \triangleq \frac{1}{2} \bar{\Pi} \, \Delta_{m-1}^{-1}, \tag{5.2}$$

and let

$$\hat{B} \triangleq \left[\; O \; \middle| \; D_m'^{-1} \; \right]' \quad : n\times p \tag{5.3}$$

$$\hat{C} \triangleq \left[\; \hat{N}_0 \; \middle| \; \hat{N}_1 \; \middle| \; \cdots\cdot \; \middle| \; \hat{N}_{m-1} \; \right] \quad : q\times n. \tag{5.4}$$

(see (4.5).)

Then it holds that

$$\left\{ \; \hat{A} = \tilde{R}_{m-1} \, A \, \tilde{R}_{m-1}^{-1} \right. \tag{5.5}$$

$$\left\{\begin{array}{ll} \hat{B} = \tilde{R}_{m-1}\ B & (5.6) \\ \hat{C} = C\ \tilde{R}_{m-1}^{-1}. & (5.7) \end{array}\right.$$

Therefore, $(\hat{A},\hat{B},\hat{C})$ is a realization of $H(s)$, which is similar to (A,B,C).

Proof: Let

$$\tilde{R}(s) \triangleq \tilde{R}_{m-1}\, T(s) = \left(R_0'(s) \,\middle|\, R_1'(s) \,\middle|\, \cdots \,\middle|\, R_{m-1}'(s) \right)' \ : n\times p.$$

Then it follows from (4.6) and (4.17) that

$$(s\, I_n - \hat{A})\, \tilde{R}(s) = \hat{B}\, D(s), \tag{5.8}$$

while it follows from (2.10) that

$$(s\, I_n - \tilde{R}_{m-1}\, A\, \tilde{R}_{m-1}^{-1})\, \tilde{R}(s) = \tilde{R}_{m-1}\, B\, D(s). \tag{5.9}$$

Comparing (5.8) with (5.9), we have (5.5) and (5.6). (5.7) comes immediately from (2.9) and (4.5).

Corollary

Let

$$\hat{X} \triangleq \text{block diag } \{\Delta_0, \Delta_1, \cdots, \Delta_{m-1}\} \quad : n\times n.$$

Then

$$\hat{A}\, \hat{X} + \hat{X}\, \hat{A}' + \hat{B}\, \Pi\, \hat{B}' = 0. \tag{5.10}$$

Proof: Obvious from (2.12),(4.4),(5.5) and (5.6).

We call $(\hat{A},\hat{B},\hat{C})$ the *controller Schwarz-form realization* (CSR) of H(s). Note that $(\hat{A},\hat{B},\hat{C})$ depends both on the choice of a right MFD (2,1) of H(s) and on the choice of a $p\times p$ positive definite matrix Π. If we start from a left MFD and a $q\times q$ positive definite matrix instead of (2.1) and Π, we can obtain the *observer Schwarz-form realization* (OSR) of H(s) in a similar way.

Digressing from the realization problem of H(s), we can see the properties of $\hat{A}$ from a matrix-theoretic viewpoint, as follows. Suppose that symmetric matrices $\{\Delta_j\}$, anti-symmetric matrices $\{\Gamma_j\}$, and a positive definite matrix $\bar{\Pi}$ are given initially, and that $\hat{A}$ is defined from these matrices by (5.1),(4.10),(4.11) and (5.2). We call $\hat{A}$ the *block-Schwarz matrix generated from* $\{\Delta_j\},\{\Gamma_j\}$ and $\bar{\Pi}$. Now, given a nonsingular matrix D_m, let $\hat{B}$ be defined by (5.3) and let $\Pi \triangleq D_m \bar{\Pi} D_m'$. Then, it is clear that $(\hat{A},\hat{B})$ is controllable, and we can verify the Lyapunov equation (5.10) by direct calculations. Hence, the block-Schwarz matrix $\hat{A}$ is stable if and only if all $\{\Delta_j\}$ are positive definite. In the scalar case, especially, it follows from the anti-symmetricity of Γ_j that $E_j=0$ for every j, and we can see that $\hat{A}$ becomes the well-known Scwarz matrix with the stability criterion

$$F_j > 0 \quad \text{for } j=1,2,\cdots,m.$$

VI. Conclusions

Starting from a given MFD $H(s)=N(s)D(s)^{-1}$, a matrix-valued inner product of polynomial matrices has been defined, and an efficient algorithm for constructing orthogonal polynomial matrices has been presented. This algorithm is regarded as a polynomial matrix version of the reversed procedure of the Routh-Hurwitz stability test for scalar polynomials. Using these results, we have derived the Schwarz-form realization of H(s) and have investigated its properties.

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