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# ON $H_\infty$ WELL POSEDNESS OF THE CAUCHY PROBLEM FOR SHRÖDINGER TYPE EQUATIONS

(Schrödinger型方程式に対する初期値問題の$H_\infty$適切性について)

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## 0. Introduction

In the present paper we consider the Cauchy problem for the equation

$$(0.1)\quad Lu(x,t) \equiv (i\partial_t + \tau\Delta + \sum_{j=1}^{m} b_j(x)\partial_{x_j} + c(x))u(x,t) = f(x,t)$$

in $R_x^m \times [0,T]$ with initial data $u_0(x)$ at $t = 0$, where $\tau$ is a real constant, and $b_j(x)$, $c(x)$ are $C^\infty$-functions whose derivatives of any order are all bounded.

If $\tau$ is a non-zero constant, the above equation (0.1) is the typical equation of non-kowalewskian type which is not parabolic. Hence, the study of the equation (0.1) is important for the study of the equations of general non-kowalewskian type.

The Cauchy problem for (0.1) was studied in the frame of $L^2$ space by J. Takeuchi [8] and S. Mizohata [7]. On the other hand, in the present paper we study in the frame of $H_\infty$ space. We note that studying (0.1) in the frame of $H_\infty$ space corresponds to studying equations of kowalewskian type in the frame of $\mathcal{E}$ space, where $\mathcal{E}$ is the usual space of $C^\infty$-functions.

In section 1 we state a sufficient condition (Theorem 1) and a necessary condition (Theorem 2) for equation (0.1) to be well posed in $H_\infty$ space, and some Remarks. Theorem 1 and Theorem 2 will be proved in section 2 and section 3, respectively.

1. Theorems

For real $s$ let $H_s$ be the Sobolev space with the usual norm $||\cdot||_s$ and let $H_\infty \equiv \bigcap_{s\in R} H_s$ be the Fréchet space with semi-norms $||\cdot||_s$ $(s = 0,\pm 1,\pm 2,\cdots)$. We say that the equation (0.1) is well posed in $H_\infty$ space on the interval $[0,T]$ $(T \neq 0)$, if for any $u_0(x) \in H_\infty$ and $f(x,t) \in \mathcal{E}_t^0([0,T];H_\infty)$ there exists a unique solution $u(x,t) \in \mathcal{E}_t^0([0,T];H_\infty)$ of (0.1) and moreover for any real constant $s$ there exist a real constant $s'$ and a constant $C_{s,s'}(T) > 0$ such that the energy inequality

$$||u(\cdot,t)||_s \leqq C_{s,s'}(T)\{||u_0(\cdot)||_{s'} + \int_0^t ||f(\cdot,\theta)||_{s'}d\theta\}$$

holds. Here, for Banach or Fréchet space $F$ $g(x,t) \in \mathcal{E}_t^0([0,T];F)$ means that the mapping : $[0,T] \ni t \longrightarrow g(\cdot,t) \in F$ is continuous in the topology of $F$.

Our aim is to prove the following two theorems. In the first theorem we consider the equation (0.1) with $\tau = 1$.

Theorem 1 ([2]). We assume that there exist constants $M$ and $N$ such that

(1.1) $$\sup_{x\in R^m,\omega\in S^{m-1}} \left|\sum_{j=1}^{m} \int_0^\rho \operatorname{Re} b_j(x + 2\theta\omega)\omega_j d\theta\right| \leqq M\log(1 + \rho) + N$$

hods for any positive $\rho$, where $S^{m-1}$ denotes the unit sphere in $R^m$. Then, we obtain

(i) The case $m = 1$. For any real $T \neq 0$ the equation (0.1) with $\tau = 1$ is well posed in $H_\infty$ space on $[0,T]$.

(ii) The case $m \geqq 2$. If besides (1.1) we assume the following (1.2) and (1.3), for any real $T \neq 0$ the equation (0.1) with $\tau = 1$ is well posed in $H_\infty$ space on $[0,T]$.

(1.2) $$\sup_{x\in R^m, \omega\in S^{m-1}} \sum_{j=1}^{m} \int_0^\infty |\partial_x^\alpha b_j(x + 2\theta\omega)| d\theta < \infty$$

holds for any multi-index $\alpha$ which is not zero.

(1.3) $$\sup_{S \in \mathcal{J}} \left| \sum_{i<j} \iint_S (\partial_{x_i} \operatorname{Re} b_j - \partial_{x_j} \operatorname{Re} b_i) dx_i \wedge dx_j \right| < \infty$$

holds, where $\mathcal{J}$ is the family of all triangles in $R^m$ and $\iint_S (\cdots) dx_i \wedge dx_j$ denotes the integral of two form over $S$.

Next, we give a necessary condition.

Theorem 2 ([3]). Assume that there exists a real constant $T \neq 0$ such that for any $u_0(x) \in H_\infty$ there exist a unique solution $u(x,t) \in \mathcal{E}_t^0([0,T];H_\infty)$ of the equation

(0.1)′ $Lu(x,t) = 0, \quad u(x,0) = u_0(x)$.

Then, we can find constants $M$ and $N$ such that

(1.4) $$\sup_{x\in R^m, \omega\in S^{m-1}} \left| \sum_{j=1}^{m} \int_0^\rho \operatorname{Re} b_j(x + 2\tau\theta\omega)\omega_j d\theta \right| \leqq M \log(1 + \rho) + N$$

holds for any positive $\rho$.

Remark 1. If $\tau = 1$, the inequality (1.4) coincides with (1.1).

So, If m = 1, the condition (1.1) is necessary and sufficient for the equation (0.1) with $\tau = 1$ to be well posed in $H_\infty$ space on $[0,T]$ for any T.

Remark 2. When $\tau$ equals zero, the equation (0.1) is kowalewskian. Then, we remark that Theorem 2 gives the so-called Lax-Mizohata theorem (Lax [5], Mizohata [6]).

## 2. Proof of Theorem 1

We use the calculations of the new type with respect to the pseudo-differential operators for the proof of Theorem 1.

$S^{\ell}_{0,0}$ denotes the set of $C^\infty$-functions $p(x,\xi)$ such that for any multi-indices $\alpha$ and $\beta$ we have

$$|\partial_\xi^\alpha \partial_x^\beta \, p(x,\xi)| \leqq C_{\alpha,\beta}(1 + |\xi|)^\ell$$

for positive constants $C_{\alpha,\beta}$. We define the pseudo-differential operator $P = p(x,D_x)$ with the symbol $\sigma(P)(x,\xi) = p(x,\xi) \in S^{\ell}_{0,0}$ by

$$P\psi(x) = \int e^{ix\cdot\xi} p(x,\xi)\hat{\psi}(\xi)đ\xi \qquad ( đ\xi = (2\pi)^{-m}d\xi )$$

for $\psi(x) \in \mathcal{S}$, where $\hat{\psi}(\xi)$ denotes the Fourier transform $\int e^{-ix\cdot\xi}\psi(x)dx$ and $\mathcal{S}$ denotes the Schwartz space of rapidly decreasing functions.

We first state the Calderón-Vaillancourt theorem, which is essentially used for the proof of Theorems 1 and 2.

Calderón-Vaillancourt theorem ([1] or [4]). If $p(x,\xi)$ belongs to $S^{0}_{0,0}$, for any real s we have

$$\|p(x,D_x)\psi(\cdot)\|_s$$
$$\leq C\Big(\sum_{|\alpha|\leq\ell_0,\,|\beta|\leq\ell_0}\ \sup_{x,\xi}\ |\partial_\xi^\alpha\partial_x^\beta\, p(x,\xi)|(1+|\xi|)^{-\ell}\Big)\|\psi(\cdot)\|_s$$

with a constant $C$ independent of $p(x,\xi)$ and $\psi$, where $\ell_0 = 2[m/2 + 1]$. For real $r$ $[r]$ denotes the largest integer not greater than $r$.

First, we note that the assumption (1.1) is equivalent to the assumption that the inequality

$$\sup_{x\in R^m,\,\omega\in S^{m-1}} \Big|\frac{1}{2}\int_{L_{x,x+2\rho\omega}} \sum_j \mathrm{Re}\, b_j\, dx_j\Big| \leq M\log(1+\rho) + N \tag{2.1}$$

with the same constants $M$ and $N$ holds for any positive $\rho$, where $\int_{L_{x,x+2\rho\omega}} (\cdots)\, dx_j$ means curvilinear integral along the straight line $L_{x,x+2\rho\omega}$ from a point $x \in R^m$ to a point $x + 2\rho\omega \in R^m$.

We shall find the solution $u(x,t)$ of (0.1) in the form

$$u(x,t) = k(x,t;D_x)v(x,t) \equiv Kv(x,t)$$

as in [7]. We define $k(x,t;\xi)$ as the solution of

$$\Big(\partial_t + 2\sum_{j=1}^m \xi_j\partial_{x_j} + \sum_{j=1}^m b_j(x)\xi_j\Big)k(x,t;\xi) = 0$$

with $k(x,0;\xi) = 1$, that is,

$$k(x,t;\xi) = \exp\Big\{\frac{1}{2}\int_{L_{x,x-2t\xi}} \sum_j b_j(x)dx_j\Big\} \equiv \exp\{\phi(x,t;\xi)\}. \tag{2.2}$$

Then, the Cauchy problem for the equation (0.1) with initial data $u_0(x)$ at $t = 0$ becomes

(2.3) $K(i\partial_t + \Delta)v(x,t) + K_1 v(x,t) = f(x,t)$

with $v(x,0) = u_0(x)$, where $K_1 = k_1(x,t;D_x)$ and

(2.4) $k_1(x,t;\xi) = (\Delta + \sum_j b_j(x)\partial_{x_j} + c(x))k(x,t;\xi).$

We can see by (2.1) that the assumption (1.1) shows

(2.5) $|\mathrm{Re}\ \phi(x,t;\xi)| \leqq M \log(1 + T|\xi|) + N \quad (t \in [0,T]).$

We can also prove that if $\alpha + \beta \neq 0$,

(2.6) $\sup_{x,\xi} |\partial_\xi^\alpha \partial_x^\beta \phi(x,t;\xi)| \leqq C_{\alpha,\beta}\, t^{|\alpha|} \quad (t \in R)$

is valid for a positive constant $C_{\alpha,\beta}$. For, if $m = 1$, we have $\phi(x,t;\xi) = F(x - 2t\xi) - F(x)$ by using the function $F(x)$ such that $\frac{dF}{dx}(x) = b_1(x)/2$. In the case $m \geqq 2$ we can also easily prove it by the assumption (1.2). We set

$$\tilde{k}(x,t;\xi) = \exp\{-\phi(x,t;\xi)\}.$$

Then, the inequalities (2.5) and (2.6) imply that $k(x,t;\xi)$ and $\tilde{k}(x,t;\xi)$ belong to $S^M_{0,0}$.

Remark 3. In general, $\tilde{k}(x,t;\xi)$ does not belong to $S^{-M}_{0,0}$. In more detail, we can prove from the form of $\phi(x,t;\xi)$ that if $k(x,t;\xi) \in S^M_{0,0}$ (resp. $S^{-M}_{0,0}$) and $\tilde{k}(x,t;\xi) \in S^{-M}_{0,0}$ (resp. $S^M_{0,0}$), $M$ must be zero.

Remark 3 states that we need the following calculations of the new type.

Lemma. We suppose the same assumptions in Theorem 1. Set $p_1(x,t;\xi) = r_1(x,\xi)\tilde{k}(x,t;\xi)$ and $p_2(x,t;\xi) = r_2(x,\xi)k(x,t;\xi)$ for any $r_j(x,\xi) \in S^0_{0,0}$ $(j = 1,2)$. Then, if we define $p(x,t;\xi)$

by the single symbol $\sigma(P_1 \circ P_2)(x,t;\xi)$ of the product of pseudo-differential operators $P_1 \circ P_2$ (that is, $p(x,t;D_x) = P_1 \circ P_2$, see [4]), $p(x,t;\xi)$ belongs to $S^0_{0,0}$ and has the estimates for $\ell = 0,1,2,\cdots$

$$\sum_{|\alpha|\leqq\ell,\,|\beta|\leqq\ell} \sup_{x,\xi} |\partial_\xi^\alpha \partial_x^\beta p(x,t;\xi)|$$
$$\leqq C_\ell(T) \prod_{j=1}^{2} \sum_{|\alpha|\leqq\ell',\,|\beta|\leqq\ell'} \sup_{x,\xi} |\partial_\xi^\alpha \partial_x^\beta r_j(x,\xi)|$$

for $t \in [0,T]$, where $\ell' = \ell + 2M + 2[m/2+1]$ and constants $C_\ell(T)$ are independent of $r_j(x,\xi)$.

<u>Proof.</u> Following [4], $p(x,t;\xi)$ is written by

$$(2.7)\quad p(x,t;\xi)$$
$$= O_s\text{-}\iint e^{-iy\cdot\eta} p_1(x,t;\xi+\eta)p_2(x+y,t;\xi)dy\,đ\eta$$
$$= O_s\text{-}\iint e^{-iy\cdot\eta} r_1(x,\xi+\eta)r_2(x+y,\xi)$$
$$\times \exp -\{\phi(x,t;\xi+\eta) - \phi(x+y,t;\xi)\}\, dy\,đ\eta.$$

If we apply the Stokes theorem, we get from (2.2)

$$2\,\mathrm{Re}\,\{\phi(x,t;\xi+\eta) - \phi(x+y,t;\xi)\}$$
$$= \int_{L_{x,x-2t(\xi+\eta)}} \omega - \int_{L_{x+y,x+y-2t\xi}} \omega$$
$$= \int_{L_{x-2t\xi,x-2t(\xi+\eta)}} \omega + \iint_{\Delta_1} d\omega$$
$$- \left( \int_{L_{x+y,x}} \omega + \int_{L_{x-2t\xi,x+y-2t\xi}} \omega - \iint_{\Delta_2} d\omega \right),$$

where $\omega = \sum_j \mathrm{Re}\, b_j(x)\, dx_j$, $d\omega$ implies the exterior derivative of $\omega$, $\Delta_1$ is the triangles whose boundary consists of the straight lines $L_{x,x-2t(\xi+\eta)}$, $L_{x-2t(\xi+\eta),x-2t\xi}$ and $L_{x-2t\xi,x}$, and also the boundary of $\Delta_2$ consists of $L_{x,x-2t\xi}$, $L_{x-2t\xi,x+y-2t\xi}$,

$L_{x+y-2t\xi,x+y}$ and $L_{x+y,x}$. We note that if $m = 1$, $d\omega$ vanishes. Hence, from the assumption (1.1) ( or (2.1)) and moreover in the case $m \geqq 2$ from the assumption (1.3) we obtain

$$(2.8)\qquad |\mathrm{Re}\ \{\phi(x,t;\xi+\eta) - \phi(x+y,t;\xi)\}|$$
$$\leqq M\log(1+T|\eta|) + 2M\log(1+|y|) + 3N + C$$

for $t \in [0,T]$, where $C$ is a positive constant.

If for (2.7) we use the integration by parts with respect to the variables $y$ and $\eta$, by (2.6) and (2.8) we can complete the proof of Lemma. Q.E.D.

Now, as in (2.7) we can see by Taylor expansion

$$\sigma(\tilde{K} \circ K)(x,t;\xi)$$
$$= 1 + \frac{1}{i} \sum_{|\alpha|=1} \int_0^1 d\theta\ O_s\text{-}\iint e^{-iy\cdot\eta}(\partial_\xi^\alpha \tilde{k})(x,t;\xi+\theta\eta)$$
$$\times (\partial_x^\beta k)(x+y,t;\xi)\ dy đ\eta,$$

where $\tilde{K} = \tilde{k}(x,t;D_x)$. Noting (2.6), we can prove in the similar way to the proof of the above Lemma that

$$\sigma(\tilde{K} \circ K)(x,t;\xi) = 1 + ts(x,t;\xi),$$

where $s(x,t;\xi)$ belongs to $S^0_{0,0}$. By the Calderón-Vaillancourt theorem we can see that $I + ts(x,t;D_x)$ is a $L^2$ bounded operator. I is an identity map. So, it follows that if $T_1$ $(0 < T_1 < T)$ is sufficienty small, there exists a inverse operator $B(t)$ of $I + ts(x,t;D_x)$ as the mapping from $L^2$ space to $L^2$ space. Therefore, the inverse operator $K^{-1}$ of $K$ as the mapping from $L^2$ space to $L^2$ space exists and has the form

$$(2.9)\qquad K^{-1} = B(t) \circ \tilde{K}.$$

Applying (2.9) to (2.3), we have

$$(2.3)' \qquad (i\partial_t + \Delta) + B(t) \circ \tilde{K} \circ K_1 v(x,t) = B(t) \circ \tilde{K} f(x,t).$$

Moreover, noting (2.4) and (2.6), we can apply Lemma in this section to $\tilde{K} \circ K_1$. That is, it follows that $B(t) \circ \tilde{K} \circ K_1$ is a $L^2$ bounded operator for $t \in [0,T_1]$. Therefore, it is easily seen in the usual way that for any $u_0(x) \in H_\infty$ and $f(x,t) \in \mathcal{E}_t^0([0,T_1];H_\infty)$ there exists a unique solution $v(x,t) \in \mathcal{E}_t^0([0,T_1];L^2)$ of (2.3)′ ( or (2.3)) with the initial data $u_0(x)$ at $t=0$ and we have the energy inequality

$$||v(\cdot,t)|| \leqq C(||u_0(\cdot)|| + \int_0^t ||f(\cdot,\theta)||_M \, d\theta)$$

for a positive constant $C$, where $||\cdot|| = ||\cdot||_0$. Here, we used the fact that $k(x,t;\xi)(1 + |\xi|^2)^{-M/2}$ belongs to $S^0_{0,0}$ and the Calderón-Vaillancourt theorem for the term $Kf(x,t) = k(x,t;D_x)\Lambda^{-M}(\Lambda^M f)(x,t)$, where $\Lambda$ is the pseudo-differential operator with the symbol $(1 + |\xi|^2)^{1/2}$. By $u(x,t) = Kv(x,t)$ we obtain

$$(2.10) \qquad ||u(\cdot,t)||_{-M} \leqq C(||u_0(\cdot)|| + \int_0^t ||f(\cdot,\theta)||_M \, d\theta)$$

for $t \in [0,T_1]$ with another constant $C$.

For any real $s$ $\Lambda^s u(x,t)$ satisfies the similar equation to (0.1), where $u(x,t)$ is the solution of (0.1) determined above. Hence, in the similar way to the proof of (2.10) we get

$$(2.11) \qquad ||u(\cdot,t)||_{s-M} \leqq C_s(||u_0(\cdot)||_s + \int_0^t ||f(\cdot,\theta)||_{s+M} \, d\theta)$$

for $t \in [0,T_1]$ with a constant $C_s$. Noting that $T_1$ is independent of the choice of the initial surface, we can complete the proof.

Remark 4. We can see from (2.11) that if $M$ in (1.1) equals zero, the solution $u(x,t)$ has no loss of regularity on $[0,T]$ for any $T$. On the other hand, in Theorem 1 we have only to assume (1.1) and (1.2) for this case ($M = 0$), because the Stokes theorem shows that (1.3) follows from (1.1). This is one of the results in [7].

## 3. Proof of Theorem 2

In this section we shall prove Theorem 2 by the energy method as in [6]. In [6] the so-called micro-localizations were fundamentally used. But, in the present paper we use the essentially different localizations. Roughly speaking, we localize the solution of (0.1)′ along the classical trajectory for the Hamiltonian $-\tau\Delta$.

The symbols $w(x,t;\xi)$ of localizing (pseudo-differential) operatos $W$ are defined by the solution of "equation of motion for the Hamilton function $-\tau|\xi|^2$ "

$$\partial_t w(x,t;\xi) = \{w(x,t;\xi), -\tau|\xi|^2\}, \tag{3.1}$$

where for $C^1$-functions $f(x,\xi)$ and $g(x,\xi)$ $\{f,g\}$ denotes the Poisson bracket $\sum_{j=1}^{m} (\partial_{x_j} f \partial_{\xi_j} g - \partial_{\xi_j} f \partial_{x_j} g)$. Then, for the solution $u(x,t)$ of (0.1)′ we can easily get by (3.1)

$$L(Wu)(x,t) = \tau(\Delta_x w)(x,t;D_x)u + [\sum_j b_j \partial_{x_j} + c, W]u, \tag{3.2}$$

where $\sigma(\Delta_x w(x,t;D_x)) = \sum_j (\partial_{x_j}^2 w)(x,t;\xi)$ and $[\cdot,\cdot]$ implies the commutator of operators. This equality (3.2) will be used fundamentally for the proof of Theorem 2.

We prove by contradiction. Then, we may assume without loss of generality

(A.1) There exists a positive $T$ such that for any $u_0(x) \in H_\infty$ there exists a unique solution $u(x,t) \in \mathcal{E}_t^0([0,T];H_\infty)$ of (0.1)′.

(A.2) The inequality (1.4) does not hold for any large constants $M$ and $N$.

Since $\mathcal{E}_t^0([0,T];H_\infty)$ is a Fréchet space with semi-norms $\max_{0\leqq t\leqq T} ||h(\cdot,t)||_s$ $(s = 0,\pm 1,\pm 2,\cdots)$, by the assumption (A.1) and the closed gragh theorem we can find a non-negative integer $q$ and a positive constant $C(T)$ such that for all solutions $u(x,t)$ of (0.1)′

$$||u(\cdot,t)|| \leqq C(T)||u_0(\cdot)||_q \tag{3.3}$$

for $t \in [0,T]$.

For the above $q$ we take a constant $M$ such that

$$M > \frac{m}{2} + 2[\frac{m}{2} + 1] + 3q \tag{3.4}$$

and fix it. For this $M$ we can take from the assumption (A.2) sequences $x^{(k)} \in R^m$, $\omega^{(k)} \in S^{m-1}$ and $\rho_k > 0$ $(k = 1,2,\cdots)$ such that

$$|\sum_j \int_0^{\rho_k} \mathrm{Re}\, b_j(x^{(k)} + 2\tau\theta\omega^{(k)})\omega_j^{(k)} d\theta| \geqq M \log(1 + \rho_k) + k.$$

Then, it is easy to see that $\rho_k$ tends to infinity as $k$ tends to infinity. Also, noting that

$$\sum_j \int_0^{\rho} \mathrm{Re}\, b_j(x + 2\tau\theta\omega)\omega_j d\theta = \frac{1}{2\tau} \sum_j \int_{L_{x,x+2\tau\rho\omega}} \mathrm{Re}\, b_j \, dx_j$$

for $\tau \neq 0$, we can assume

$$(3.5)\quad \begin{cases} \rho_k \longrightarrow \infty \quad \text{as} \quad k \longrightarrow \infty, \\ \int_0^{\rho_k} b(x^{(k)} + 2\tau\theta\omega^{(k)}; \omega^{(k)})d\theta \geqq M \log(1 + \rho_k) + k, \\ \int_0^{t} b(x^{(k)} + 2\tau\theta\omega^{(k)}; \omega^{(k)})d\theta \geqq 0 \quad (t \in [0, \rho_k]) \end{cases}$$

by taking another sequences, if necessary, where we set

$$(3.6)\qquad b(x;\xi) = -\sum_j \mathrm{Re}\ b_j(x)\xi_j .$$

Though the proof is easy, we omit it. We fix these sequences.

Let $h(x)$ be the $C^\infty$-function which takes the value 1 in the set $\{x;\ |x| \leqq 1/4\}$ and takes the value 0 in $\{x;\ |x| \geqq 1/2\}$. Let $\delta$ be a sufficiently small positive constant such that

$$(3.7)\qquad M > \frac{m}{2} + 2[\frac{m}{2} + 1] + (3 + \delta)q$$

and set by using the above sequence $\rho_k$

$$n = n(k) = \rho_k^{3+\delta} .$$

Now we define $w_k(x,t;\xi)$ $(k = 1,2,\cdots)$ by the solution of (3.1) with the initial value $\rho_k^{m/2}h(\rho_k(x - x^{(k)}))h(\rho_k^2(\xi - n\omega^{(k)})/n)$ at $t = 0$, by using the above sequences. That is,

$$(3.8)\quad w_k(x,t;\xi) = \rho_k^{m/2}h(\rho_k(x - x^{(k)} - 2\tau t\xi))h(\rho_k^2(\xi - n\omega^{(k)})/n).$$

We set for any multi-indices $\alpha$ and $\beta$

$$(3.8)'\quad w_k^{\alpha,\beta}(x,t;\xi) = \rho_k^{m/2}(\partial_x^\alpha h)(x)(\partial_\xi^\beta h)(\xi)\Big|_{\substack{x=\rho_k(x-x^{(k)}-2\tau t\xi), \\ \xi=\rho_k^2(\xi-n\omega^{(k)})/n}} .$$

We note that $w_k^{0,0}(x,t;\xi) = w_k(x,t;\xi)$.

Next, we shall define the initial value $\psi_k(x)$ $(k = 1,2,\cdots)$

of the equation (0.1)′ corresponding to the localizing operator $W_k = w_k(x,t;D_x)$ as follows. Take a $C^\infty$-function $\psi(x)$ such that $\psi(0) = 2$ and the support of $\hat{\psi}(\xi)$ is included in the set $\{\xi;\ h(\xi) = 1\}$. We define

$$\hat{\psi}_k(\xi) = e^{-ix^{(k)}\cdot\xi}\hat{\psi}(\xi - n\omega^{(k)}) \tag{3.9}$$

and let $u_k(x,t)$ be the solution of (0.1)′ with the initial value $\psi_k(x)$ at $t = 0$. Then, we can easily have for $k = 1,2, \ldots$

$$||W_k u_k(\cdot,t)|_{t=0}|| \geqq ||h(\cdot)|| > 0 \tag{3.10}$$

and also have from (3.3) and (3.9)

$$||u_k(\cdot,t)|| \leqq C_1(T)n^q \qquad (n = \rho_k^{3+\delta}) \tag{3.11}$$

for $t \in [0,T]$ with a positive constant $C_1(T)$ independent of $k$.

Hereafter, we consider the variable $t$ only in the interval $[0,\rho_k/n]$. Of course, we assume that $k$ is large enough so that $\rho_k/n = \rho_k^{-(2+\delta)} \leqq T$. Now, take a positive integer $s$ so that

$$\delta[\frac{s+2}{2}] \geqq \frac{m-2}{2} + 2[\frac{m}{2} + 1] + (3 + \delta)(q + 1) \tag{3.12}$$

holds and fix it. Set by the localized solution $W_k^{\alpha,\beta}u_k(x,t)$

$$\sigma_k(t) = \sum_{0 \leqq |\alpha+\beta| \leqq s} (\rho_k^3/n)^{[(|\alpha+\beta|+1)/2]} ||W_k^{\alpha,\beta}u_k(\cdot,t)||. \tag{3.13}$$

Then, we obtain

Proposition 1. We have

$$\sigma_k(t) \leqq C_0\ \rho_k^{m/2 + 2[m/2 + 1] + (3 + \delta)q} \tag{3.14}$$

for $t \in [0,\rho_k/n]$, where $C_0$ is a constant independent of $k$.

Proposition 2. For large $k$ we get

(3.15) $$\sigma_k(\rho_k/n) \geqq C_1(1+\rho_k)^M$$

with a constant $C_1 > 0$ independent of $k$.

Since we have determined constant $\delta > 0$ so that (3.7) holds, (3.14) and (3.15) is not compatible for large $k$. Thus, we can prove Theorem 2.

Proof of Proposition 1. If we apply the Calderón-Vaillancourt theorem to the term $W_k^{\alpha,\beta}u_k(x,t)$, we can see by (3.11) that

(3.16) $$||W_k^{\alpha,\beta}u_k(\cdot,t)|| \leqq C_{\alpha,\beta}\ \rho_k^{m/2+2[m/2+1]+(3+\delta)q}$$

for $t \in [0,\rho_k/n]$, which complete the proof. Here, we used

(3.17) $$0 \leqq \rho_k t \leqq \rho_k^2/n = \rho_k^{-(1+\delta)} \quad (t \in [0,\rho_k/n]).$$

Q.E.D.

Proof of Proposition 2. We first note from (3.2) that

(3.18) $$\begin{aligned} & L(W_k u_k)(x,t) \\ & = f_k(x,t) \\ & \equiv \{\sum_j [b_j(x)\partial_{x_j} + c(x), W_k] + \tau(\Delta_x w_k)(x,t;D_x)\}u_k(x,t). \end{aligned}$$

Now, it is easily seen from (3.8) for the support $\operatorname{supp} w_k^{\alpha,\beta}(\cdot, t;\cdot)$ of the function $w_k^{\alpha,\beta}(x,t;\xi)$ with respect to the variables $x$ and $\xi$ that we have for $t \in [0,\rho_k/n]$

(3.19) $$\begin{aligned} & \operatorname{supp} w_k^{\alpha,\beta}(\cdot,t;\cdot) \\ & \subset \{(x,\xi);\ |x-(x^{(k)}+2n\tau t\omega^{(k)})| \leqq 2/\rho_k, \\ & \qquad |\xi/n - \omega^{(k)}| \leqq 1/(2\rho_k^2)\}. \end{aligned}$$

By using (3.19) and (3.11) we get the estimates

$$(3.20)\qquad \|\{b_j(x)(\tfrac{1}{i}\,\partial_{x_j}) - b_j(x^{(k)} + 2n\tau t\omega^{(k)};n\omega_j^{(k)})\}W_k u_k(\cdot,t)\| \leqq \text{const.}\ \frac{n}{\rho_k}\ \|W_k u_k(\cdot,t)\| + \text{const.}$$

for $t \in [0,\rho_k/n]$, where const. means a positive constant independent of $k$ and hereafter we shall use the symbol const. in the same sense. We omit the detail proof of (3.20). Hence, from (3.18) we can easily have

$$(3.21)\qquad \frac{1}{2}\ \frac{d}{dt}\ \|W_k u_k(\cdot,t)\|^2 \geqq \{b(x^{(k)} + 2n\tau t\omega^{(k)};n\omega^{(k)}) - \text{const.}\ (1 + \frac{n}{\rho_k})\} \times \|W_k u_k(\cdot,t)\|^2 - \|f_k(\cdot,t)\| \times \|W_k u_k(\cdot,t)\| - \text{const.}\ \|W_k u_k(\cdot,t)\|$$

for $t \in [0,\rho_k/n]$.

We shall estimate $\|f_k(\cdot,t)\|$. We first note that if $|\alpha + \beta| \geqq s + 1$ for $s$ defined so that (3.12) holds,

$$(3.22)\qquad \frac{n}{\rho_k}\ (\rho_k^3/n)^{[(|\alpha + \beta| + 1)/2]}\|W_k^{\alpha,\beta}u_k(\cdot,t)\| \leqq C_{\alpha,\beta} < \infty$$

are obtained for any $k$ and $t \in [0,\rho_k/n]$ from (3.16). Now, it is easy to see that

$$(3.23)\qquad \|(\Delta_x w_k)(x,t;D_x)u_k(\cdot,t)\| \leqq \text{const.}\ \frac{n}{\rho_k}\ \sum_{|\alpha+\beta|=2} (\rho_k^3/n)\|W_k^{\alpha,\beta}u_k(\cdot,t)\|.$$

Next, following [4], the symbol $\frac{1}{i}\,\sigma([b_j(x)\partial_{x_j},\ W_k])(x,t;\xi)$ is written by

$$\frac{1}{i}\,b_j(x)\partial_{x_j}w_k(x,t;\xi) - \sum_{1\leqq|\gamma|\leqq\nu} \frac{1}{\gamma!}(\frac{1}{i})^{|\gamma|}(\partial_x^\gamma b_j)(\partial_\xi^\gamma w_k)\xi_j +$$

$+$ " the remainder term "

for any $\nu = 1,2,\cdots$. Here, though the detail proof is omitted, if we take a positive integer $p$ such that $(1+\delta)(p+1) \geqq m/2 + 4[m/2+1] + (3+\delta)(q+1)$ and we use

$$\partial_\xi^\gamma w_k(x,t;\xi) = \sum_{\alpha+\beta=\gamma} \frac{\gamma!}{\alpha!\beta!} (-2\tau t\rho_k)^{|\alpha|}(\rho_k^2/n)^{|\beta|} w_k^{\alpha,\beta}(x,t;\xi)$$

and (3.17), we can get

$$\begin{aligned} &||[b_j(x)\partial_{x_j}, W_k]u_k(\cdot,t)|| \\ &\leqq \text{const. } \frac{n}{\rho_k} \sum_{1\leqq|\alpha+\beta|\leqq p+1} \rho_k(\rho_k^2/n)^{|\alpha+\beta|}||W_k^{\alpha,\beta}u_k(\cdot,t)|| \\ &\qquad + \text{const.} \end{aligned}$$

for $t \in [0,\rho_k/n]$. Similarly, we can estimate $||[c(x), W_k]u_k(\cdot,t)||$. Hence, noting (3.22), we obtain together with (3.23)

$$(3.24) \qquad ||f_k(\cdot,t)|| \leqq \text{const. } \frac{n}{\rho_k}\sigma_k(t) + \text{const.},$$

which shows by (3.21)

$$\begin{aligned} (3.25) \qquad &\frac{d}{dt}||W_k u_k(\cdot,t)|| \\ &\geqq \{b(x^{(k)} + 2n\tau t\omega^{(k)}; n\omega^{(k)}) - \text{const. } (1+\frac{n}{\rho_k})\} \\ &\times ||W_k u_k(\cdot,t)|| - \text{const. } \frac{n}{\rho_k}\sigma_k(t) - \text{const.}. \end{aligned}$$

In the same way, the similar inequality to (3.25) for $\frac{d}{dt}||W_k^{\alpha,\beta}u_k(\cdot,t)||$ $(1 \leqq |\alpha+\beta| \leqq s)$ holds. Finally, we obtain

$$\begin{aligned} (3.26) \qquad \frac{d}{dt}\sigma_k(t) &\geqq \{b(x^{(k)} + 2n\tau t\omega^{(k)}; n\omega^{(k)}) - \text{const. } (1+\frac{n}{\rho_k})\} \\ &\times \sigma_k(t) - \text{const.} \end{aligned}$$

for $t \in [0,\rho_k/n]$.

If we integrate (3.26) with respect to the variable $t$ from $0$ to $\rho_k/n$ and then we use (3.5) and (3.10), we can easily get (3.15). Q.E.D.

Remark 5. In more detail, we can see from the proof of Theorem 2 that the following is necessary in order that there exists a constant $T \neq 0$ such that for any initial data $u_0(x) \in H_\infty$ a unique solution $u(x,t) \in \mathcal{E}_t^0([0,T];H_\infty)$ of (0.1)' exists and the inequality (3.3) holds for some $q$. For any greater than $m/2 + 2[m/2 + 1] + 3q$ there exists a constant $N$ such that the inequality (1.4) holds.

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