An Approach to the Subgraph Homeomorphism Problem*

Takao ASANO

Department of Mathematical Engineering and Instrumentation Physics,
Faculty of Engineering, University of Tokyo, Bunkyo-ku, Tokyo 113

Abstract

The subgraph homeomorphism problem for a fixed pattern graph H is
stated as follows: given an input graph G=(V,E), determine whether G has
a subgraph homeomorphic to H. We show that the subgraph homeomorphism
problem for the fixed graph $K_{3,3}$ is solvable in polynomial time, where
$K_{3,3}$ is the Thomsen graph, one of the Kuratowski graphs used to
characterize planar graphs. Specifically, we present an O(V)-time
algorithm for this problem. This problem was suspected to be NP-complete
by Fortune, Hopcroft and Wyllie. We also present several pattern graphs
for each of which an O(V)-time algorithm exists.

1. Introduction

The subgraph homeomorphism problem for a fixed pattern graph H is
stated as follows: given an input graph G=(V,E), determine whether G has
a subgraph homeomorphic to H, i.e., a subgraph isomorphic to a graph
obtained from H by a sequence of subdivisions of edges. This problem was
one of the most popular open problems in computational complexity [7].
This problem remains open, although some significant new subcases have
been settled [12].

The "fixed-vertex" version of the problem (the input specifies
exactly which vertex of G is to correspond to each vertex of H) has been
completely classified for directed graphs by Fortune, Hopcroft and Wyllie
[6]: it is polynomial-time solvable if H is a fixed graph all of whose
arcs share a common tail, or all of whose arcs share a common head; and

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* A preliminary version of this paper was presented at the meeting of the
Technical Group on Automata and Languages of the Institute of Electronics
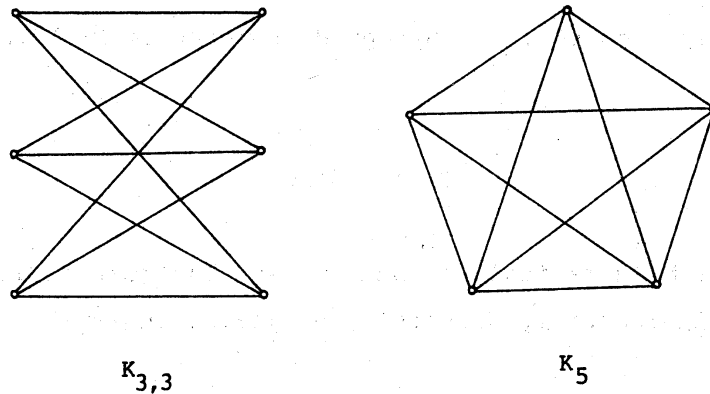and Communication Engineers of Japan, Tokyo, June, 1983 [2].

Fig. 1. The Kuratowski graphs $K_{3,3}$ and K_5.

it is NP-complete for all other fixed graphs H. With respect to the "undirected and fixed-vertex" version of the problem, several complicated polynomial-time algorithms have been found for particular values of H, such as a triangle [15] and two independent edges [19,21,22,23].

With respect to the "non-fixed-vertex" versions of the problem, any polynomial-time maximum cardinality matching algorithms in [20] solve the problem for k independent edges, where k is any fixed interger. Fortune, Hopcroft and Wyllie [6] have classified a number of H (both polynomially solvable and NP-complete) in the directed case, and Liu and Geldmacher [16] have presented a polynomial-time algorithm for the complete graph with four vertices, i.e., for $H=K_4$ in the undirected case. However, much remains open [12]. For example, with respect to the "undirected and non-fixed-vertex" version of the problem, it seems that a polynomial-time algorithm has not been presented explicitly for any 2-connected graph with four more vertices except the graph K_4. Note that a polynomial-time fixed-vertex algorithm for H implies a polynomial-time non-fixed-vertex algorithm for H, but not vice versa.

In this paper we attack the "undirected and non-fixed-vertex" version of the subgraph homeomorphism problem for a fixed pattern graph H. We first show that the subgraph homeomorphism problem for the fixed graph $K_{3,3}$ is solvable in polynomial time, where $K_{3,3}$ is the Thomsen graph, one of the Kuratowski graphs used to characterize planar graphs (Fig. 1). Specifically, we show that, for any input graph $G=(V,E)$, there is an $O(V)$-time algorithm for this problem. Next, we present an $O(V^2)$-time algorithm for actually finding a subgraph of G homeomorphic to $K_{3,3}$ if G has such a subgraph. These results might be interesting, because it was suspected by Fortune, Hopcroft and Wyllie [6] that, for a specific
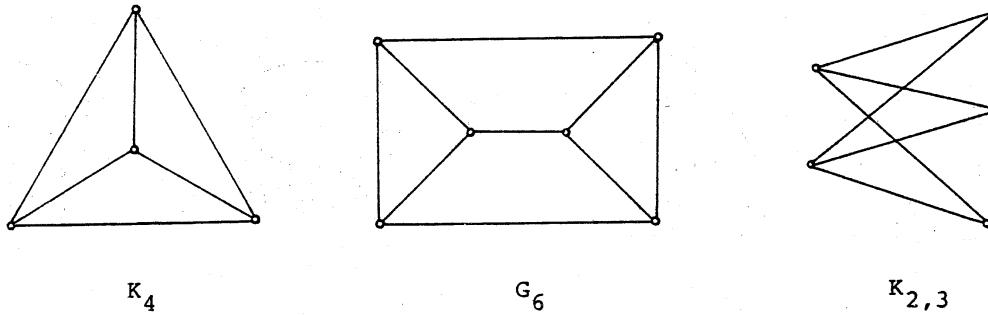
Fig. 2. Graphs K_4, G_6 and $K_{2,3}$.

Kuratowski graph H, the subgraph homeomorphism problem for H is NP-complete even though there is a polynomial planarity testing algorithm.

We use an efficient algorithm in [10] for decomposing a graph into 3-connected components in order to obtain the above results. This approach will be useful in designing an efficient algorithm for the subgraph homeomorphism problem for a fixed 3-connected pattern graph, because we show that, for a 3-connected graph H, a graph G has a subgraph homeomorphic to H if and only if some 3-connected component of G has a subgraph homeomorphic to H, and we may expect that 3-connected graphs have some specific properties.

As applications of this approach, we first present several pattern graphs for each H of which there is an O(V)-time algorithm to determine whether an input graph G=(V,E) has a subgraph homeomorphic to H. Included among them are the graphs K_4, G_6 and $K_{2,3}$ shown in Fig. 2 two of which are used to characterize outerplanar graphs. Our O(V)-time algorithm for H=K_4, which also finds a subgraph homeomorphic to K_4 if an input graph G=(V,E) has such a subgraph, may compare favorably with the previous known O(E)-time algorithm in [16]. Then, we present a new characterization of outerplanar graphs together with an O(V)-time algorithm to determine whether an input simple graph G=(V,E) is outerplanar.

2. Preliminaries

For terminology on graph theory, we follow [9]. A graph considered in this paper is a finite undirected graph. For a graph G we denote by V(G) and E(G) the vertex set and the edge set of G, respectively. For U⊂ V(G), we denote by G-U the subgraph of G obtained from G by deleting all vertices in U and all edges incident with vertices in U. For S ⊂ E(G), we
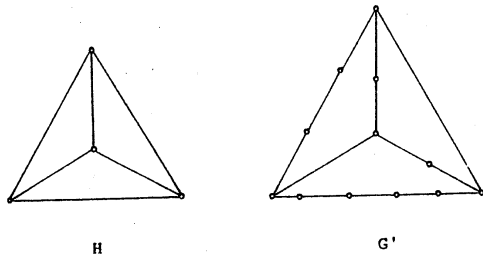
Fig. 3. A graph G' homeomorphic to a graph H.
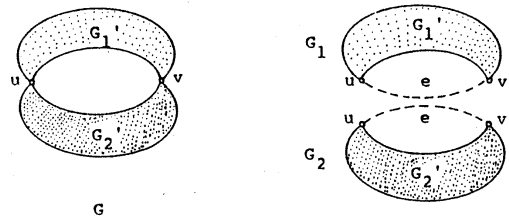
Fig. 4. A separation pair {u,v} of a graph G and split graphs G_1 and G_2 with a virtual edge e.

denote by G-S the <u>subgraph</u> of G obtained by deleting all edges in S and by G/S the <u>contraction</u> of G obtained by contracting all edges in S. For two disjoint subsets S_c and S_d of E(G), $(G-S_d)/S_c$ is a <u>subcontraction</u> of G. A graph G' is <u>homeomorphic to</u> a graph H if G' is obtained from H by a sequence of subdivisions of edges (Fig. 3). Formally, the <u>subgraph homeomorphism problem for a fixed pattern graph</u> H is defined as follows.

<u>Instance</u>: A simple graph G.

<u>Question</u>: Does G contain a subgraph homeomorphic to H?

A connected graph G is <u>2-connected</u> if, for each two distinct edges e and e' of G, there is a cycle of G containing e and e'. A maximal connected (resp., 2-connected) subgraph of G is a <u>connected</u> (resp., <u>2-connected</u>) <u>component</u> of G. An unordered pair {u,v} of distinct vertices in G is a <u>separation pair</u> of G if there exist two subgraphs G_1' and G_2' satisfying the following:

(a) V(G)= V(G_1') ∪ V(G_2'), V(G_1') ∩ V(G_2')= {u,v};

(b) E(G)= E(G_1') ∪ E(G_2'), E(G_1') ∩ E(G_2')=ϕ, $|E(G_1')| \geq 2$, $|E(G_2')| \geq 2$;

(c) For some $e_1 \in E(G_1')$ and $e_2 \in E(G_2')$, there is a cycle of G containing e_1 and e_2.

Graphs G_1' and G_2' are called <u>separation graphs</u> with respect to the separation pair {u,v}. Define G_i (i=1,2) as the graph obtained from G_i' by adding a new edge e=(u,v). Graphs G_1 and G_2 are called <u>split graphs</u> of G with respect to {u,v} and the common new edge is called a <u>virtual edge</u> (Fig. 4). Decomposing a graph G into two split graphs G_1 and G_2 is called <u>splitting</u>. For two split graphs G_1 and G_2 with the common virtual edge e=(u,v), define a graph G as follows:

V(G)=V(G_1) ∪ V(G_2) and E(G)=(E(G_1) ∪ E(G_2)) -{e}.

The graph G is called a <u>merged graph</u> of G_1 and G_2 obtained by <u>merging</u> the virtual edge e. Merging is the inverse of splitting. A 2-connected
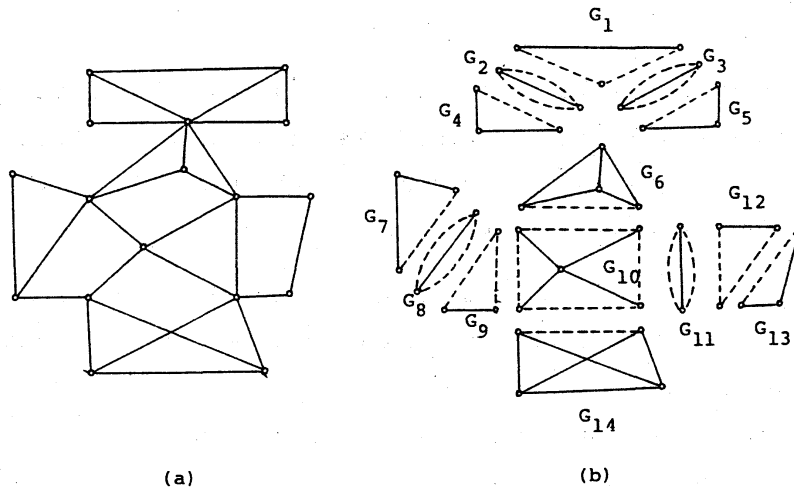
Fig. 5. Decomposition D of a graph G into 3-connected components.

(a) graph G   (b) decomposition D

graph G is called 3-connected if G has no separation pair. A 3-connected graph with three or more vertices is a simple graph. Maximal 3-connected subcontractions of G are called 3-connected components of G. A decomposition of a graph G into 3-connected components is obtained as follows.

(i) Divide G into 2-connected components $D=\{G_1,G_2,\ldots,G_k\}$.

(ii) For each G_i in D, if G_i is not 3-connected then decompose G_i into split graphs G_{i1} and G_{i2} with respect to a separation pair $\{u,v\}$ of G_i and $D:=D-\{G_i\}+\{G_{i1},G_{i2}\}$.

(iii) If all G_i in D are 3-connected then stop, otherwise go to (ii).

Note that there is an $O(|E(G)|)$-time algorithm for decomposing a graph G into 3-connected components [10,25]. Each 3-connected component of a graph G is a simple graph or isomorphic to the multigraph K_3^* with three parallel edges (Fig. 5). Furthermore, if D_1 and D_2 are two decompositions of a graph G into 3-connected components, then there is a bijection $f:D_1 \rightarrow D_2$ such that G_i is isomorphic to $f(G_i)$ for any G_i in D_1 [17]. Thus, the number of 3-connected components of a graph G, i.e., the number of 3-connected components in a decomposition of G into 3-connected components does not depend on decompositions.

3. Subgraph Homeomorphism Problem for the Fixed Pattern Graph $K_{3,3}$

The main object in this section is to show that, for an arbitrarily given input simple graph G, the subgraph homeomorphism problem for $K_{3,3}$,

which was suspected to be NP-complete by Fortune, Hopcroft and Wyllie [6], is solvable in $O(|V(G)|)$ time. We first give some preliminary lemmas. The following lemma is easy to obtain but it plays the most essential role throughout this paper.

Lemma 1. For a 3-connected graph H, a graph G has a subgraph homeomorphic to H if and only if there is a 3-connected component of G that has a subgraph homeomorphic to H.

Proof. We shall show the lemma by induction on the number of 3-connected components of G, i.e., on the number of 3-connected components in a decomposition of G into 3-connected components. If G is 3-connected then the lemma is trivially true. Thus, we may assume that G is not 3-connected. Note that, since H is a 3-connected graph, G has a subgraph homeomorphic to H if and only if there is a 2-connected component of G that contains a subgraph homeomorphic to H. Thus, we can assume that G is 2-connected. Let {u,v} be any separation pair of G. Let G_1' and G_2' be separation graphs of G with respect to the separation pair {u,v} . Let G_1 and G_2 be the split graphs of G corresponding to the separation graphs G_1' and G_2', respectively. Then, we have only to prove the following (i) because each of G_1 and G_2 has fewer 3-connected components than G.

    (i)   A graph G has a subgraph homeomorphic to H if and only if G_1 or G_2 has a subgraph homeomorphic to H.

(i) can be obtained by the following observations.

    (ii)   If G has a subgraph homeomorphic to a graph G' and G' has a subgraph homeomorphic to a graph G", then G has a subgraph homeomorphic to G".

    (iii) G has a subgraph homeomorphic to each G_i (i=1,2).

    (iv) If G has a subgraph homeomorphic to H then G_1 or G_2 has a subgraph homeomorphic to H.

(ii) is trivial by the definition. (iii) is almost evident, because each of the separation graphs G_1' and G_2' has a path connecting the two vertices u and v. (iv) is obtained by the 3-connectedness of H, because if G has a subgraph homeomorphic to H and none of G_1 and G_2 had a subgraph homeomorphic to H then it is easily shown that H would not be 3-connected. Thus, we have (i) and consequently, the lemma by the

induction. □

Since the Thomsen graph $K_{3,3}$ is 3-connected, we have the following.

Corollary 1. A graph G has a subgraph homeomorphic to $K_{3,3}$ if and only if there is a 3-connected component of G that has a subgraph homeomorphic to $K_{3,3}$.

The following lemma is a famous characterization of planar graphs.

Lemma 2 [14]. A graph G is nonplanar if and only if G has a subgraph homeomorphic to K_5 or $K_{3,3}$. (K_n is the complete graph with n vertices. The graphs K_5 and $K_{3,3}$ are called the Kuratowski graphs.)

Since 3-connected graphs are restricted graphs in some sense, they may have some specific properties. The following lemma, which is easy to derive, is an example of such properties and plays a crucial role in this paper. Hall [8] first obtained this but his proof seems to have a very small hole. We give a brief constructive proof of this lemma, because the algorithm described in the next section is based on the proof.

Lemma 3 [8]. A 3-connected graph with six or more vertices is nonplanar if and only if G has a subgraph homeomorphic to $K_{3,3}$.

Proof. Since the sufficiency is evident by Lemma 2, we consider only the necessity. Suppose that G is nonplanar and has a subgraph G' homeomorphic to K_5. If $G'=K_5$, then, for any vertex v of G which is not contained in G', there are three vertex-disjoint paths in G from the vertex v to three distinct vertices of G', because G is 3-connected (Fig. 6(a)). It is easily seen that G has a subgraph homeomorphic to $K_{3,3}$ (Fig. 6(b)). Thus we may assume $G' \neq K_5$. Let u and v be two vertices of degree 4 in G' such that there is a path, say P(u,v), of length ≥ 2 in G' which not only connects the two vertices u and v but also contains no other vertices of degree 4 in G'. Since G is 3-connected, for some vertex w (w≠u,v) on the path P(u,v) and some
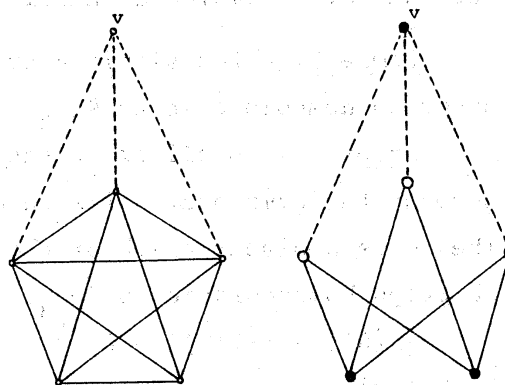
(a)   (b)

Fig. 6.

A subgraph of a graph G homeomorphic to $K_{3,3}$.
(a) subgraph $G'=K_5$ and three vertex-disjoint paths from a vertex v to three vertices of (
(b) subgraph homeomorphic to $K_{3,3}$

vertex x of G' which is not on the path $P(u,v)$, there is a path $P(w,x)$ in G connecting the two vertices w and x such that any internal vertex on the path $P(w,x)$ is not contained in G'. By symmetry, we have only to consider the cases shown in Fig. 7(a). Thus, by an easy inspection, one can see that G has a subgraph homeomorphic to $K_{3,3}$ (Fig. 7(b)). □

By Corollary 1 and Lemma 3, we have the following lemma.

Lemma 4. A graph G has no subgraph homeomorphic to $K_{3,3}$ if and only if every 3-connected component of G is a planar graph or the graph K_5.

The following lemma gives an upper bound on the number of edges of a simple graph which has no subgraph homeomorphic to $K_{3,3}$. It can be obtained by a simple calculation.

(a)   Fig. 7.   (b)

A subgraph of a graph G homeomorphic to $K_{3,3}$.

Lemma 5. If a simple graph G with two or more vertices has no subgraph homeomorphic to $K_{3,3}$, then $|E(G)| \leq 3|V(G)|-5$.
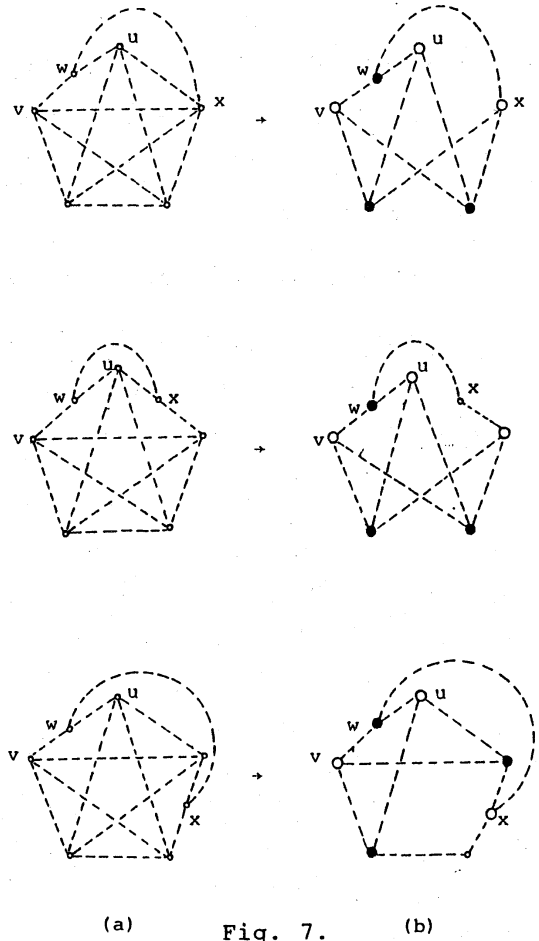
Proof. We shall prove the lemma by induction on the number of 3-connected components of such a graph G. Suppose that G is 3-connected. Then G is a planar graph or the graph K_5 by Lemma 3, because G has no subgraph homeomorphic to $K_{3,3}$. If G is planar then

$$|E(G)| \leq 3|V(G)| - 6$$

by Euler's formula for planar graphs [9], and if $G=K_5$ then

$$|E(G)| = 3|V(G)| - 5.$$

Thus the lemma is true for such 3-connected graphs. Suppose that the lemma is true for all such graphs with k or fewer 3-connected components. Let G be such a graph with k+1 3-connected components. Assume that G has a separation pair $\{u,v\}$. Let G_1 and G_2 be split graphs of G with respect to the separation pair $\{u,v\}$. Clearly,

$$|E(G)| = |E(G_1)| + |E(G_2)| - 2 \quad \text{and} \quad |V(G)| = |V(G_1)| + |V(G_2)| - 2.$$

By the proof of Lemma 1, none of G_1 and G_2 has a subgraph homeomorphic to $K_{3,3}$. Thus, if both G_1 and G_2 are simple then by the inductive hypothesis we have

$$|E(G_i)| \leq 3|V(G_i)| - 5$$

for each G_i (i=1,2) since G_i has fewer 3-connected components than G, and consequently, we have

$$|E(G)| = |E(G_1)| + |E(G_2)| - 2 \leq 3|V(G_1)| + 3|V(G_2)| - 12 = 3|V(G)| - 6.$$

If G_1 or G_2, say G_1, is not simple, then $G_1 - \{(u,v)\}$ and G_2 are both simple. By the inductive hypothesis, we have

$$|E(G_1)| \leq 3|V(G_1)| - 4 \quad \text{and} \quad |E(G_2)| \leq 3|V(G_2)| - 5$$

and consequently, we have

$$|E(G)| \leq 3|V(G)| - 5.$$

If G is not 3-connected and G has not a separation pair, then each 2-connected component is a 3-connected graph. In this case let G' be a 2-connected component. Let G'' be the graph obtained from $G - E(G')$ by deleting all isolated vertices. We can always choose G' in such a way that G' and G'' have at most one vertex in common. Thus,

$$|E(G)| = |E(G')| + |E(G'')| \quad \text{and} \quad |V(G)| \geq |V(G')| + |V(G'')| - 1.$$

By the same argument as described above, we have

$$|E(G)| \leq 3|V(G)| - 7.$$

Thus we obtain the lemma. □

Now we can design an $O(|V(G)|)$-time algorithm to determine whether a given input simple graph G has a subgraph homeomorphic to $K_{3,3}$ as follows.

Algorithm A.

Step 0. For a given input simple graph G, if $|V(G)| \geq 2$ and $|E(G)| \geq 3|V(G)| - 4$ then return "yes" (G has a subgraph homeomorphic to $K_{3,3}$ by Lemma 5).

Step 1. Decompose G into 3-connected components $D(G) = \{G_1, G_2, \ldots, G_k\}$ by using a linear-time decomposition algorithm [10].

Step 2. For each G_i in $D(G)$, determine whether G_i is nonplanar or not, by using a linear-time planarity testing algorithm [11].

Step 3. If some G_i is nonplanar and distinct from K_5, then return "yes" (G has a subgraph homeomorphic to $K_{3,3}$ by Lemma 4). Otherwise return "no".

The correctness of the Algorithm A is evident from Lemmas 4 and 5. As for the time complexity, the graph G in Step 1 has no more edges than $3|V(G)|-5$. Thus Step 1 requires only $O(|V(G)|)$ time. Similarly, Steps 2 and 3 require only $O(|V(G)|)$ time. Thus we have the following theorem.

Theorem 1. Algorithm A correctly determines whether an input simple graph G has a subgraph homeomorphic to $K_{3,3}$ in $O(|V(G)|)$ time.

4. Algorithm for Finding a Subgraph Homeomorphic to $K_{3,3}$

Algorithm A can be modified in such a way that it actually finds a subgraph of a graph G homeomorphic to $K_{3,3}$, if G has such a subgraph. To describe an algorithm, we need only one lemma.

Lemma 6. Let D be a decomposition of a graph G into 3-connected components. Let $\{e_1,e_2,\ldots,e_r\}$ be the set of all virtual edges of a 3-connected component G' in D. Then there is a set of vertex-disjoint paths in G, say $\{P_1,P_2,\ldots,P_r\}$, such that each P_j $(j=1,2,\ldots,r)$ connects the two end-vertices of e_j and contains no edge of G'.

Proof. By the definition of split graphs, each virtual edge is contained in exactly two 3-connected components in the decomposition D. Furthermore, two 3-connected components have one or zero virtual edge in common. Let $D' = \{F_1,F_2,\ldots,F_h\}$ be the set of graphs obtained from the set of 3-connected components in D by merging all virtual edges except the edges e_1, e_2, $\ldots$, e_r. Then G' is clearly contained in D' and each graph F_j in D' distinct from G' has one or zero virtual edge. Thus, $r \leq h-1$ in general, and $r=h-1$ if G is 2-connected. We can assume, without loss of generality, that $F_h=G'$ and, for each e_j $(j=1,2,\ldots,r)$, the graph F_j contains e_j. By the definition of a separation pair, the subgraph $F_j-\{e_j\}$ of F_j contains a path P_j connecting the two end-vertices of e_j. Clearly, the paths P_1, P_2, $\ldots$, P_r are vertex-disjoint paths of G and contain no edge of G'. $\square$

Now, we can obtain the following algorithm.

Algorithm MAF.

Step 0. For a given input simple graph G, if $|V(G)| \geq 2$ and $|E(G)| \geq 3|V(G)|-4$ then let G' be any subgraph of G consisting of $3|V(G)|-4$ edges and set $G:=G'$. (G' has a subgraph homeomorphic to $K_{3,3}$ by Lemma 5.)

Step 1. Decompose G into 3-connected components $D=\{G_1,G_2,\ldots,G_k\}$ by

using a linear-time decomposition algorithm [12].

  <u>Step 2</u>. For each G_i in D, determine whether G_i is nonplanar by using a linear-time planarity testing algorithm [13].

  <u>Step 3</u>. If there is no nonplanar graph different from K_5, then return "no" (G has no subgraph homeomorphic to $K_{3,3}$ by Lemma 4). Otherwise, let G_i be a nonplanar graph different from K_5.

  <u>Step 4</u>. Find a minimal nonplanar subgraph G_i' of G_i. (G_i' is homeomorphic to K_5 or $K_{3,3}$ by Lemma 2.)

  <u>Step 5</u>. If G_i' is homeomorphic to K_5 then find a subgraph G' of G_i homeomorphic to $K_{3,3}$ by the same technique as used in the proof of Lemma 3. Otherwise, set $G':=G_i'$. (G' is homeomorphic to $K_{3,3}$.)

  <u>Step 6</u>. For all the virtual edges $e_1=(u_1,v_1)$, $e_2=(u_2,v_2)$, ..., $e_q=(u_q,v_q)$ in E(G'), find q vertex-disjoint paths P_1, P_2, ..., P_q in G such that each P_j (j=1,2,...,q) connects the two end-vertices u_j and v_j of the edge e_j and contains no edges of G'.

  <u>Step 7</u>. Return "the graph H'" obtained from G' by replacing each virtual edge $e_j=(u_j,v_j)$ (j=1,2,...,q) of G' with the corresponding path P_j.

  The correctness of the algorithm immediately follows from Lemmas 2-6. Therefore, we concentrate on the time complexity of the algorithm. Steps 0-3 require only $O(|V(G)|)$ time, because the graph G after Step 0 contains at most $3|V(G)|-4$ edges. By Lemma 6, Step 6 can be implemented by the so-called path finding algorithm which requires only linear time [1]. Thus, Steps 6 and 7 require only $O(|V(G)|)$ time.

  As for Step 5, we can obtain a subgraph G' of G_i homeomorphic to $K_{3,3}$ in $O(|V(G)|)$ time by using a network flow algorithm as follows [5]. Suppose that G_i' is homeomorphic to K_5. Let s and t be two new vertices.

  Now assume that $G_i'=K_5$. We first identify the vertex s with an arbitrarily chosen vertex v of G_i which is not in G_i', and next add to G_i five new edges each connecting a distinct vertex of G_i' and the vertex t. Let G_a be the resulting graph. Then it is clear that G_i has three vertex-disjoint paths from the vertex v to three distinct vertices of G_i' if and only if G_a has three vertex-disjoint paths connecting the vertices s and t. For G_a, consider the following directed graph G_{ad} obtained by first splitting each vertex u into u^- and u^+ with making an arc (u^+,u^-), and then making arcs (u^-,v^+) and (v^-,u^+) if and only if there is an edge (u,v) in G_a. Then it is easy to see that G_a has three vertex-disjoint

paths connecting the two vertices s and t if and only if G_{ad} has three edge-disjoint paths from s^- to t^+. Now consider the network N_{ad} with the entrance s^- and the exit t^+, which is obtained from G_{ad} by associating all arcs unit capacities. Then, G_{ad} has three edge-disjoint paths from s^- to t^+ if and only if N_{ad} has a flow of value three from s^- to t^+. Thus, G_i has three vertex-disjoint paths from v to three distinct vertices of G_i' if and only if N_{ad} has a flow F of value three from s^- to t^+. Since G_i is 3-connected, G_i has always such three vertex-disjoint paths. Thus, we can find such a flow F in $O(|E(G_i)|)$ time by using a network flow algorithm in [20], since N_{ad} has $O(|E(G_i)|)$ edges and F is of value three. It is trivial to obtain such three vertex-disjoint paths from the flow F. Thus, we can obtain the subgraph G' of G_i homeomorphic to $K_{3,3}$ in $O(|E(G_i)|)$ time if $G_i'=K_5$ (see Fig. 6).

Next assume that $G_i' \neq K_5$. Let u and v be two vertices of degree 4 in G_i' such that there is a path $P(u,v)$ of length ≥ 2 of G_i' which not only connects the vertices u and v but also contains no other vertices of degree 4 in G_i'. For each vertex y of G_i', if y is on the path $P(u,v)$ then we add a new edge (s,y), otherwise we add a new edge (t,y). Then we delete two vertices u and v. Let G_a be the resulting graph obtained from G_i. From a shortest path P in G_a connecting s and t, we can easily find a path of G_i such that it not only connects a vertex w $(\neq u,v)$ on the path $P(u,v)$ and a vertex x of G_i' which is not on the path $P(u,v)$ but also contains no vertex of G_i' except w and x. Such a shortest path of G_a can be obtained in $O(|E(G_i)|)$ time by the so-called breadth-first search algorithm [1]. Thus, a subgraph G' of G_i homeomorphic to $K_{3,3}$ is obtained in $O(|E(G_i)|)$ time even if $G_i' \neq K_5$.

Therefore, Step 5 requires only $O(|V(G)|)$ time.

These observations imply that if Step 4 requires $g(|V(G)|)$ time then the algorithm MAF requires only $O(g(|V(G)|))$ time. The following simple method achieves $g(|V(G)|)=O(|V(G)|^2)$.

```
begin
    let G_i' be any subgraph of G_i consisting of 3|V(G_i)|-5 edges;
    {G_i' is nonplanar by Euler's formula [9]}
    G_i":=G_i';
    for each edge e of G_i" do
        if G_i'-{e} is nonplanar then G_i':=G_i'-{e}
end;
```

In this method we are trying to delete edges of G_i one by one without violating the nonplanarity. If an edge cannot be deleted without violating, then it is called a <u>critical</u> edge. Critical edges are not deleted and they do not lose their criticality when other edges are deleted. Therefore, we end up with a nonplanar subgraph of G_i which consists entirely of critical edges, that is, a minimal nonplanar subgraph of G_i is obtained. By Kuratowski's Theorem (Lemma 2), it is homeomorphic to K_5 or $K_{3,3}$. Each edge is treated once in this method, and each such treatment calls a linear-time planarity testing [11]. Hence, this method requires only $O(|V(G_i)|^2)$ time.

Thus we have the following theorem.

<u>Theorem 2</u>. For a given input simple graph G, Algorithm MAF not only correctly determines whether G has a subgraph homeomorphic to $K_{3,3}$ but also finds a subgraph of G homeomorphic to $K_{3,3}$ in $O(|V(G)|^2)$ time if G has such a subgraph. Furthermore, the time complexity of the algorithm depends only on that of Step 4, that is, if Step 4 requires $g(|V(G)|)$ time then the whole algorithm also requires only $O(g(|V(G)|))$ time.

5. Applications

The technique used in the previous section can be applied to many graph problems including the subgraph homeomorphism problems for other fixed pattern graphs. Recall that we could obtain an efficient algorithm for the subgraph homeomorphism problem for the graph $K_{3,3}$ by the following facts.

(i)   A graph G has a subgraph homeomorphic to a 3-connected graph H if and only if there is a 3-connected component of G that has a subgraph homeomorphic to H.

(ii) There is an efficient algorithm for decomposing a graph into 3-connected components [10].

(iii) There is an efficient algorithm to determine whether a 3-connected graph has a subgraph homeomorphic to $K_{3,3}$.

These suggest that, for a fixed 3-connected pattern graph H, if there is an efficient algorithm for 3-connected graphs then we can obtain an efficient algorithm for any input graphs. In this section we present this type of applications.

5.1. Subgraph Homeomorphism Problems for Other Fixed Pattern Graphs

We present several pattern graphs H for each of which there is an $O(|V(G)|)$-time algorithm not only to determine whether an input simple graph G has a subgraph homeomorphic to H but also to find a subgraph homeomorphic to H if G has such a subgraph. Included among them are the graphs K_4, G_6 and $K_{2,3}$ (Fig. 2). We first consider the graph K_4 as a fixed pattern graph H. By an argument similar to the one described before we have the following lemmas.

Lemma 7. A 3-connected graph G with four or more vertices has a subgraph homeomorphic to K_4.

Proof. Let v be any vertex of a 3-connected graph G with four or more vertices. Since the subgraph $G-\{v\}$ of G is 2-connected, for any edge $e=(v_0,v_1)$ and any vertex w not incident with e, $G-\{v\}$ has a cycle containing e and w [10]. Let $C = v_0,v_1,\ldots,v_p(=w),v_{p+1},\ldots,v_m(=v_0)$ be such a cycle. Since G is 3-connected, there are three vertex-disjoint paths from v to three distinct vertices on the cycle C. We can assume, without loss of generality, that two of the three vertices are on the arc $C[v_1,v_p]$ of C connecting v_1 and $v_p(=w)$ and the other on the arc $C[v_{p+1},v_0]$ connecting v_{p+1} and v_0. Thus, by an easy inspection, one can see that the graph obtaind from the cycle C and the three vertex-disjoint paths is homeomorphic to K_4. □

Since the graph K_4 is 3-connected, we have the following lemma by Lemmas 1 and 7.

Lemma 8. A graph G has a subgraph homeomorphic to K_4 if and only if there is a 3-connected component of G with four or more vertices.

Thus, we have the following lemma by a calculation similar to the one in the proof of Lemma 5.

Lemma 9. If a simple graph G with two or more vertices has no subgraph homeomorphic to K_4, then $|E(G)| \leq 2|V(G)|-3$.

By Lemmas 7-9, we can obtain the following algorithm, in the same way as before, which not only determines whether any input simple graph G has a subgraph homeomorphic to K_4 but also finds a subgraph homeomorphic to K_4 if G has such a subgraph.

Algorithm BF.

Step 0. For a given input simple graph G, if $|V(G)|\geq 2$ and

$|E(G)| \geq 2 |V(G)| -2$ then let G' be any subgraph of G consisting of $2|V(G)|-2$ edges and set $G:=G'$. (G' has a subgraph homeomorphic to K_4 by Lemma 9).

Step 1. Decompose G into 3-connected components $D(G)=\{G_1,G_2,\ldots,G_k\}$ by using a linear-time decomposition algorithm.

Step 2. If there is no 3-connected component with four or more vertices, then return "no" (G has no subgraph homeomorphic to K_4 by Lemma 8). Otherwise let G_i be a graph in $D(G)$ with four or more vertices. (G_i has a subgraph homeomorphic to K_4 by Lemma 7.)

Step 3. Find a subgraph G' of G_i homeomorphic to K_4 by the method described in the proof of Lemma 7.

Step 4. For all the virtual edges $e_1=(u_1,v_1)$, $e_2=(u_2,v_2)$, $\ldots$, $e_q=(u_q,v_q)$ in $E(G')$, find q vertex-disjoint paths P_1, P_2, $\ldots$, P_q in G such that each P_j ($j=1,2,\ldots,q$) connects the two end-vertices u_j and v_j of the edge e_j and contains no edges of G'. Return "the graph H'" obtained from G' by replacing each virtual edge $e_j=(u_j,v_j)$ of G' with the corresponding path P_j.

By the same argument as in Section 4, we have the following theorem.

Theorem 3. For a given input simple graph G, Algorithm BF not only correctly determines whether G has a subgraph homeomorphic to K_4 but also finds a subgraph homeomorphic to K_4 if G has such a subgraph in $O(|V(G)|)$ time.

Similarly, for the graph G_6 shown in Fig. 2, we can obtain an $O(|V(G)|)$-time algorithm which not only determines whether an input simple graph G has a subgraph homeomorphic to G_6 but also finds a subgraph homeomorphic to G_6 if G has such a subgraph. The algorithm is based on the following lemma. A proof of this lemma and details of the algorithm are left to the readers.

Lemma 10. For a 3-connected graph G with n ($n \geq 6$) vertices, G has a subgraph homeomorphic to G_6 if and only if G is not any of the following graphs: the wheel W_{n-1} of order $n-1$, the complete bipartite graph $K_{3,n-3}$ with three left vertices and $n-3$ right vertices, and the graphs $(P_2 \cup K_1)+\overline{K}_{n-3}$, $P_3+\overline{K}_{n-3}$ and $K_3+\overline{K}_{n-3}$ obtained from $K_{3,n-3}$ by adding one, two and three edges among the left vertices, respcetively (see Fig. 8). (A wheel of order $n-1$, denoted by W_{n-1}, is the graph obtained from a cycle C_{n-1} of length $n-1$ by adjoining one new vertex v and $n-1$ new links joining v and the $n-1$ vertices of C_{n-1}.)
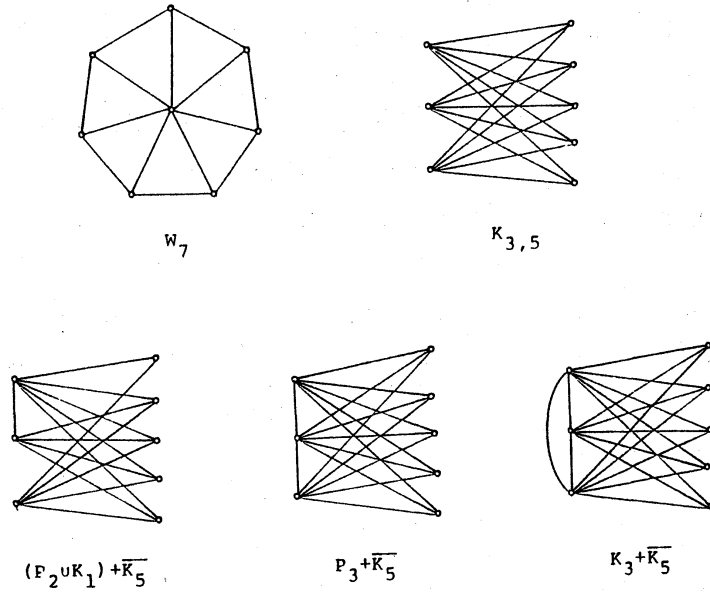
W_7 $K_{3,5}$

$(P_2 \cup K_1) + \overline{K_5}$ $P_3 + \overline{K_5}$ $K_3 + \overline{K_5}$

Fig. 8. Graphs W_{n-1}, $K_{3,n-3}$, $(P_2 \cup K_1) + \overline{K_{n-3}}$, $P_3 + \overline{K_{n-3}}$
and $K_3 + \overline{K_{n-3}}$ (in the case of n=8).

Although we have considered only 3-connected graphs as a pattern graph, the technique can be applied to some 2-connected graphs. Now we consider the graph $K_{2,3}$ (Fig. 2) as a fixed pattern graph H. Clearly, $K_{2,3}$ is not 3-connected.

Lemma 11. A 3-connected graph G with five or more vertices has a subgraph homeomorphic to $K_{2,3}$.

Proof. By Lemma 7, such a graph G has a subgraph homeomorphic to K_4. By an argument similar to the one in the proof of Lemma 3, one can easily prove that G has a subgraph homeomorphic to $K_{2,3}$. □

In the proof of Lemma 6, if G is a simple graph then we can always obtain, for any r distinct virtual edges $e_1 = (u_1, v_1)$, $e_2 = (u_2, v_2)$, ..., $e_r = (u_r, v_r)$ of the same 3-connected component G', r vertex-disjoint paths P_1, P_2, ..., P_r in G such that each P_j (i) is of length ≥ 2, (ii) connects the two end-vertices u_j and v_j of the edge e_j, and (iii) contains no edge of G'. By this observation, we can obtain the following lemma.

Lemma 12. A graph G has a subgraph homeomorphic to $K_{2,3}$ if and only if there is a 3-connected component satisfying one of the following.

(i) It has five or more vertices.

(ii) It is the graph K_4 with one or more virtual edges.

(iii) It is the graph K_3^* with three virtual edges.

Proof. (Sufficiency) Since we can consider virtual edges of the same 3-connected component as vertex-disjoint paths of length ≥ 2, it is easily seen that if G has a 3-connected component satisfying (ii) or (iii) then G has a subgraph homeomorphic to $K_{2,3}$. If G has a 3-connected component satisfying (i) then the 3-connected component has a subgraph homeomorphic to $K_{2,3}$ by Lemma 11, and consequently, G has a subgraph homeomorphic to $K_{2,3}$.

(Necessity) If any 3-connected component of G satisfies none of (i), (ii) and (iii), then every 3-connected component of G satisfies one of the following.

(a) It is K_4 with no virtual edge.

(b) It is K_3.

(c) It has two or fewer vertices with two or fewer virtual edges.

If a 3-connected component of G satisfies (a) then it is a 2-connected component whose edges are not contained in any subgraph of G homeomorphic to $K_{2,3}$. Therefore, we can assume that no 3-connected component of G satisfies (a). Suppose that every 3-connected component of G satisfies (b) or (c). Then it can be easily seen that G has no subgraph homeomorphic to $K_{2,3}$. $\Box$

By the same calculation as before, we have the following lemma.

Lemma 13. If a simple graph G with two or more vertices has no subgraph homeomorphic to $K_{2,3}$ then $|E(G)| \leq 2|V(G)| - 2$.

By Lemmas 11-13, we can obtain the following algorithm.

Algorithm CF.

Step 0. For a given input simple graph G, if $|V(G)| \geq 2$ and $|E(G)| \geq 2|V(G)| - 1$ then let G' be any subgraph of G consisting of $2|V(G)| - 1$ edges and set G:=G'. (G' has a subgraph homeomorphic to $K_{2,3}$ by Lemma 13).

Step 1. Decompose G into 3-connected components $D(G) = \{G_1, G_2, \ldots, G_k\}$ by using a linear-time decomposition algorithm.

Step 2. If there is no 3-connected component satisfying (i), (ii) or (iii) of Lemma 12, then return "no" (G has no subgraph homeomorphic to $K_{2,3}$ by Lemma 12). Otherwise let G_i be a graph satisfying (i), (ii) or (iii) of Lemma 12. (G has a subgraph homeomorphic to $K_{2,3}$ by Lemma 12.)

Step 3. Find a subgraph of G homeomorphic to $K_{2,3}$ by the method described in the proof of Lemma 11.

By the same argument as before, we have the following theorem.

Theorem 4. For a given input simple graph G, Algorithm CF not only correctly determines whether G has a subgraph homeomorphic to $K_{2,3}$ but also finds a subgraph homeomorphic to $K_{2,3}$ if G has such a subgraph in $O(|V(G)|)$ time.

Similarly, for each pattern graph H of graphs C_4 and C_5 (Fig. 9), we can obtain an $O(|V(G)|)$-time algorithm. Details are left to the readers. Note that, for the complete bipartite graphs $K_{2,p}$ shown in Fig. 8, one can easily design a polynomial-time (for example, $O(p|V(G)|^4)$-time) algorithm using a network flow algorithm (see [5]).
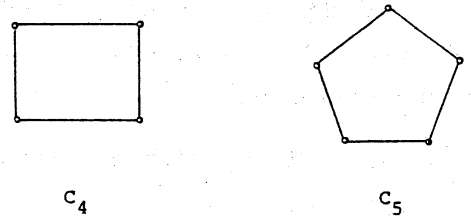

C_4 C_5

Fig. 9.   Graphs C_4 and C_5.

5.2. Testing Graph Properties

Some graph property π is characterized in terms of excluded homeomorphic subgraphs: a graph G satisfies π if and only if there is a set $S(\pi)$ of graphs such that G has no subgraph homeomorphic to a graph in $S(\pi)$. "Planarity", "outerplanarity", etc. are examples of such a property. In this section we show that the previous arguments can be applied to testing such a property. First consider the "outerplanarity". Since G is outerplanar if and only if G has no subgraph homeomorphic to $K_{2,3}$ or K_4 [9], we can easily design an $O(|V(G)|)$-time algorithm to determine whether an input simple graph G is outerplanar by Theorems 3 and 4. However, by Lemmas 8 and 12, we can also design a more simple $O(|V(G)|)$-time algorithm based on the following lemma which is a new characterization of outerplanar graphs.

Lemma 14. A graph G is outerplanar if and only if every 3-connected component of G is the graph K_3 or has two or fewer vertices with two or fewer virtual edges.

Similarly, since a graph G is series-parallel if and only if G has no subgraph homeomorphic to K_4 [4], we can obtain $O(|V(G)|)$-time algorithm to determine whether a simple graph G is series-parallel. However, there have already been $O(|V(G)|)$-time algorithms for testing these properties [18,24].

6. Concluding Remarks

In this paper, we have shown that subgraph homeomorphism problem for the fixed graph $K_{3,3}$ is solvable in polynomial time. To obtain the result, we have first observed that, for any 3-connected graph H, a graph G has a subgraph homeomorphic to H if and only if there is a 3-connected component of G that has a subgraph homeomorphic to H, and then employed an efficient algorithm for decomposing a graph into 3-connected components and an efficient planarity testing algorithm. As applications of this technique, we have first presented several pattern graphs, such as K_4, G_6, $K_{2,3}$, C_4 and C_5 (Figs. 2 and 9), for each H of which, there is an $O(|V(G)|)$-time algorithm to find a subgraph homeomorphic to H. Then we have presented an $O(|V(G)|)$-time algorithm to determine whether a graph G satisfies a property π, such as π = "outerplanar graph" and "series-paralell graph".

As for finding a minimal nonplanar subgraph of a nonplanar graph G, Mr. Kaschube informed us recently that Professor Williamson has an $O(|V(G)|)$-time algorithm [26]. Thus, the time complexity of our algorithm MAF for finding a subgraph of G homeomorphic to $K_{3,3}$ becomes $O(|V(G)|)$ (see Theorem 2 and Step 4 of Algorithm MAF in Section 4). Mr. Kaschube also imformed us that he independently obtained the similar result for the subgraph homeomorphism problem for $K_{3,3}$ [13]. It remains open whether the subgraph homeomorphism problem for the fixed graph K_5 is polynomial-time solvable.

Even if the subgraph homeomorphism problem for a fixed 3-connected graph H were solvable in polynomial time, the problem of finding a maximum subgraph that has no subgraph homeomorphic to H is NP-complete [3]. Thus, the problem of finding a maximum subgraph that has no subgraph homeomorphic to H is NP-complete for each H, $H=K_{3,3}$, K_4, G_6. Similarly, the problem of finding a maximum subgraph that has no subgraph homeomorphic to H is NP-complete for each H, $H=K_{2,3}$, K_4 and G_6, even if it is restricted to planar graphs [2].

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