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A Note on The Invariant Holonomic System

by

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INTRODUCTION. In this report, I shall prove a theorem which is an interpretation of the main result of [S]. I have studied in [S] the holonomic system which governs an invariant spherical hyperfunction (= ISH) on the tangent space of a symmetric space. In particular, if the tangent space in question satisfies a condition introduced there, I have shown that there exists no non-zero singular ISH on it (cf. Theorem 5.2 in [S]). This result is interpreted in terms of $\underline{\underline{D}}$-modules. This will be shown in Theorem 1 of this report.

§1. Let $\underline{g}$ be a complex semisimple Lie algebra and let σ be its complex linear involution. Then we obtain a direct sum decomposition $\underline{g} = \underline{h} + \underline{q}$, where $\underline{h} = \{X \in \underline{g}; \sigma X = X\}$ and $\underline{q} = \{X \in \underline{g}; \sigma X = -X\}$. Let $S(\underline{q})$ be the symmetric algebra over $\underline{q}$. If $\underline{q}^*$ is the dual of $\underline{q}$, then $S(\underline{q})$ is regarded as the polynomial algebra on $\underline{q}^*$. On the other hand, a constant coefficient differential operator $\partial(P)$ on $\underline{q}$ is associated with every element $P \in S(\underline{q})$. For example, if $A \in \underline{q}$, then $\partial(A)$ is the vector field on $\underline{q}$ defined by $(\partial(A)f)(X) = \frac{d}{dt} f(X + tA)|_{t=0}$. Let H be a connected Lie group with Lie algebra $\underline{h}$. We may assume that H acts on $\underline{q}$ as $\underline{h}$

does. Let $I(\underline{q})$ be the subalgebra of $S(\underline{q})$ consisting of H-invariant elements. Then $\underline{\underline{N}}(\underline{q}) = \{X \in \underline{q};\ P(X) = P(0)$ for all $P \in I(\underline{q})\}$ is called the nilpotent subvariety of $\underline{q}$. An element X of $\underline{q}$ is contained in $\underline{\underline{N}}(\underline{q})$ if and only if $ad_{\underline{g}}(X)$ is a nilpotent matrix. Take a Cartan subspace $\underline{a}$ of $\underline{q}$. Then, by definition, almost all elements of $\underline{q}$ are H-conjugate to those of $\underline{a}$. Let $\underline{q}'$ be the totality of $\underline{q}$-regular semisimple elements of $\underline{q}$ and put $\underline{q}_s = \underline{q} - \underline{q}'$. There is an H-invariant homogeneous polynomial $D(X)$ on $\underline{q}$ called the discriminant such that $\underline{q}_s = \{X \in \underline{q};\ D(X) = 0\}$.

Since the notations introduced above are not familiar to non-exparts in this area, it seems worthwhile to give an example. Take $\underline{g} = \underline{sl}(n, \mathbb{C})$ and let σ be the involution of $\underline{g}$ defined by $\sigma X = -{}^tX$. In this case, $\underline{h} = \underline{so}(n, \mathbb{C})$ (= the totality of skew-symmetric matrices with trace zero) and $\underline{q} = \{X \in \underline{g};\ {}^tX = X\}$. We may take $H = SO(n, \mathbb{C})$. In this case, the totality of diagonal matrices with trace zero is one of Cartan subspaces of $\underline{q}$. So we take this as $\underline{a}$. As to the algebra $I(\underline{q})$, it is a polynomial algebra with $(n-1)$ generators $P_2, \ldots, P_n$ which are defined as follows :

$$\det(t + X) = t^n + P_2(X)t^{n-2} + \ldots + P_n(X).$$

Here we have identified $\underline{q}^*$ with $\underline{q}$ by the Killing form $B(X,Y) = tr(XY)$ $(X, Y \in \underline{q})$. Let $D(X)$ be the discriminant of $\det(t + X)$. Then $\underline{q}_s = \{X \in \underline{q};\ D(X) = 0\}$. By definition, an element X of $\underline{q}$ is contained in $\underline{q}_s$ if and only if at least two eigenvalues of X coincide. Last the nilpotent subvariety $\underline{\underline{N}}(\underline{q})$ consists of symmetric nilpotent matrices.

§2. Return to the general situation.

As usual, let $\underline{\underline{O}}_{\mathfrak{g}}$ and $\underline{\underline{D}}_{\mathfrak{g}}$ be the sheaf of holomorphic functions on $\underline{\mathfrak{g}}$ and that of differential operators with coefficients in $\underline{\underline{O}}_{\mathfrak{g}}$, respectively.

For any $A \in \underline{h}$, define a vector field $\tau(A)$ on $\underline{\mathfrak{g}}$ by $(\tau(A)f)(X) = \frac{d}{dt} f(X + t[X,A])|_{t=0}$. A function $f(X)$ on $\underline{\mathfrak{g}}$ is invariant if $f(h \cdot X) = f(X)$ for any $h \in H$. In this case, it follows that $(\tau(A)f)(X) = 0$ for any $A \in \underline{h}$. Noting this, we define that a function $f(X)$ on an open subset U of $\underline{\mathfrak{g}}$ is locally invariant if $\tau(A)f = 0$ for any $A \in \underline{h}$.

For any $\lambda \in \underline{\mathfrak{g}}^*$, we define the sheaf $\underline{\underline{J}}_{\lambda}$ of left ideals of $\underline{\underline{D}}_{\mathfrak{g}}$ as follows:

$$\underline{\underline{J}}_{\lambda} = \sum_{P \in I(\underline{\mathfrak{g}})} \underline{\underline{D}}_{\mathfrak{g}}(\partial(P) - P(\lambda)) + \sum_{A \in \underline{h}} \underline{\underline{D}}_{\mathfrak{g}}\tau(A)$$

Let $\underline{\underline{M}}_{\lambda} = \underline{\underline{D}}_{\mathfrak{g}}/\underline{\underline{J}}_{\lambda}$ be the sheaf of left $\underline{\underline{D}}_{\mathfrak{g}}$-modules associated to $\underline{\underline{J}}_{\lambda}$. It is known that $\underline{\underline{M}}_{\lambda}$ is coherent as the sheaf of left $\underline{\underline{D}}_{\mathfrak{g}}$-modules. Let u be the canonical generator of $\underline{\underline{M}}_{\lambda}$. Then $\underline{\underline{M}}_{\lambda} = \underline{\underline{D}}_{\mathfrak{g}}u$ and u satisfies the system of differential equations

$$(1) \qquad \begin{cases} (\partial(P) - P(\lambda))u = 0 & \text{for all } P \in I(\mathfrak{g}) \\ \tau(A)u = 0 & \text{for all } A \in \underline{h}. \end{cases}$$

By definition, u is regarded as a locally invariant function on $\underline{\mathfrak{g}}$ which is a joint eigenfunction of the differential operators $\partial(P)$ $(\forall P \in I(\underline{\mathfrak{g}}))$.

Since $\underline{\mathfrak{g}}$ is an affine space, the cotangent bundle $T^*\underline{\mathfrak{g}}$ is identified with $\underline{\mathfrak{g}} \times \underline{\mathfrak{g}}^* \simeq \underline{\mathfrak{g}} \times \underline{\mathfrak{g}}$ (by the Killing form). Then it follows from the definition that $Ch(\underline{\underline{M}}_\lambda)$ is contained in the analytic subset

$$\Lambda = \{ (X,\Xi) \in \underline{g} \times \underline{g} \; ; \; [X,\Xi] = 0, \quad \Xi \in \underline{N}(\underline{g}) \}$$

of $T^*\underline{g}$. Since $\dim \Lambda = \dim \underline{g}$, we find that $\underline{\underline{M}}_\lambda$ is holonomic in the sense of Sato-Kashiwara.

The purpose of this report is to prove the next theorem.

Theorem 1. Assume the following. There is a real semisimple Lie algebra $\underline{g}_0$ and it maximal compact subalgebra $\underline{k}_0$ such that $\underline{g}_0$ is a normal real form of $\underline{g}$ and that $\underline{h}$ is a complexification of $\underline{k}_0$. Then, for any $\lambda \in \underline{g}^*$, $\underline{\underline{M}}_\lambda$ has no non-trivial coherent quotient $\underline{\underline{L}}$ such that $\mathrm{Supp}\, \underline{\underline{L}} \subseteq \underline{g}_s$.

The assumption in Theorem 1 holds for the case where $\underline{g} = \underline{sl}(n, \mathbb{C})$ and $\underline{h} = \underline{so}(n, \mathbb{C})$.

This connects with Theorem 6.7.2 in [HK]. Hotta and Kashiwara mentioned that Theorem 6.7.2 in [HK] is shown by an argument similar to a result of Harish-Chandra in [H]. I will give in §4 a proof of Theorem 1 which is an analogue of Theorem 6.7.2 in [HK].

§3. Let X be a complex manifold of dim = n and let Y be a closed submanifold of X with dim Y n-k. As usual, let $\underline{\underline{D}}_X$ denote the sheaf of differential operators on X.

Let us consider the algebraic local cohomology $\underline{\underline{H}}^k_{[Y]}(\underline{\underline{O}}_X)$ which is a sheaf of $\underline{\underline{D}}_X$-modules. This sheaf is usually denoted by $\underline{\underline{B}}_{Y|X}$. If $k = 0$, this is nothing but the structure sheaf $\underline{\underline{O}}_X$. But if $k \neq 0$, this is not coherent over $\underline{\underline{O}}_X$. In spite of this, it is known that $\underline{\underline{B}}_{Y|X}$ has the structure of left $\underline{\underline{D}}_X$-modules.

It is easy to describe $\underline{\underline{B}}_{Y|X}$ by using local coordinate systems. Let $x = (x_1,\dots,x_n)$ be a local coordinate system on an open subset U of X. Assume that $Y \cap U = \{x \in U;\ x_1 = \dots = x_k = 0\}$. Then, by definition, $\underline{\underline{B}}_{Y|X}$ coincides with

$$\underline{\underline{D}}_X / \sum_{i=1}^{k} \underline{\underline{D}}_X x_i + \sum_{i=k+1}^{n} \underline{\underline{D}}_X \frac{\partial}{\partial x_i} .$$

Note that $\underline{\underline{B}}_{Y|X}$ is an example of holonomic systems and that $Ch(\underline{\underline{B}}_{Y|X})$ coincides with $T^*_Y X$ which is the conormal bundle of Y. The next lemma concerns the converse of this property.

<u>Lemma 2 ([K])</u>. Let $\underline{\underline{M}}$ be a holonomic system on X. Assume that $Ch(\underline{\underline{M}})$ coincides with $T^*_Y X$. Then $\underline{\underline{M}}$ is isomorphic to the direct sum of a finite number of copies of $\underline{\underline{B}}_{Y|X}$.

Let $\underline{\underline{M}}$ be a regular holonomic system on $X - Y$. Assume that $\underline{\underline{M}}$ is extendable to X, namely that there exists a holonomic system $\tilde{\underline{\underline{M}}}$ on X such that $\tilde{\underline{\underline{M}}}|(X-Y) \simeq \underline{\underline{M}}$ as sheaves of left $\underline{\underline{D}}_{X-Y}$-modules.

<u>Definition 3 ([HK])</u>. A holonomic system $\underline{\underline{M}}'$ on X is a minimal extension of $\underline{\underline{M}}$ if the following conditions hold:

(i) $\underline{\underline{M}}'|(X-Y) \simeq \underline{\underline{M}}$ as $\underline{\underline{D}}_{X-Y}$-modules.

(ii) Let $\underline{\underline{L}}$ be a holonomic system on X. Assume that $\underline{\underline{L}}$ is a subquotient of $\underline{\underline{M}}'$ and that $\mathrm{Supp}(\underline{\underline{L}}) \subset Y$. Then $\underline{\underline{L}} = 0$.

Note that minimal extensions of $\underline{\underline{M}}$ are unique up to isomorphism. So we denote by ${}^{\pi}\underline{\underline{M}}$ the minimal extension of $\underline{\underline{M}}$.

§4. Proof of Theorem 1. Let $\underline{\underline{L}}$ be a coherent sheaf of $\underline{\underline{D}}_X$-modules. Assume that $\underline{\underline{L}}$ is a quotient of $\underline{\underline{M}}_\lambda$ and that $\mathrm{Supp}(\underline{\underline{L}})$ is contained in $\underline{g}_s$.

Let $\underline{\underline{C}}_1, \dots, \underline{\underline{C}}_r$ be the totality of mutually distinct H-orbits of $\underline{\underline{N}}(\underline{g})$. For each i, let Λ_i be the closure of the set $\{(X,Y) \in \Lambda \, ; \, Y \in \underline{\underline{C}}_i\}$ and put $Y_i = \{X \in \underline{g}; \, (X,Y) \in \underline{\underline{C}}_i$ for some $Y \in \underline{\underline{N}}(\underline{g})\}$. Then each Y_i is a closed analytic subset of $\underline{g}$. Since $\mathrm{Ch}(\underline{\underline{M}}_\lambda) \subset \Lambda$, it follows from the definition that $\mathrm{Supp}(\underline{\underline{L}})$ coincides with the union $Z = \bigcup_{i \in A} Y_i$ for some subset A of $\{1,\dots,r\}$. The assumption of $\underline{\underline{L}}$ implies that each Y_i $(i \in A)$ is contained in $\underline{g}_s$. Since Z is a locally closed, there exists an $i \in A$ such that Y_i contains an open subset of Z. Take an open subset U of $\underline{g}$ such that $U \cap Y_i$ is a closed submanifold of U. We may assume from the first that $\underline{\underline{L}}|U \neq 0$. For the sake of simplicity, put $\underline{\underline{L}}' = \underline{\underline{L}}|U$ and $Y = U \cap Y_i$. Then it follows that $\mathrm{Ch}(\underline{\underline{L}}') = T^*_Y U$. Hence, by means of Lemma 2, we find that $\underline{\underline{L}}' \simeq \overset{d}{\oplus} \underline{\underline{B}}_{Y|U}$ for some integer d. Then

$$(2) \qquad \underline{\underline{\mathrm{Hom}}}_{\underline{\underline{D}}_U}(\underline{\underline{M}}_\lambda|U, \underline{\underline{L}}') \simeq \overset{d}{\oplus} \underline{\underline{\mathrm{Hom}}}_{\underline{\underline{D}}_U}(\underline{\underline{M}}_\lambda|U, \underline{\underline{B}}_{Y|U}).$$

Now we recall the proof of Theorem 5.2 in [S]. By an argument similar to that, we obtain the next lemma.

Lemma 4. Retain the above notation and the assumption in Theorem 1. Then $\underline{\underline{Hom}}_{\underline{\underline{D}}_U}(\underline{\underline{M}}_\lambda|U, \underline{\underline{B}}_{Y|U}) = 0$.

This lemma combined with (2) implies that $\underline{\underline{L}}$ is not a quotient of $\underline{\underline{M}}_\lambda$. This contradicts the assumption and the theorem is proved.

§5. I proposed in [S] the next conjecture.

Conjecture I. Retain the notation and the assumption in Theorem 1. Then $\underline{\underline{M}}_\lambda$ is a minimal extension of $\underline{\underline{M}}_\lambda|\underline{g}'$.

In virtue of Theorem 1, Conjecture I is reduced to the next one.

Conjecture I'. Retain the same situation in Conjecture I. Let $\underline{\underline{L}}$ be a holonomic system on $\underline{g}$. Assume that $\underline{\underline{L}}$ is a subsheaf of $\underline{\underline{M}}_\lambda$ and that $\mathrm{Supp}(\underline{\underline{L}}) \subset \underline{g}_s$. Then $\underline{\underline{L}} = 0$.

I have no idea to prove this conjecture at present. But I have some examples which agree Cojecture I'. For example, Conjecture I' is true for the cases $(\underline{g}, \underline{h}) = (\underline{sl}(n, \mathbb{C}), \underline{so}(n, \mathbb{C}))$, $n = 2, 3$. The case $n = 2$ is easy to prove but the case $n = 3$ is slightly complicated to prove.

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