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3-dimensional singularities with resolutions whose exceptional sets are toric divisors

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Introduction. It is known by Mumford et al [1] that cusp singularities which appear in compactifications of quotient spaces of tube domains have resolutions whose exceptional sets are toric divisors (see Definition 1.1). In this paper, we consider whether the converse is true in the 3-dimensional case. Namely, if an isolated singularity (V,p) has a resolution whose exceptional set is a toric divisor, then is (V,p) a cusp singularity in the sense of [4]? If a 2-dimensional singularity has a resolution whose exceptional set is a cycle of rational curves, then it is well-known to be a cusp singularity. Hence the answer to the above problem is affirmative in the 2-dimensional case. Moreover, 2-dimensional cusp singularities are taut, i.e., a 2-dimensional isolated singularity is uniquely determined from its weighted dual graph, if the exceptional set is a cycle of rational curves. We also consider whether 3-dimensional cusp singularities have the same property. We only give a partial answer to these problems.

In Section 1, we introduce the weighting of the dual graph of a toric divisor. We show next that it must satisfy some condition (Lemma 1.4). From this fact, we see that the weighted dual graph of a toric divisor which is the exceptional set of a 3-dimensional isolated singularity agrees with that of a cusp singularity.

In Section 2, we show that the isomorphism classes $I(\Delta)$ of 2-dimensional toric divisors which are exceptional sets and which have a fixed weighted dual graph Δ are parametrized by a subset of the cohomology group $H^1(\Gamma,T)$ of a group action. (Proposition 2.4.) In Particular, if the fundamental group of Δ is abelian, then there is a one to one correspondence between $I(\Delta)$ and a finite group $H^1(\Gamma,T)$ and each element of $I(\Delta)$ has a representative isomorphic to the exceptional set of a resolution of a Hilbert modular cusp singularity.

In Section 3, we show that under some assumptions, some neighborhoods U and U' of two isomorphic toric divisors X and X' are formally equivalent (Theorem 3.1). Then by the theorem of Hironaka and Rossi [2], U and U' are actually equivalent. Hence any isolated singularity with a resolution whose exceptional set is isomorphic to a toric divisor X is isomorphic to the isolated singularity (V,p) we obtain by contracting X in U, if (U,X) satisfies the condition of Theorem 3.1.

1. **The weighted dual graph of a toric divisor.** We first recall that a torus embedding Z is an algebraic variety containing an algebraic torus $T = (C^*)^r$ as a Zariski open set such that T acts on Z extending the natural action on itself defined by multiplications. We call a 2-dimensional torus embedding, a toric surface. The union of the 1-dimensional orbits of a compact toric surface by the action of the algebraic torus is a cycle of rational

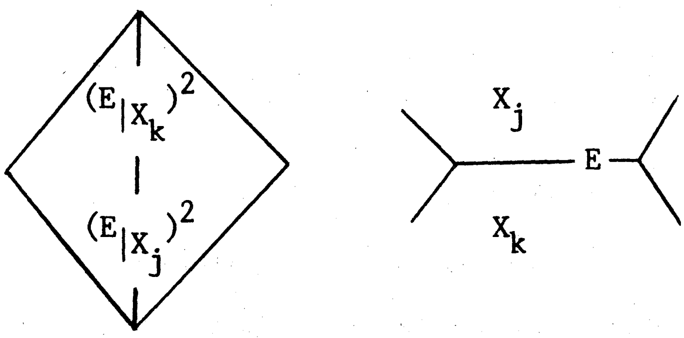
curves.

Definition 1.1. A reduced effective divisor $X = X_1+X_2+ \dots +X_s$ of a 3-dimensional complex manifold is said to be toric if X has only normal croosings as singularities, each irreducible component X_j of X is isomorphic to a compact toric surface Z and the union $\bigcup_{k\neq j} X_j \cap X_k$ of the double curves $X_j \cap X_k$ on X_j coincides with the union of the 1-dimensional orbits.

Let X be a 2-dimensional toric divisor and let Δ be its dual graph. Namely, to each irreducible component X_j of X corresponds to a vertex v_j of Δ so that X_j and X_k (resp. X_j, X_k and X_ℓ) intersect along a curve (resp. at a point) if and only if v_j and v_k are joined by an edge of Δ (resp. v_j, v_k and v_ℓ form a triangle in Δ). Then Δ is a triangulation of a compact topological surface.

Definition 1.2. A weighting of the dual graph of a toric divisor is a pair of integers on both sides of each edge of Δ attached in the following way: Let X_j and X_k be irreducible components of X intersecting along a double curve E. Let v_j, v_k and e be the vertices and the edge of Δ corresponding to X_j, X_k and E, respectively. Then we attach the self-intersection numbers $(E_{|X_j})^2 = (X_j \cdot X_k^2)$ and $(E_{|X_k})^2 = (X_j^2 \cdot X_k)$ of E on X_j and X_k to the sides of the vertices v_j and v_k of the edges e, respectively.

We call Δ with weighting as above a weighted dual graph.



Clearly we have

Lemma 1.3. Let X be a toric divisor of a 3-dimensional complex manifold M and let X_j and X_k be two irreducible components of X intersecting along a double curve E. Then

$$(E_{|X_j})^2 = \deg(O[X_k]_{|E}) \quad \text{and} \quad (E_{|X_k})^2 = \deg(O[X_j]_{|E}).$$

Definition 1.4. We say a weighted dual graph Δ satisfies the monodromy condition at a vertex v of Δ, if the following conditions are satisfied: Let $v_1, v_2, \dots, v_s$ be the vertices of the link of v going around v in this order. We first attach three elements n, n_1 and n_2 of a basis of $\mathbf{Z}^3$ to the vertices v, v_1 and v_2, respectively. Then we can attach the elements $n_3, \dots, n_s, n_{s+1}$ and n_{s+2} of $\mathbf{Z}^3$ to the vertices $v_3, \dots v_s, v_1$ and v_2, respectively, by the equalities

$$(*) \quad n_{i-1} + a_i n_i + n_{i+1} + b_i n = 0 \qquad i = 3, \dots, s+1,$$

where a_i (resp. b_i) is the weight on the side of v_i (resp. v) of the edge joinning v and v_j and $a_{s+1} = a_1$ (resp. b_{s+1}

$= b_1$). Then we require that $n_{s+1} = n_1$, $n_{s+2} = n_2$ and $p(n_1)$, $p(n_2)$, ... , $p(n_s)$ go around $p(n)$ once in this order, where $p : \mathbf{R}^3\setminus\{0\} \to S^2$ is the natural projection onto sphere $S^2 = (\mathbf{R}^3\setminus\{0\})/\mathbf{R}_{>0}$.

Lemma 1.5. Let X be a toric divisor of a 3-dimensional complex manifold. Then the weighting of the dual graph of X satisfies the monodromy condition at all the vertices of Δ.

Proof. Let V be an irreducible component of X and let $D=D_1+D_2+ \dots +D_s$ be the double curves on V. Here, we may assume that D_i and D_{i+1} intersect at a point for each i in $\mathbf{Z}/s\mathbf{Z}$. Since V is a toric surface and D is the union of the 1-dimensional orbits of V, we have $O[V]_{|V} = d_1D_1 + d_2D_2 + \dots + d_sD_s$, for some integers $d_1, d_2, \dots , d_s$ [3, Proposition 6.1]. Then we have $b_i = \deg(O[V]_{|D_i}) = O[V]_{|V}\cdot D_i = d_ia_i + d_{i-1} + d_{i+1}$

by Lemma 1.3, where $b_i = (D_{i|X_i})^2$ and $a_i = (D_{i|V})^2$. Here X_i is the irreducible component of X intersecting V along D_i. On the other hand, associated to a toric surface V, we have s elements n_1, n_2, ... and n_s of Z^2, going around the origin exactly once in this order, each adjacent pair $\{n_i,n_{i+1}\}$ of which form a basis of $\mathbf{Z}^2$ with the relations $n_{i-1} + a_in_i + n_{i+1} = 0$. Let $\bar{n}_i = (n_i,d_i)$ and $\bar{n} = (0,0,-1)$ be the elements of $\mathbf{Z}^3 = \mathbf{Z}^2 \oplus \mathbf{Z}$. Then $\bar{n}_{i-1} + \bar{n}_{i+1} + a_i\bar{n}_i + b_i\bar{n} = 0$ and $\{\bar{n}_i,\bar{n}_{i+1},\bar{n}\}$ form a basis of $\mathbf{Z}^3$ for each i in $\mathbf{Z}/s\mathbf{Z}$. This is nothing but a monodromy condition at the vertex corresponding to V. q.e.d.

Let $X=X_1+X_2+ \dots +X_s$ be a toric divisor of a 3-dimensional complex manifold U with its dual graph Δ. Let S be the compact topological surface of which Δ is a triangulation. Let $\tilde{\Delta}$ be the triangulation of the universal covering space $\tilde{S}$ of S induced from Δ. Choose a basis $\{n_1,n_2,n_3\}$ of $\mathbf{Z}^3$ and a triangle of $\tilde{\Delta}$. Let $\sigma(v_1) = n_1$, $\sigma(v_2) = n_2$ and $\sigma(v_3) = n_3$, where v_1, v_2 and v_3 are the vertices of the triangle. Then by the above lemma, we obtain a map σ : { all vertices of $\tilde{\Delta}$ } $\to \mathbf{Z}^3$ by the equality (*). Moreover, we have a homomorphism $\rho : \pi_1(S) \to GL(3,\mathbf{Z})$ by $\rho(\gamma)\cdot\sigma(v) = \sigma(\gamma\cdot v)$ for each element γ of $\pi_1(S)$ and for the vertices of $\tilde{\Delta}$. Now assume that X in U is contractible to a point and let $\pi : (U,X) \to (V,p)$ be the contraction. Let $[\pi^*f]$ be the zero divisor of the pull-back holomorphic function π^*f on U for a holomorphic function f on V vanishing at p. Then we can write $[\pi^*f] = \sum m_i(f)X_i + Y$ so that the support of each irreducible component of Y is not contained in X. Let m_i be the smallest number among $m_i(f)$'s with f running through all elements of the maximal ideal $\mathfrak{m}$ of O_V at p, and let $X^+ = \sum m_iX_i$. Then we have a canonical linear map

$$h : \mathfrak{m}/\mathfrak{m}^2 \longrightarrow H^0(X,O_X(-X^+)),$$

sending $f + \mathfrak{m}^2$ to $(\pi^*f/g)_{|W\cap X}$, if g is a defining equation for X on an open set W.

Lemma 1.6. After blowing up U along double curves and triple points on X and then replacing X by its total transform, we may assume that each double curve of X is not contained in the fixed locus of the image of h.

Proof. Let $I : (V,p) \to (B,0)$ be an embedding of (V,p) into an open set B of $\mathbf{C}^N$ with $I(p) = 0$ and let $F = \mathrm{Hom}(\mathbf{C}^N,\mathbf{C})$ be the vector space consisting of the linear functions of $\mathbf{C}^N$. We show that there exist a composition $\Pi : W \to U$ of blowing ups along double curves and triple points of the total transforms of X and an open set C of F satisfying the following property:

(P) $(\Pi|_{W(p)}\cdot\pi)^*f = y_1^{a_1}\cdot y_2^{a_2}\cdot y_3^{a_3}\cdot f_p$ with $f_p(0) \neq 0$, $a_1, a_2, a_3 \in \mathbf{Z}_{>0}$, for any function f in C on some neighborhood $W(p)$ of each triple point p of $X' = \Pi^{-1}(X)$ with a local coordinate (y_1,y_2,y_3) such that $W(p) \cap X'$ is defined by $y_1\cdot y_2\cdot y_3 = 0$. Then we have an effective divisor $X^!$ of W supported by X' and a homomorphism $h : \mathfrak{m}/\mathfrak{m}^2 \to H^0(X',\mathcal{O}_{X'}(-X^!))$ such that the zero divisors of $h(f+\mathfrak{m}^2)$ contain no double curve of X' for all f in C and thus we have the assertion of the theorem.

We denote by $\Gamma_+(f)$, the Newton's polyhedra $\Gamma_+(K)$ of $K = \{ n \in \mathbf{Z}^3_{\geq 0} \mid a_n \neq 0 \}$, i.e., the convex hull of $\bigcup_{n \in K} (n+\mathbf{R}^3_{\geq 0})$ for a germ $f = \sum_{n \in \mathbf{Z}^3_{\geq 0}} a_n x^n$ of a holomorphic function of $\mathbf{C}^3$ at 0. We use the following lemma of Varchenko [5, Lemma 2.13] by replacing $\mathbf{R}^k$ by $\mathbf{C}^3$.

Lemma. There exists a finite nonsingular r.p.p. decomposition (Z^3,Σ) of Z^3 with $|\Sigma| := \bigcup_{\sigma\in\Sigma}\sigma = R^3_{\geq 0}$ such that we can write $\tau_\sigma^* f = y_1^{a_1}\cdot y_2^{a_2}\cdot y_3^{a_3}\cdot f_\sigma$ with $f_\sigma(0) \neq 0$, for any 3-dimensional cone $\sigma = R_{\geq 0}\, n_1 + R_{\geq 0}\, n_2 + R_{\geq 0}\, n_3$ of Σ and for any holomorphic function f with $\Gamma_+(f) = \Gamma_+(K)$. Here $\Gamma_+(K)$ is a Newton's polyhedra and τ_σ is the restriction of $\tau : \mathrm{Temb}(\Sigma) \to \mathrm{Temb}(\{\text{ faces of } R^3_{\geq 0}\}) = C^3$ to an open neighborhood U_σ of $\mathrm{orb}(\sigma)$ and (y_1,y_2,y_3) is a coordinate of U_σ such that $U_\sigma \cap (\mathrm{Temb}(\Sigma)\setminus T)$ is defined by $y_1\cdot y_2\cdot y_3 = 0$.

Here we note that this lemma also holds, although we replace Σ by any nonsingular subdivision of Σ. Let $t_1, t_2, \ldots, t_\ell$ be the triple points of X and let U_i be an open neighborhood of t_i with a local coordinate (z_i^1,z_i^2,z_i^3) such that $X\cap U_i$ is defined by $z_i^1\cdot z_i^2\cdot z_i^3 = 0$. We choose an open set C of F so that the Newton's polyhedra $\Gamma_+(\pi^* f)$ associated with the coordinate (z_i^1,z_i^2,z_i^3) coincide for all f in C, on each open set U_i. Assume that we have a composition $\Pi_{j-1} : W_{j-1} \to U$ of blowing ups along double curves and triple points of the total transforms of X satisfying the property (P) at each triple point of $X^{(j-1)} = \Pi_{j-1}^{-1}(X)$ contained in $\Pi_{j-1}^{-1}(U_k)$ with $k \leq j-1$. Then we have an r.p.p. decomposition (Z^3,Λ) with $|\Lambda| = R^3_{\geq 0}$ satisfying the following commutative diagram:

$$\begin{array}{ccc} \Pi_{j-1}^{-1}(U_j) & \longrightarrow & U_j \\ \big\downarrow & & \big\downarrow \\ \mathrm{Temb}(\Lambda) \longrightarrow & \mathrm{Temb}(\{\text{ faces of } \mathbb{R}^3_{\geq 0}\}) & = \mathbb{C}^3. \end{array}$$

Let Σ_o be an r.p.p. decomposition of R^3 as in the above lemma for $\Gamma_+(\pi|_{U_j}{}^* f)$ with f in C. We have a subdivision Σ of Σ_o obtained by a succession of divisions of Λ corresponding to blowing ups along double curves and triple points of the total transform of the union of the 2-dimensional orbits of $\mathrm{Temb}(\Lambda)$ (see [3, Corollarly 7.6 and §8]). Then as associated to the above subdivision Σ of Λ, we have a composition $\Pi_j' : W_j \to W_{j-1}$ of blowing ups of W_{j-1} satisfying the following commutative diagram:

$$\begin{array}{ccc} (\Pi_j'\circ\Pi_{j-1})^{-1}(U_j) & \longrightarrow & \Pi_{j-1}(U_j) \\ \big\downarrow & & \big\downarrow \\ \mathrm{Temb}(\Sigma) & \longrightarrow & \mathrm{Temb}(\Lambda). \end{array}$$

Then the composition $\Pi_j = \Pi_j'\circ\Pi_{j-1}$ of Π_j' and Π_{j-1} satisfies (P) at each triple points of $X^{(j)} = \Pi_j^{-1}(X)$ contained in $\Pi_j^{-1}(U_k)$ with $k \leq j$. Thus we have a desirable map $\Pi = \Pi_\ell : W = W_\ell \to U$, since each triple point of $X' = \Pi^{-1}(X)$ is contained in one of $\Pi^{-1}(U_j)$'s q.e.d.

Remark. Let $U' \to U$ be a blowing up along a double curve or at a triple point of X and let X' be the total transform of X. Then X' is also a toric divisor and the dual graph Δ'

of X' is a subdivision of the dual graph Δ of X. Moreover, the weighting of Δ' induces a Z^3-weighting σ' : { all vertices of $\tilde{\Delta}'$ } $\to Z^3$ such that the restriction of σ' to the vertices of $\tilde{\Delta}$ agrees with σ, where $\tilde{\Delta}'$ is the triangulation of $\tilde{S}$ induced from Δ'.

Lemma 1.7. Under the above notations and the assumption of Lemma 1.6, we have

$$m_i a + m_j b + m_k + m_\ell \leqq 0$$

for each double curve $D = X_i \cdot X_j$, where D intersects irreducible components X_k and X_ℓ of X, $a = (D_{|X_j})^2$ and $b = (D_{|X_i})^2$. The equality holds if and only if $\deg O_D(-X^+) = 0$.

Proof. Under the assumption of Lemma 1.6 we see that $-X^+ \cdot D = \deg O_D(-X^+) \geq 0$. On the other hand, we have

$$\begin{aligned} -X^+ \cdot D &= -(m_i X_i + m_j X_j + m_k X_k + m_\ell X_\ell) \cdot D \\ &= -(m_i a + m_j b + m_k + m_\ell), \end{aligned}$$

by Lemma 1.3. q.e.d.

Let $\bar{\sigma}(v) = m_v^{-1} \cdot \sigma(v)$ for each vertex v of $\tilde{\Delta}$, where $m_v = m_i$ if the irreducible component X_i of X corresponds to the vertex $[v]$ of Δ which is the image of v by the projection $\tilde{\Delta} \longrightarrow \Delta$. Let u, v, w and x be the vertices of $\tilde{\Delta}$ such that two triples (u,v,w) and (u,v,x) form two adjacent triangles. Then we have

$$a \cdot m_u \cdot \bar{\sigma}(u) + b \cdot m_v \cdot \bar{\sigma}(v) + m_w \cdot \bar{\sigma}(w) + m_x \cdot \bar{\sigma}(x) = 0,$$

since $a \cdot \sigma(u) + b \cdot \sigma(v) + \sigma(w) + \sigma(x) = 0$. Hence the point $\bar{\sigma}(x)$ is on or above the plane of $\mathbf{R}^3$ containing the points $\bar{\sigma}(u)$, $\bar{\sigma}(v)$ and $\bar{\sigma}(w)$ according as $a \cdot m_u + b \cdot m_v + m_w + m_x$ is zero or negative. We call the edge joining u and v, a proper edge, if the inequality $a \cdot m_u + b \cdot m_v + m_w + m_x < 0$ holds. Let $f : \tilde{S} \to S^2$ be an extension of $p \circ \sigma$ such that the image of each triangle of $\tilde{\Delta}$ is a spherical triangle, where $p : \mathbf{R}^3 \setminus \{0\} \to S^2 = (\mathbf{R}^3 \setminus \{0\})/\mathbf{R}_{>0}$ is the natural projection.

Proposition 1.8. Under the above notations and the assumption of Lemma 1.6, f is injective and the closure of the image $f(\tilde{S})$ of $\tilde{S}$ is contained in an open hemisphere of S^2.

Proof. Noting the above facts and that the image of $\bar{\sigma}$ is contained in $m^{-1} \cdot \mathbf{Z}^3$ with $m = m_1 m_2 \ldots m_s$, we have the first assertion and the fact that $f(\tilde{S})$ is contained in a hemisphere of S^2, in the same manner as in the proof of Theorem 4.5 in [4]. Hence we only show that the closure of $f(\tilde{S})$ is strictly contained in an open hemisphere of S^2. We can classify the vertices of $\tilde{\Delta}$ into three types i), ii) and iii), according as there is no, exactly two and more than two proper edges meeting at a vertex. If there exists at least one vertex of type iii) among the vertices of $\tilde{\Delta}$, then the closure of $f(\tilde{S})$ is contained in an open hemisphere. If all vertices of $\tilde{\Delta}$ are of type i), then by Lemma 1.7, $O_D(-X^+)$ is trivial for each double curve D of X and hence $\mathcal{O}_{X_j}(-X^+)$ is trivial for each irreducible component X_j. This contradicts the assumption that X is contractible. Hence we may only consider the case where there are no vertex of type iii) and at least one vertex of type ii). In this case, all proper edges of $\tilde{\Delta}$ form a disjoint union $\cup L_j$ of lines which are mapped to great semicircles of S^2. These great semicircles have common boundrary points s and n, since $f(\tilde{S})$ must be contained in a hemisphere. The image of $\rho : \pi_1(S) \to GL(3,\mathbf{Z})$ must act on $f(\tilde{S})$ properly discontinuously and without fixed points and must fix s and n. Moreover, the quotient $f(\tilde{S})/\rho(\pi_1(S))$ is compact. But by an easy calculation, we see that such a subgroup of $GL(3,\mathbf{Z})$ cannot exist. q.e.d.

Theorem 1.9. Assume that $(U,X) \to (V,p)$ is a resolution of a 3-dimensional isolated singularity (V,p) whose exceptional set X is a toric divisor. Then there exist a cusp singularity $(V_o,p_o) = \mathrm{Cusp}(C,\Gamma)$ belonging to $\mathcal{T}$ [4] and a resolution π_o : $(U_o,X_o) \to (V_o,p_o)$ of (V_o,p_o) such that the weighted dual graph of the exceptional set X_o coincides with that of X.

Proof. We have a map $f : \tilde{S} \to S^2$ and a homomorphism ρ : $\pi_1(S) \to GL(3,\mathbf{Z})$ in the above way from the weighted dual graph of X. Let $C = p^{-1}(f(\tilde{S})) = \mathbf{R}_{>0}\cdot f(\tilde{S})$ and let $\Gamma = \rho(\pi_1(S))$. Then by the above proposition and [4, Proposition 4.3], we have a cusp singularity $(V_o,p_o) = \mathrm{Cusp}(C,\Gamma)$ and a resolution π_o : $(U_o,X_o) \to (V_o,p_o)$ of (V_o,p_o) such that the weighted dual graph of the exceptional set X_o coincides with that of X. Here U_o

and X_o are the quotient spaces by Γ of an open set $\tilde{U}_o = \mathrm{ord}^{-1}(C) \cup \tilde{X}_o$ and the union of the 2-dimensional orbits $\tilde{X}_o$ of the torus embedding $\mathrm{Temb}(\Sigma) = T \cup \tilde{X}_o$, respectively, where $\Sigma = \{ R_{\geq 0} \cdot f(\lambda) \mid \lambda$ are triangles, edges and vertices of $\tilde{\Delta} \} \cup \{\{0\}\}$ and $\mathrm{ord} = -\log|\ | : (C^*)^3 \to R^3$. q.e.d.

2. A classification of the toric divisors with the same weighted dual graph. We write U, X, V and p for U_o, X_o, V_o and p_o in Theorem 1.9, dropping the subscript o. Let Δ be the weighted dual graph of X and let $\tilde{\Delta}$ be the triangulation induced from Δ of the universal covering space of the compact topological surface of which Δ is a triangulation. Then $\tilde{\Delta}$ is the dual graph of $\tilde{X}$. Let $H^1(\Gamma,T)$ be the first cohomology group for the group Γ with coefficients in the Γ-module $T \simeq (C^*)^3$. Take a 1-cocycle σ representing an element $[\sigma]$ of $H^1(\Gamma,T)$. Namely, $\sigma : \Gamma \to T$ is a map from Γ to T satisfying the 1-cocycle condition $\sigma(\gamma\delta) = \sigma(\gamma)\sigma(\delta)^{\gamma}$. Since both Γ and T act on $\mathrm{Temb}(\Sigma)$ and $\tilde{X}$, the set $\Gamma_\sigma = \{ \sigma(\gamma)\circ\gamma \mid \gamma \in \Gamma \}$ is a subgroup of $\mathrm{Aut}(\mathrm{Temb}(\Sigma))$ isomorphic to Γ by the cocycle condition. The quotient $X_\sigma = \tilde{X}/\Gamma_\sigma$ of $\tilde{X}$ by Γ_σ is a compact analytic space with the weighted dual graph which coincides with Δ, since each element α of T maps each irreducible component of $\tilde{X}$ onto itself. Let τ be another representative of $[\sigma]$. Then there exists an element α of T with $\tau(\gamma) = \sigma(\gamma)\alpha^{\gamma^{-1}}$ for each element γ of Γ. Then we have the equality $\alpha\sigma(\gamma)\circ\gamma = \tau(\gamma)\circ\gamma\circ\alpha$ for any element γ of Γ. Hence α induces an isomorphism from X_σ to $X_\tau = \tilde{X}/\Gamma_\tau$. Assume that we can take a 1-cocycle σ so that $\{ \sigma(\gamma) \mid \gamma \in \Gamma \}$ is contained in the compact real torus $CT = \{ z \in C^* \mid |z| = 1 \}^3$ of T. Then Γ_σ also acts on $\tilde{U}$ properly discontinuously and without fixed points, since the map $\mathrm{ord}_{|\tilde{U}\setminus\tilde{X}} : \tilde{U}\setminus\tilde{X} \to C$ is also Γ_σ-equivariant. Hence we obtain a pair (U_σ, X_σ) of a 3-dimensional complex manifold $U_\sigma = \tilde{U}/\Gamma_\sigma$ and a toric divisor $X_\sigma = \tilde{X}/\Gamma_\sigma$. We see that X_σ is contractible to a point and thus we obtain a 3-dimensional isolated singularity V_σ, by the same reason as in the proof of [4, Proposition 1.7].

Proposition 2.1. If Γ is an abelian group, then $H^1(\Gamma,T)$ is a finite group and for each element $[\sigma]$ of $H^1(\Gamma,T)$, we can take a 1-cocycle σ representing $[\sigma]$ so that $\{ \sigma(\gamma) \mid \gamma \in \Gamma \}$ is contained in the compact real torus CT of T.

Proof. Since Γ is a fundamental group of a compact topological surface, Γ is a free abelian group of rank 2. Let γ and δ be generators of Γ. Then $\det(\gamma-1) \neq 0$, since γ has three real eigenvalues which are not equal to 1 as we see in the proof of [4, Theorem 3.1]. Hence for any element β of T, there exists an element α of T with $\alpha^{\gamma-1} = \beta$. Then we can take a 1-cocycle σ representing $[\sigma]$ with $\sigma(\gamma) = 1$ for any element $[\sigma]$ of $H^1(\Gamma,T)$. Since Γ is abelian, we have $\sigma(\varepsilon)^{\gamma-1} = \sigma(\gamma)^{\varepsilon-1} = 1$, by the cocycle condition for any element ε of Γ. Hence $\sigma(\varepsilon)$ is contained in the kernel $\ker(\gamma-1)$ of the map $T \ni \alpha \longrightarrow \alpha^{\gamma-1} \in T$. Clearly $\ker(\gamma-1)$ is contained in CT and is a finite set. Hence $\{ \sigma(\gamma) \mid \gamma \in \Gamma \} \subset CT$ and $H^1(\Gamma,T)$ is a finite group, since a representative σ is uniquely determined from $\sigma(\gamma)$

and $\sigma(\delta)$. q.e.d.

Remark. When Γ is an abelian group, $\mathrm{Cusp}(C,\Gamma)$ is a Hilbert modular cusp singularity [4, Proposition 3.1 and Corollary 3.2]. Let σ be a 1-cocycle representing an element of $H^1(\Gamma,T)$ such that $\{ \sigma(\gamma) \mid \gamma \in \Gamma \} \subset CT$. We easily see that the set $\Gamma' = \{ \gamma \in \Gamma \mid \sigma(\gamma) = 1 \}$ is a subgroup of Γ of finite index. Then Γ and Γ_σ are commensurable, i.e., Γ' is also a subgroup of Γ_σ of finite index. Hence V_σ is also a Hilbert modular cusp singularity.

In the following, we show conversely that any toric divisor X' of a 3-dimensional complex manifold with the weighted dual graph which coincides with Δ is isomorphic to X_σ for some element $[\sigma]$ of $H^1(\Gamma,T)$. Let $\tilde{X}' \to X'$ be the unramified covering space of X' induced from the covering map $\tilde{\Delta} \to \Delta$ of the dual graph.

Proposition 2.2. Under the above notations, we have an isomorphism: $\tilde{X}' \simeq \tilde{X}$.

Proof. For each vertex v of $\tilde{\Delta}$, let (X_v,D_v) (resp. (X'_v,D'_v)) be the pair of the irreducible component of X (resp. X') corresponding to v and the union of the double curves on it. Since each irreducible component and the double curves on it are uniquely determined up to isomorphisms from the weighted dual graph, we have an isomorphism $(X'_v,D'_v) \simeq (X_v,D_v)$ of pairs. Take a vertex v and fix one such isomorphism.

$$I_v : (X'_v, D'_v) \simeq (X_v, D_v).$$

Let w be a vertex which is connected to v by an edge e of $\tilde{\Delta}$. Let $D = X_v \cdot X_w$ and $D' = X'_v \cdot X'_w$ be the double curves of X_v and X'_v, respectively, corresponding to e. Then we can take an isomorphism

$$I_w : (X'_w, D'_w) \simeq (X_w, D_w)$$

in such a way that the restriction $I_{w|D'}$ of I_w to D' is equal to that of I_v, i.e., $I_{w|D'} = I_{v|D'}$. Next let u be a vertex of Δ such that u, v and w are vertices of a triangle. Then the restrictions of I_v and I_w to the double curves $X'_u \cdot X'_v$ and $X'_u \cdot X'_w$, respectively can be extended to a nuique isomorphism

$$I_u : (X'_u, D'_u) \simeq (X_u, D_u)$$

of pairs. Hence the following lemma complete the proof, since $\tilde{\Delta}$ is simply connected.

Lemma 2.3. Let v be a vertex of $\tilde{\Delta}$ and $w_1, w_2, \dots,$ w_s be the vertices adjacent to v going around v in this order. Fix isomorphisms $I_v : X'_v \simeq X_v$ and $I_{w_1} : X'_{w_1} \simeq X_{w_1}$ with $I_{v|D'_1} = I_{w_1|D'_1}$, where $D'_1 = X'_1 \cdot X'_{w_1}$. We have isomorphisms $I_{w_2} : X'_{w_2} \longrightarrow X_{w_2}$, $I_{w_3} : X'_{w_3} \longrightarrow X_{w_3}$, ... and $I_{w_s} : X'_{w_s} \longrightarrow X_{w_s}$, successively, with $I_{v|D'_j} = I_{w_j|D'_j}$ and $I_{w_{j-1}|E'_j} = I_{w_j|E'_j}$, by the above way, where $D'_j = X'_v \cdot X'_{w_j}$ and $E'_j = X'_{w_{j-1}} \cdot X'_{w_j}$. Then $I_{w_s}|E'_1 = I_{w_1}|E'_1$. Namely, we have an isomorphism from $X'_v + X'_{w_1} + \dots X'_{w_s}$ to $X_v + X_{w_1} + \dots + X_{w_s}$.

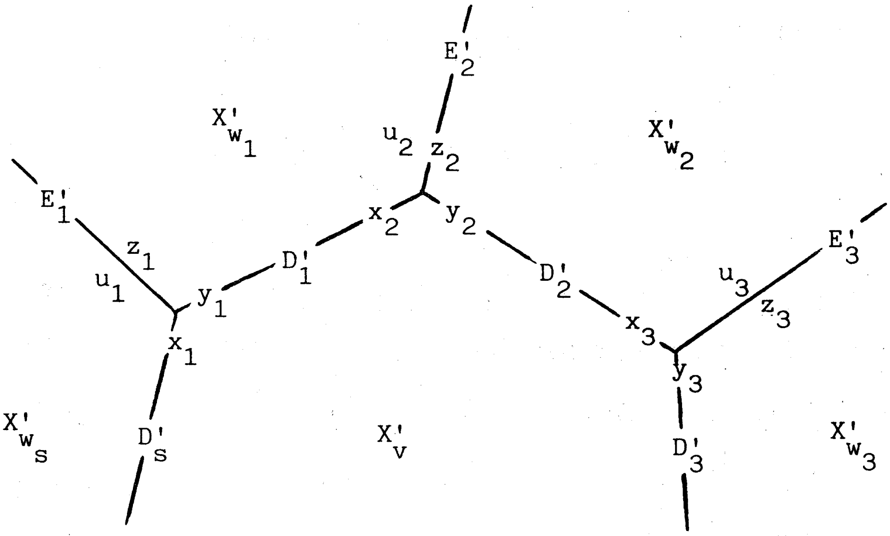


Figure 2.1

Proof. Take global coordinates (x_i,y_i), (z_i,y_i) and (x_i,u_i) for open sets of X'_v, X'_{w_i} and $X'_{w_{i-1}}$, respectively, for each i of $\mathbf{Z}/s\mathbf{Z}$ so that E'_i is defined by $y_i = 0$ and also by $x_i = 0$, that D'_i is defined by $z_i = 0$ and also by $u_{i+1} = 0$ on X'_{w_i}, that D'_i is defined by $y_{i+1} = 0$ and also by $x_i = 0$ on X'_v and that the relations $y_i = x_{i+1}^{-1}$, $x_i = y_{i+1}\, x_{i+1}^{-a_i}$ and $z_i = u_{i+1}\, x_{i+1}^{-b_i}$ hold. (See Figure 2.1.) We may assume that $u_i = z_i$ on each double curve E'_i for $i = 2, 3, \dots, s$. Then $u_1 = tz_1$ on E'_1 for a nonzero complex number t. It is sufficient to show that $t = 1$. Take an open covering $\{U_i\}_{i=1,2,\dots,s}$ of X'_v so that each U_i is a Stein neighborhood of the triple point $X'_v \cdot X'_{w_{i-1}} \cdot X'_{w_i}$. Let f_i be a defininig equation for $X'_v \cap U_i$ on U_i. Let $g_{ij} = (f_i/f_j)_{|X'_v}$. Then the collection $\{g_{ij}\}$ of the transition functions g_{ij} defines the normal bundle $[X'_v]_{|X'_v}$ of X'_v. On the other hand, the line bundle $[X'_v]_{|X'_v}$ is linearly equivalent to $d_1 \cdot D_1 + d_2 \cdot D_2 + \ldots + d_s \cdot D_s$ for some integers d_1, d_2, ... and d_s. Hence we have a collection $\{h_i\}$ of nowhere vanishing holomorphic functions h_i on $X'_v \cap U_i$ with

$$\bar{g}_{ij} := h_i \cdot g_{ij} \cdot h_j^{-1} = (y_i^{d_{i-1}} x_i^{d_i})(y_j^{d_{j-1}} x_j^{d_j})^{-1}. \text{ In}$$

particular, $\bar{g}_{ii+1} = x_{i+1}^{-b_i}$, since $b_i = d_{i-1} + a_i \cdot d_i + d_{i+1}$. Now let $\bar{f}_i = f_i \cdot \bar{h}_i$ for a holomorphic function $\bar{h}_i$ which is an extension of h_i to U_i. Then $(\bar{f}_i/\bar{f}_{i+1})_{|X'_v} = x_{i+1}^{-b_i}$ and hence $\{(\bar{f}_i/z_i)(\bar{f}_{i+1}/u_{i+1})^{-1}\}_{|D'_i} = 1$. So $(\bar{f}_i/z_i)$ and $(\bar{f}_{i+1}/u_{i+1})$ define a nonzero holomorphic function F_i on D'_i. Since $u_i = z_i$ on E'_i, $F_{i-1} = F_i$ on the triple point $D'_{i-1} \cdot D'_i$. Since the double curves D_i are compact, the collection $\{F_i\}$ is a nonzero constant function on $D'_1 + D'_2 + \ldots + D'_s$. Hence $t = u_1/z_1 = (\bar{f}_1/z_1) \cdot (\bar{f}_1/u_1)^{-1} = 1$. q.e.d.

Let $\mathrm{Aut}_o(\tilde{X})$ be the subgroup of $\mathrm{Aut}(\tilde{X})$ consisting of the automorphisms of $\tilde{X}$ which map each irreducible component of $\tilde{X}$ to itself, i.e., the induced action of $\mathrm{Aut}_o(\tilde{X})$ on $\tilde{\Delta}$ is trivial. Clearly T acts on $\tilde{X}$ effectively and hence is a subgroup of $\mathrm{Aut}_o(\tilde{X})$.

Proposition 2.4. $\mathrm{Aut}_o(\tilde{X}) = T$.

Proof. Let u, v and w be the vertices of a triangle of $\tilde{\Delta}$. Let (x_u,y_u), (x_v,y_v) and (x_w,y_w) be coordinates of toric surfaces X_u, X_v and X_w, respectively, such that $X_u \cdot X_v$ (resp. $X_v \cdot X_w$, resp. $X_w \cdot X_u$) is defined by $y_u = 0$ (resp. $y_v = 0$, resp. $y_w = 0$) and also by $x_v = 0$ (resp. $x_w = 0$, resp. $x_u = 0$) and that $x_u = y_v$ (resp. $x_v = y_w$, resp. $x_w = y_u$) on $X_u \cdot X_v$ (resp. $X_v \cdot X_w$, resp. $X_w \cdot X_u$). (See Figure 2.2.) Then the restriction of any element g of $\mathrm{Aut}_0(\tilde{X})$ to X_t for $t = u$, v and w can be written as $(x_t,y_t) \to (\alpha_t \cdot x_t, \beta_t \cdot y_t)$ for some nonzero complex numbers α_t and β_t. Clearly $\alpha_u = \beta_v$, $\alpha_v = \beta_w$ and $\alpha_w = \beta_u$. On the other hand, the action of any element β of T on a neighborhood in $\tilde{U}$ of the triple point $X_u \cdot X_v \cdot X_w$ can be written as $(x,y,z) \to (\beta_1 x, \beta_2 y, \beta_3 z)$ for a coordinate (x,y,z) of $\tilde{U}$ which is $(y_v,x_v,0)$, $(0,y_w,x_w)$ and $(x_u,0,y_u)$ on X_v, X_w and X_u, respectively. Hence there exists a unique element α of T, the restriction of which to $X_u + X_v + X_w$ agrees with that of g. Then as in the proof of Proposition 2.2, we have $g = \alpha$ on $\tilde{X}$. q.e.d.

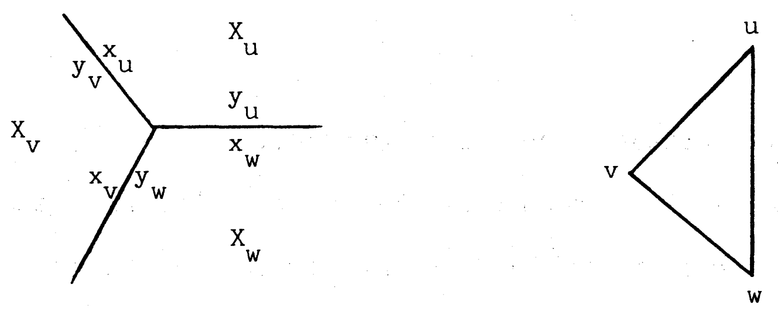


Figure 2.2

By Propositions 2.2 and 2.4, there exist a subgroup Γ' of $\mathrm{Aut}(\tilde{X})$ with $X' \simeq \tilde{X}/\Gamma'$ and an isomorphism $h : \Gamma \simeq \Gamma'$ with

$\gamma^{-1} \cdot h(\gamma) \in T$ for any element γ of Γ. Let $\sigma(\gamma) = h(\gamma) \cdot \gamma^{-1}$. Then by an easy calculation we see that the map $\sigma : \Gamma \to T$ satisfies the 1-cocycle condition $\sigma(\gamma\delta) = \sigma(\gamma)\sigma(\delta)^{\gamma}$ for any element γ and δ of Γ. Hence the map $\sigma : \Gamma \to T$ defines an element $[\sigma]$ of $H^1(\Gamma, T)$. Here assume that $\tilde{X}/\Gamma' \simeq \tilde{X}/\Gamma''$ with $\Gamma'' = \{ \tau(\gamma) \cdot \gamma \mid \gamma \in \Gamma \}$ for some $\tau : \Gamma \to T$ satisfying the 1-cocycle condition. Then there exists an element β of T satisfying $\beta \cdot (\sigma(\gamma) \cdot \gamma) = (\tau(\gamma) \cdot \gamma) \cdot \beta$ for any element γ of Γ. Hence we have $\sigma(\gamma)/\tau(\tau) = \beta^{\gamma^{-1}}$. Therefore, γ and τ defines the same element of $H^1(\Gamma, T)$ and such element is uniquely determined from an isomorphism classes of toric divisors. Thus we obtain:

Proposition 2.5. There exists an injective map from the set of isomorphism classes of toric divisors whose weighted dual graph coincides with Δ and which are exceptional sets of 3-dimensional isolated singularities to $H^1(\Gamma, T)$, which sends each toric divisor X' to the element $[\sigma]$ of $H^1(\Gamma, T)$ with $X' \simeq X_{\sigma}$.

When Γ is abelian, the map in Proposition 2.5 is bijective, by Proposition 2.1.

3. **The pseudo-tautness of cusp singularities.** Let (U, X) and (U', X') be pairs of 3-dimensional complex manifolds and their toric divisors such that X and X' are isomorphic. We denote by ${}_nX$, the analytic space whose reduced space is X and whose structure sheaf is $O_U/O_U(-(n+1)X)$.

Theorem 3.1. Under the above notations, if $H^1(X,\Theta_U\otimes O_X(-pX)) = 0$ for all positive integers p, then we have an isomorphism ${}_nX \simeq {}_nX'$ for each positive integer n, where Θ_U is the holomorphic tangent sheaf of U.

Take an open Stein covering $\{U_A\}$ (resp. $\{U'_A\}$) of X (resp. X') with local coordinates (x_A,y_A,z_A) (resp. (x'_A,y'_A,z'_A)) such that X (resp. X') is defined by $x_A\cdot y_A\cdot z_A = 0$ (resp. $x'_A\cdot y'_A\cdot z'_A = 0$) on each open set U_A (resp. U'_A). Here for an isomorphism $i : X' \simeq X$ fixed once for all we may assume that $i(U'_A\cap X') = U_A\cap X$, that $x'_A(p) = x_A(i(p))$, that $y'_A(p) = y_A(i(p))$ and that $z'_A(p) = z_A(i(p))$ for any point p of $U'_A\cap X'$. Let

$$(x_A,y_A,z_A) = (f^0_{AB}(x_B,y_B,z_B),g^0_{AB}(x_B,y_B,z_B),h^0_{AB}(x_B,y_B,z_B))$$

and

$$(x'_A,y'_A,z'_A) = (f'_{AB}(x'_B,y'_B,z'_B),g'_{AB}(x'_B,y'_B,z'_B),h'_{AB}(x'_B,y'_B,z'_B))$$

for each pair (A,B) with $U_A\cap U_B \neq \emptyset$. Then

$$\begin{aligned}&(f'_{AB}(x_B,y_B,z_B),g'_{AB}(x_B,y_B,z_B),h'_{AB}(x_B,y_B,z_B))\\ &= (f^0_{AB}(x_B,y_B,z_B),g^0_{AB}(x_B,y_B,z_B),h^0_{AB}(x_B,y_B,z_B))\\ &\qquad (\text{ mod. } (x_B\cdot y_B\cdot z_B)).\end{aligned}$$

Now let

$$(x_B \cdot y_B \cdot z_B)\cdot(f^1_{AB}, g^1_{AB}, h^1_{AB})$$
$$= (f'_{AB}(x_B,y_B,z_B), g'_{AB}(x_B,y_B,z_B), h'_{AB}(x_B,y_B,z_B)) - (f^0_{AB}, g^0_{AB}, h^0_{AB}),$$

and let

$$\vartheta^1_{AB} = \frac{\partial}{\partial x_A}\cdot f^1_{AB} + \frac{\partial}{\partial y_A}\cdot g^1_{AB} + \frac{\partial}{\partial z_A}\cdot h^1_{AB}.$$

Then by an easy calculation, we have

$$(x_C\cdot y_C\cdot z_C)\cdot\vartheta^1_{AC} = (x_B\cdot y_B\cdot z_B)\cdot\vartheta^1_{AB} + (x_C\cdot y_C\cdot z_C)\cdot\vartheta^1_{BC}$$
$$(\text{ mod. } (x_C\cdot y_C\cdot z_C)^2),$$

if $U_A \cap U_B \cap U_C \neq \emptyset$. Hence the collection $\{\vartheta^1_{AB|X}\}$ of the restrictions of ϑ^1_{AB} to X defines an element of $Z^1(X, \Theta_U \otimes O_X(-X))$. By the assumption of the theorem, we have a collection $\{\vartheta^1_A\}$ of vector fields

$$\vartheta^1_A = \frac{\partial}{\partial x_A}\cdot f^1_A + \frac{\partial}{\partial y_A}\cdot g^1_A + \frac{\partial}{\partial z_A}\cdot h^1_A$$

on U_A with $(x_B\cdot y_B\cdot z_B)\cdot\vartheta^1_{AB} = (x_A\cdot y_A\cdot z_A)\cdot\vartheta^1_A - (x_B\cdot y_B\cdot z_B)\cdot\vartheta^1_B$ (mod. $(x_B\cdot y_B\cdot z_B)^2$). Let $x^1_A = x_A + (x_A\cdot y_A\cdot z_A)\cdot f^1_A$, $y^1_A = y_A + (x_A\cdot y_A\cdot z_A).g^1_A$ and $z^1_A = z_A + (x_A\cdot y_A\cdot z_A)\cdot h^1_A$. Then by an easy calculation, we have

$$(x^1_A, y^1_A, z^1_A) = (f'_{AB}(x^1_B,y^1_B,z^1_B), g'_{AB}(x^1_B,y^1_B,z^1_B), h'_{AB}(x^1_B,y^1_B,z^1_B))$$
$$(\text{ mod. } (x_B\cdot y_B\cdot z_B)^2).$$

Thus by the assumption of the theorem, we can define inductively new coordinates (x_A^n, y_A^n, z_A^n) for any positive integer n such that

$$(x_A^n, y_A^n, z_A^n) = (f'_{AB}(x_B^n, y_B^n, z_B^n), g'_{AB}(x_B^n, y_B^n, z_B^n), h'_{AB}(x_B^n, y_B^n, z_B^n)),$$
$$(\text{ mod. } (x_B \cdot y_B \cdot z_B)^{n+1}).$$

Thus we complete the proof of Theorem 3.1.

We have the following theorem immediately from Theorem 3.1 and [2, Theorem].

Theorem 3.2. Let X and X' be toric divisors of 3-dimensional complex manifolds U and U', respectively. Assume that $X \simeq X'$, that X is contractible to a point and that $H^1(\Theta_U \otimes O_X(-pX)) = 0$ for all positive integers p. Then some neighborhoods of X and X' are isomorphic.

The exsamples in [4, 5. Exsamples (I) and (II)] satisfy the condition of Theorem 3.1. Let $(U,X) \to (V,p)$ be a resolution of a 3-dimensional isolated singularity (V,p) such that the exceptional set X is a toric divisor. If (U,X) satisfies the condition of Theorem 3.1 and $H^1(\Gamma,T)$ is trivial, then (V,p) is taut and is a cusp singularity in the sense of [4], by Theorem 1.9, Proposition 2.5 and Theorem 3.2, where Γ is the fundamental group of the dual graph of X and acts on $T \simeq (\mathbf{C}^*)^3$ through the map $\rho : \Gamma \to GL(3,\mathbf{Z})$ we obtain by the way as in Section 1.

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