数理解析研究所講究録
第 535 巻 1984 年 95-109

Normal Surfaces and Intersection Theory

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In this note we develop geometry of normal surfaces by using the intersection theory introduced by Mumford [4]. We shall study the contraction criterion, the projection formula, the Noether formula, the vanishing theorem, the minimal model, the Miyaoka inequality, etc. Details will be discussed elsewhere.

Notation

A surface will mean an irreducible reduced compact complex space of dimension 2. A divisor will mean a Weil divisor (i.e., a linear combination of irreducible curves) unless otherwise specified. We use "birational morphism" instead of bimeromorphic morphism.

Y : a normal surface

X : a resolution of singularities of Y

Div(Y) : the group of divisors on Y

An element of $\mathrm{Div}(Y,\mathbb{Q})=\mathrm{Div}(Y)\otimes\mathbb{Q}$ is called a $\mathbb{Q}$-divisor. Given a $\mathbb{Q}$-divisor $D=\Sigma\alpha_i C_i$ where the C_i are irreducible curves and $\alpha_i\in\mathbb{Q}$, we write as

$[D] = \Sigma[\alpha_i]C_i$ ($[\alpha]$ is the greatest integer $\leqq\alpha$)

$\{D\} = \Sigma\{\alpha_i\}C_i$ ($\{\alpha\}$ is the least integer $\geqq\alpha$)

1. Contraction criterion

Let Y be a normal surface. The intersection pairing $Div(Y)\times Div(Y) \to \mathbb{Q}$ is defined as follows ([4]). Let $\pi:X \to Y$ be a resolution of singularities and let $A=\cup E_i$ denote the exceptional set of π. For a divisor D on Y we define the *inverse image* π^*D as

$$\pi^*D=\bar{D}+\Sigma\alpha_i E_i$$

where $\bar{D}$ is the strict transform of D and the rational numbers α_i are uniquely determined by the equations: $\bar{D}E_j+\Sigma\alpha_i E_i E_j=0$ for all j. For two divisors D and D' the *intersection number* DD' is defined to be the rational number $(\pi^*D)(\pi^*D')$.

A divisor D on Y is numerically equivalent to zero, denoted by $D\approx 0$, if DC=0 for all curves C on Y. Two divisors D and D' are numerically equivalent, $D\approx D'$, if $D-D'\approx 0$. Set $N(Y,\mathbb{Q})=(Div(Y)/\approx)\otimes\mathbb{Q}$. The *Picard number* $\rho(Y)$ of Y is the rank of the $\mathbb{Q}$-vector space $N(Y,\mathbb{Q})$. We have the equality: $\rho(Y)=\rho(X)-\rho(\pi)$ where $\rho(\pi)$ is the number of irreducible components of A.

The following is the normal surface version of the Grauert's contraction criterion theorem.

Theorem (1.1)(Contraction Criterion). Let $C_i,\ldots,C_k$ be irreducible curves on a normal surface Y. Then the union $\cup C_i$ can be contracted to normal points if and only if the intersection matrix (C_iC_j) is negative definite.

Proof. By definition $\pi^*C_i=\overline{C}_i+Z_i$ with $\mathrm{Supp}(Z_i)\subset A$. Let $G=\Sigma\alpha_i\overline{C}_i+Z$ be a $\mathbb{Q}$-divisor on X such that $\mathrm{Supp}(Z)\subset A$. Write $G=\pi^*(\Sigma\alpha_iC_i)+Z'$ where $Z'=Z-\Sigma\alpha_iZ_i$. We find that $G^2=(\Sigma\alpha_iC_i)^2+Z'^2$. As a consequence the Grauert's theorem applied to X proves the assertion. Q.E.D.

We now consider a birational morphism $f:Y'\to Y$ between normal surfaces Y' and Y. We denote by A_f the exceptional set of f. Write $A_f=\cup C_i$. For a divisor D on Y the inverse image f^*D is also defined. As a corollary of the above criterion, we can write as

(1.2) $$f^*D=\overline{D}+\Sigma\beta_iC_i$$

where $\overline{D}$ is the strict transform of D by f and the rational numbers β_i are determined by the equations: $\overline{D}C_j+\Sigma\beta_iC_iC_j=0$ for all j. We can prove that $\rho(Y')=\rho(Y)+\rho(f)$.

Definition (1.3). Let D be a $\mathbb{Q}$-divisor on a normal surface. We say that D is nef (numerically effective) if $DC\geq 0$ for all curves C on Y and that D is pseudo effective if $DP\geq 0$ for all nef divisors P on Y.

2. Projection formula

A coherent sheaf F on Y is reflexive if $F^{\vee\vee}\simeq F$ where $F^{\vee}$ is the dual sheaf $\mathrm{Hom}(F,O_Y)$. A reflexive sheaf of rank one is called a divisorial sheaf. Set $Y_0=Y\setminus\mathrm{Sing}\ Y$ with the inclusion

$i:Y_0 \to Y$. A coherent sheaf F on Y_0 is said to be <u>extendible</u> if it extends to a coherent sheaf on Y. It is proved by Serre (Ann.Inst.Fourier 16) that if F is an extendible reflexive sheaf on Y_0, then i_*F is a reflexive sheaf on Y, which is unique as a reflexive extension of F.

For a divisor D on Y the invertible sheaf $O(D_{|Y_0})$ on Y_0 is extendible. Indeed the coherent sheaf $\pi_*O(\bar{D})$ is an extension. It follows that the sheaf $i_*O(D_{|Y_0})$ is a divisorial sheaf on Y. We denote it by O(D). Clearly $i_*i^*O(D) \simeq O(D)$. When Y is Moišezon, every divisorial sheaf is defined by a divisor. For a $\mathbb{Q}$-divisor D we understand that $O(D)=O([D])$. Two $\mathbb{Q}$-divisors D and D' are <u>linearly equivalent</u>, denoted by $D \sim D'$, if the difference $D-D'$ is a principal divisor of a non-zero meromorphic function. We have the equivalence: $D \sim D' \Longleftrightarrow$ (i) $D-D'$ is integral, (ii) $O(D) \simeq O(D')$.

The following result connects the cohomological invariants of Y with those of X.

<u>Theorem</u>(2.1)(Projection Formula). Let D be a $\mathbb{Q}$-divisor on a normal surface Y. Let $\pi: X \to Y$ be a resolution. Then

$$\pi_*O(\pi^*D) \simeq O(D).$$

<u>Outline of Proof</u>. It is sufficient to consider the local situation. Let (V,y) be a normal surface singularity with a resolution $\pi: U \to V$. As before let $A=\cup E_i$ denote the exceptional set of π. There is an exact sequence originated by Laufer:

$$0 \to H^0(U,O(\pi^*D)) \to H^0(U \setminus A,O(\pi^*D)) \to H^1_c(U,O(\pi^*D)).$$

Since $H^0(U\setminus A, O(\pi^*D)) \simeq H^0(V\setminus y, O(D)) \simeq H^0(V, O(D))$, the assertion follows from the vanishing: $H_c^1(U, O(\pi^*D))=0$. By duality we have $H_c^1(U, O(\pi^*D)) \simeq H^1(U, O(K+\{-\pi^*D\}))$ where the K is a canonical divisor of U. So we can complete the proof by the following

<u>Theorem</u> (2.2)(Local Vanishing Theorem). Let D be a $\mathbb{Q}$-divisor on U. Suppose that $DE_j \geq 0$ for all j. Then

$$R^1\pi_* O(K+\{D\})=0.$$

<u>Remark</u>. In the algebraic context, Theorems (2.1) and (2.2) hold in all characteristics.

<u>Theorem</u> (2.3)(Generalized Projection Formula). Let $f: Y' \to Y$ be a birational morphism of normal surfaces. Let D be a $\mathbb{Q}$-divisor on Y and let Z be an effective $\mathbb{Q}$-divisor supported on the exceptional set A_f. Then

$$f_* O(f^*D+Z) \simeq O(D).$$

3. $\mathbb{Q}$-divisor Δ

We study the inverse image of a canonical divisor. Let (V,y) be a normal surface singularity with a resolution $\pi: U \to V$. Let $A=\cup E_i$ denote the exceptional set. If K is a canonical divisor of U, then $K_V=\pi_* K$ is a canonical divisor of V. Now define a $\mathbb{Q}$-divisor $\Delta=\Sigma\delta_i E_i$ by the equations: $KE_j+\Sigma\delta_i E_i E_j=0$ for all j. We infer from the definition in Sect.1 that

(3.1) $$\pi^* K_V=K+\Delta.$$

When π is the minimal resolution in the sense that there is no exceptional curve of the first kind in A, it can be shown that $\Delta \geqq 0$ and that $\Delta=0 \Longleftrightarrow y$ is a rational double point. We introduce the following numerical invariants:

(3.2)
$$h(y)=\dim R^1\pi_* O_U \quad \text{(the genus)}$$
$$\mu(y)=e(A)+\Delta^2-1+12h(y) \quad \text{(the Milnor number)}$$

where $e(A)$ is the Euler number of A. Note that $\mu(y)\in\mathbb{Q}$.

Example (3.3). Let us examine the case in which the weighted dual graph of the exceptional set of the minimal good resolution is a star and only the central curve (if exists) may have positive genus. This is the case if the singularity has a good $\mathbb{C}^*$-action (Orlik-Wagreich).

(a) cyclic quotient singularity.
The weighted dual graph is a chain of $\mathbb{P}^1$'s.

E_1 E_2 E_n
o———o- - - - - - -o
$-a_1$ $-a_2$ $-a_n$

Define $d/e=[a_1,\ldots,a_n]=a_1-\cfrac{1}{a_2-\cfrac{1}{\ddots-\cfrac{1}{a_n}}}$

Consider the equations: $X_{k+1}=a_kX_k-X_{k-1}$. Let $\{c_{ik}\}$, $\{c'_{ik}\}$ be two solutions as

$$c_0=d,\ c_1=e,\ldots,\ \text{then } c_n=1,\ c_{n+1}=0,$$

$$c'_0=0,\ c'_1=1,\ldots,\ \text{then } c'_n=e',\ c'_{n+1}=d \quad (ee'=1 \bmod d).$$

We have $c_k\geq 0$, $c'_k\geq 0$. By a calculation (cf.Knöller, Math.Ann.213),

(3.4)
$$\Delta=\sum(1-(c_k+c'_k)/d)E_k$$
$$\mu=n+4-\nu-(e+e'+2)/d$$

where ν is the multiplicity of y, which is equal to $\Sigma(a_i-2)+2$.

(b) a star with a central curve E_0 with genus g. There are finite number of branches of chains of $\mathbb{P}^1$'s, E_{ij} $j=1,\ldots,n_i$. Let $E_0^2=-a_0$, $E_{ij}^2=-a_{ij}$ $(a_{ij}\geq 2)$. Define $d_i/e_i=[a_{i1},\ldots,a_{in_i}]$, $\{c_{ik}\}$, $\{c'_{ik}\}$ as above. The negative definiteness of the intersection matrix implies $a_0-\Sigma e_i/d_i>0$. With these notation we get

(3.5)
$$\Delta=\sum(1-((1-\delta_0)c_{ik}+c'_{ik})/d_i)E_{ik}+\delta_0E_0,$$

where

$$\delta_0=1+\frac{\Sigma(1-1/d_i)+2g-2}{a_0-\Sigma e_i/d_i}.$$

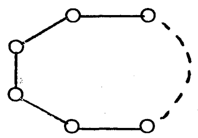
<u>Example</u> (3.6). Assume π is the minimal good resolution. Those singularities having the property: $\delta_i\leq 1$ for all i, have been classified by K.Watanabe (Math.Ann. 250) and by Y.Kawamata (in somewhat different context, Lecture Notes in Math. 732,

Springer). We give the list. Here o denotes a non-singular rational curve and ● denotes a non-singular rational curve with self-intesection -2. Cf. Wagreich (Topology 11).

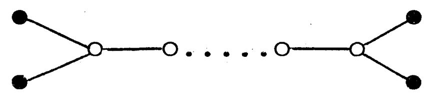
Table (3.7)(Singularities with $\delta_i \leq 1$ for all i).

(1) smooth point (1)* quotient singularities

(2) cusp singularities

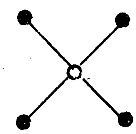


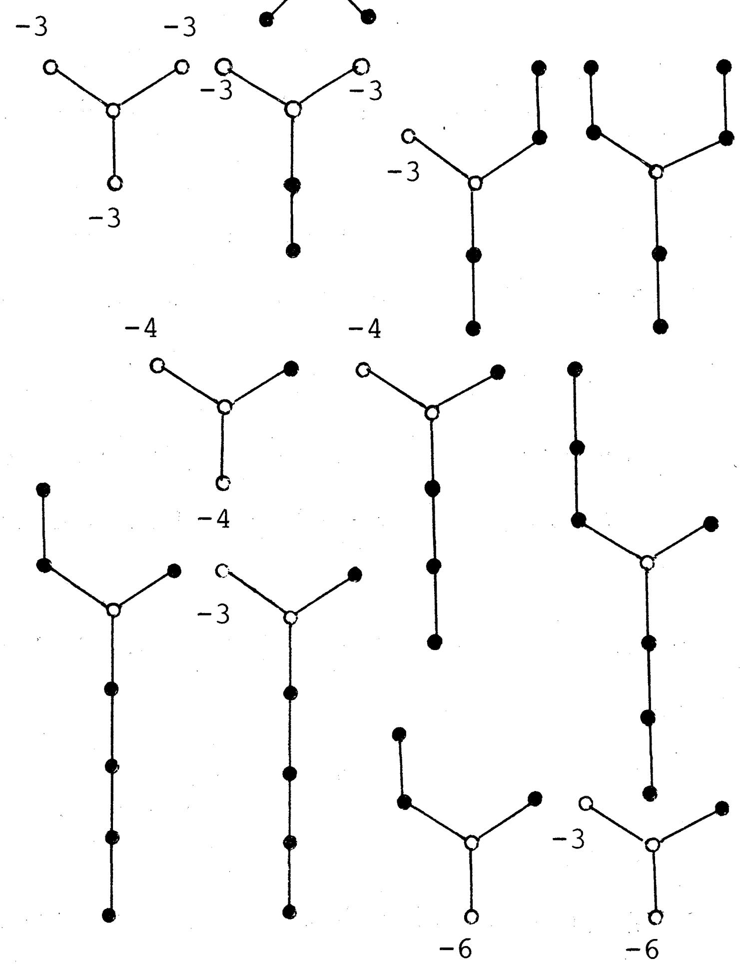
(2)*



(3) simple elliptic singularities

(3)*





(quotients of simple elliptic singularities)

The proof proceeds as follows. It turns out that π coincides with the minimal resolution except the case  -1. If A is a single curve, it is either $\mathbb{P}^1$ or an elliptic curve. We consider the case in which A has more than one component. Assume y is not a rational double point. If A' is a proper subset of A, letting Δ' be the $\mathbb{Q}$-divisor associated to A', then we must have $\Delta > \Delta'$. We infer from this that every component of A is $\mathbb{P}^1$. Next one shows that A is a star except the cases (2), (2)*. In case A is a chain, every coefficient of Δ is less that one (cf.(3.4)). In case A is a star with a central curve, we deduce from (3.5) that $\delta_0 \leqq 1 \Longleftrightarrow \Sigma(1-1/d_i) \leqq 2$. Looking in the coefficients of Δ the condition $\delta_0 \leqq 1$ implies that all other coefficients are less than one. The inequality $\Sigma(1-1/d_i) \leqq 2$ has finite possibilities of d_i, which correspond to the cases (1)* and (3)*:

(1)* (2,2,d), (2,3,3), (2,3,4), (2,3,5)

(3)* (2,2,2,2), (3,3,3) (2,4,4), (2,3,6).

Remark. The above singularities (3) and (3)* have appeared as ball cusp singularities ([2]). The case (1)* $\Leftrightarrow$ $\delta_i < 1$ for all i.

4. Noether formula, vanishing theorem

We come back to study normal surfaces. Let Y be a normal surface and let $\pi : X \to Y$ be a resolution with A the exceptional set. For the sake of simplicity, we assume that X has a canonical divisor K. This is the case if X is projective, or equivalently if Y is Moišezon. In general we have to deal with the canonical line bundle. For this argument, we refer to [6].

Since $\pi_* K$ becomes a canonical divisor of Y, we denote it by K_Y. If Sing $Y=\cup y_i$, let Δ_i be the $\mathbb{Q}$-divisor associated to y_i supported on $A_i=\pi^{-1}(y_i)$. Write $\Delta=\Sigma\Delta_i$. By (3.1) we have $\pi^* K_Y=K+\Delta$ and hence

(4.1) $$K_Y^2=K^2-\Delta^2.$$

<u>Theorem</u> (4.2) (Noether Formula). Let Y be a normal surface. Then

$$\chi(O_Y)=\frac{1}{12}(K_Y^2+e(Y)+\Sigma\,\mu(y_i))$$

where e(Y) is the Euler number of Y.

<u>Proof</u>. Recall the Noether formula for X:

$$\chi(O_X)=\frac{1}{12}(K^2+e(X)).$$

We have the following relations of Euler numbers:

$$e(X)=e(X\backslash A)+e(A)=e(Y\backslash \mathrm{Sing}\ Y)+e(A)=e(Y)+\Sigma(e(A_i)-1).$$

On the other hand $\chi(O_Y)=\chi(O_X)+\dim R^1\pi_* O_X$. Combining these with (4.1) and the definition of μ, we get the required result. Q.E.D.

For the Riemann-Roch formula for divisorial sheaves, see [1]. We shall state the vanishing results.

<u>Theorem</u> (4.3) (Generalized Ramanujam Vanishing Theorem). Let Y be a normal Moišezon surface. Let D be a nef $\mathbb{Q}$-divisor with $D^2>0$ on Y. Then

$$H^i(Y,O(K_Y+\{D\}))=0 \quad \text{for } i>0.$$

Proof. This follows from the corresponding vanishing theorem for X, combined with the local vanishing theorem and the projection formula (for details see [6]). Q.E.D.

The local vanishing theorem can be generalized as follows.

Theorem (4.4). Let $f:Y \to Y'$ be a birational morphism of normal surfaces. If D is a relatively nef $\mathbb{Q}$-divisor on Y, then

$$R^1 f_* O(K_Y + \{D\}) = 0.$$

Corollary. In particular we have $R^1 f_* O(K_Y)=0$, which is a generalization of the Grauert-Riemenschneiders's vanishing theorem.

5. Minimal model

Let Y be a normal surface and D a divisor on Y. For every positive integer m we infer from the projection formula that $\dim H^0(Y,O(mD))=\dim H^0(X,O(m\pi^*D))$. We define the D-dimension of Y, denoted by $\kappa(D,Y)$, to be $\kappa(\pi^*D,X)$.

Definition.

$P_m(Y)=\dim H^0(Y,O(mK_Y))$ (the arithmetic m-genus)

$\kappa(Y)=\kappa(K_Y,Y)$ (the arithmetic Kodaira dimension)

Let (Y,D) be a pair of a normal surface Y and a $\mathbb{Q}$-divisor D on Y. Such a pair is called a normal pair. We say that (Y,D) is (relatively) minimal if Y contains no irreducible curves C with DC<0, $C^2<0$. A birational morphism $f:(Y,D) \to (Y',D')$ is a birational morphism $f:Y \to Y'$ satisfying $f_*D=D'$. Write as $D=f^*D'+R$ where $\mathrm{Supp}(R)\subset A_f$. We say that f is totally discrepant if every irreducible component of A_f appears in R with positive coefficient. Given a normal pair (Y,D), a minimal normal pair (Y',D') is called its minimal model if there is a totally discrepant birational morphism $f:(Y,D) \to (Y',D')$. In this case, by the projection formula (2.3) we get $H^0(Y,O(mD))\simeq H^0(Y',O(mD'))$ for every positive integer m, hence $\kappa(D,Y)=\kappa(D',Y')$.

Theorem (5.1). Every normal pair has a minimal model. Furthermore, if D is pseudo effective, then (Y,D) admits a unique minimal model (Y',D') and D' is nef.

Proof. Let (Y,D) be a normal pair. Suppose it is not minimal. Then it contains an irreducible curve C with DC<0, $C^2<0$. Let $\varphi:Y \to Y_1$ be the contraction of C. If we put $D_1=\varphi_*D$, by (1.2) we find that $D=\varphi^*D_1+(DC/C^2)C$. It follows from the hypothesis that $D>\varphi^*D_1$. Note that $\rho(Y_1)=\rho(Y)-1$. Thus by a finite number of successive such contractions we arrive at a minimal model (for the latter assertion see [6]). Q.E.D.

Corollary (Zariski Decomposition). Let (Y,D) be a normal pair. Suppose D is pseudo effective. Let (Y',D';f) be its minimal model. If we write $P=f^*D'$, then the decomposition

$$D=P+N$$

satisfies the following properties: (i) P is nef, (ii) N is effective and Supp(N) is contracted by f. Furthermore, such decomposition is unique.

We talk of a pair (X,K+D) where X is a smooth surface and D is a reduced curve with normal crossings. If (Y,K_Y+B) is its minimal model, then Y has only quotient singularities (cf.[8]). Indeed, write as $K+D=f^*(K_Y+B)+R$, $\Delta=\Delta^+-\Delta^-$, then $D+\Delta^-=\Delta^++f^*B+R$. Since f is totally discrepant, every coefficient of $\Delta^+<1$.

For normal surfaces a birational morphism $f:Y \to Y'$ is <u>totally discrepant</u> if $f:(Y,K_Y) \to (Y',K_{Y'})$ is totally discrepant in the above sense. In this case we have $P_m(Y)=P_m(Y')$ for m>0 and $\kappa(Y)=\kappa(Y')$. We say that Y is <u>minimal</u> if the pair (Y,K_Y) is minimal. Also Y' is a minimal model of Y if (i) Y' is minimal, (ii) there is a totally discrepant birational morphism $f:Y \to Y'$. Theorem (5.1) asserts that every normal surface has a minimal model. We are thus reduced to study minimal normal surfaces. If Y is minimal, then either (i) K_Y is not pseudo effective, or (ii) K_Y is nef. For further discussions and classification theory, we refer to [7] (for the Gorenstein case see [5]).

<u>Example</u> (5.2). Let B be a non-singular curve of genus $g \geqq 2$. Let $X=\mathbb{P}(E)$ be a ruled surface defined by a rank 2 vector bundle E on B. Suppose E is normalized as in the book of Hartshorne. Set $\mathfrak{e}=\det E$, $e=-\deg \mathfrak{e}$. There is a base section b with $b^2=-e$. Suppose e>0. Let $\pi:X \to Y$ be the contraction of b. Since $\rho(Y)=1$, Y is of course minimal. We have $\pi^*K_Y=K+\Delta=((2g-2-e)/e)b+p^*(\mathbb{k}+\mathfrak{e})$ where $p:X \to B$ is the projection map and $\mathbb{k}$ denotes a canonical divisor of B. It follows that $K_Y^2=(2g-2)^2/e \geqq 0$ and $e(Y)=3-2g<0$.

There occur three cases: (i) K_Y is nef (if $e<2g-2$), (ii) $K_Y \approx 0$ (if $e=2g-2$), (iii) $-K_Y$ is nef (if $e>2g-2$).

Finally we mention about the Miyaoka inequality. We recall the following recent result (Miyaoka [3]): Let X be a smooth projective surface and D a divisor having normal crossings on X. Suppose K+D is pseudo effective and let K+D=P+N be the Zariski decomposition. Then

(5.3) $$(K+D)^2-\frac{1}{4}N^2 \leqq 3e(X \backslash D).$$

We deal with normal surfaces whose singularities are contained in Table (3.7). Notice that there {(2), (3)} are elliptic singularities and {(1)*, (2)*, (3)*} are rational singularities. We want to point out two facts.

(5.4)
(i) $\kappa(Y) \geqq 0$ if and only if K_Y is pseudo effective.

(ii) If K_Y is nef, then

$$\frac{3}{2}\#\ \text{rat.Sing}\ Y+3\#\ \text{ellip.Sing}\ Y+K_Y^2 \leqq 3e(Y).$$

In particular, we have $e(Y) \geqq 0$.

We show (ii). Let $\pi: X \to Y$ be the minimal resolution. As noticed in Example (3.6), the exceptional set $A=\bigcup E_i$ has normal crossings. If we write $D=\Sigma E_i$, then $D-\Delta \geqq 0$. The pseudo effectiveness of K_Y implies that of K+D. Clearly, $e(X \backslash D)=e(Y)-\#\ \text{Sing}\ Y$. On the other hand $(K+D)^2=(K+\Delta)^2+(D-\Delta)^2$ and

$$(D-\Delta)^2=(K+D)D-\Delta(D-\Delta) \geqq (K+D)D=-2\#\ \text{rat.Sing}\ Y.$$

If K_Y is nef, we get $P=\pi^*K_Y$ and so $N=D-\Delta$. By (5.3) we get (ii).

When Y has worse singularities, this is not necessarily the case. For instance in Example (5.2), if $e<2g-2$, then K_Y is nef and $\kappa(Y)=2$, but $e(Y)<0$.

In the case of quotient singularities, a more precise result can be found in [3].

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