数理解析研究所講究録
第 536 巻 1984 年 4-26

Expansive properties of Lorenz attractors

M.KOMURO 小室元政

ABSTRACT. Expansiveness of Lorenz attractors is discussed. We introduce two conceptions of expansiveness for real-time flows, which are called K- and K^*-expansiveness, and prove that Lorenz attractors are not K-expansive but K^*-expansive. This means that if the orbit topology is suited, two points of Lorenz attractor which are not close in the orbit topology can be separated under the flow.

Introduction.

E.Lorenz studied in [10] the following system of differential equations in connection with the problems in hydrodynamics;

$$\dot{x} = -10x+10y \qquad \dot{y} = 28x-y-xz \qquad \dot{z} =-8/3z+xy .$$

J.Gukenheimer introduced in [3] a geometric description of a flow which seems to have the qualitative dynamics of the solutions of the Lorenz equation. This geometric Lorenz flow has a complicated attractor. R.F.Williams described in [13] this attractor, which is called the geometric Lorenz attractor, as the inverse limit of a semi-flow on a 2-dimensional branched manifold. We call the geometric Lorenz attractor simply the Lorenz attractor. The Lorenz attractors do not satisfy Smale's Axiom A (see [12] for the definition). That is because the attractors have non-isoleted fixed point, and the orbit can be "slowed down" for an arbitrarily long time; this spoils some uniformity in the hyperbolicity required for Axiom A to hold. However, as Guckenheimer mentions in [3], the Lorenz attractors seems to preserve as much hyperbolicity as they possibly could without satisfying Axiom A.

There is a well developed "statistical mechanics" for attractors satisfying Axiom A ([1]). It is an interesting problem how much of this statistical theory can be extended to apply to the Lorenz attractors. In order to solve this problem, it must be clarified whether the Lorenz attractors have the basic properties of Axiom A systems (ex. the pseudo-orbit tracing property, expansiveness, specification, etç). Auther mentioned the pseudo-orbit tracing property of Lorenz attractors in [9]. The aim of this paper is to study an expansiveness of Lorenz attractors.

The concept of expansiveness for homeomorphism plays important role in the study of discrete flows. R.Bowen and P.Waltes gave in [2] a definition of expansiveness for real-time flows, which is called C-expansiveness ([6]). The basic idea of their definition is that two points which are not close in the orbit topology induced by the real can be separated at the same time even if one allows a continuous time lag. The fixed points of C-expansive flow must be isolated (Proposition 1 in [2]). The Lorenz attractor is not C-expansive because it has a non-isoleted fixed point.

H.B.Keynes and M.Sears introduced in [6] the idea of restriction of the time lag, and gave several definitions of expansiveness weaker than C-expansiveness. When one allows only the time lag which is given by an increasing surjective homeomorphuism of the real, K-expansiveness is defined. It is unknown yet whether the fixed points of K-expansive flow are isoleted . So it is a question whether the Lorenz attractor is K-expansive. In Theorem (II) we will give the negative answer for this question. The reason is not that the restriction of time lag is insufficient, but that the topology induced by the real is unsuited to measure the closeness of two points in same orbit. Taking this fact into consideration, we give a definition of K^*-expansiveness ,and prove that the Lorenz attractor is K^*-expansive (Theorem (I)).

The concept of K^*-expansiveness is enough to show that two points which do not lie on a same orbit can be separated. However it may not be the best one which clarifies the expansive property of the Lorenz attractors. An attempt to clarify such a concept will be mentioned in a later paper.

§1 Definitions and Theorem.

Throughout this paper the symboles $\mathbb{R}$ and $\mathbb{Z}$ denote the set of all real numbers and the set of all integers respectively.

Let X be a compact metric space with a distance function d. We denote

$$d(Y_1,Y_2) = \inf \{ d(y_1,y_2 : y_i \in Y_i \ (i=1,2)\}$$

for any subsets $Y_1, Y_2 \subset X$. A flow $\psi = \{\psi^t\}_{t\in\mathbb{R}}$ on X is a continuous map

$$\psi : X\times\mathbb{R} \longrightarrow X; (x,t) \longrightarrow \psi(x,t) = \psi^t(x)$$

such that $\psi^{t+s} = \psi^t \circ \psi^s$ holds for every $s,t \in \mathbb{R}$ and ψ^0 is the identity map (clearly ψ^t is a homeomorphism on X for each $t \in \mathbb{R}$). This is often denoted by (X,ψ). For each $x \in X$ and $t \in \mathbb{R}$ we denote $x\cdot t = \psi^t(x)$. Also for any subsets $Y \subset X$ and $J \subset R$ we denote $Y\cdot J = \{x\cdot t : x \in Y,\ t \in J\}$.

Definition 1. Let $C(\mathbb{R})$ be the set of all continuous functions from $\mathbb{R}$ to itself. Define

$$C = \{g \in C(R) : g(0)=0\},$$

$$K = \{g \in C : g(\mathbb{R})=\mathbb{R},\ g(s)>g(t)\ (s>t)\}.$$

A flow (X,ψ) is said to be C-expansive ([2],[6]) (resp. K-expansive) if it satisfies the followig ;for every $\varepsilon>0$ there exists an $\varepsilon>0$ such that if for some $g \in C$ (resp. $g \in K$) $d(x\cdot t, y\cdot g(t)) \leqq \delta$ for all $t \in \mathbb{R}$, then $y \in x\cdot[-\varepsilon,\varepsilon]$.

A flow (X,ψ) is said to be K^*-expansive if it satisfies the following; for every $\varepsilon>0$ there exists a $\delta>0$ such that if for some $g \in K$

$$d(x\cdot t, y\cdot g(t)) \leqq \delta \qquad \text{for all } t \in \mathbb{R},$$

then $y\cdot g(t_0) \in x\cdot[t_0-\varepsilon, t_0+\varepsilon]$ for some $t_0 \in \mathbb{R}$.
Clearly the following relation holds in general;

$$\text{C-expansive} \Longrightarrow \text{K-expansive} \Longrightarrow K^*\text{-expansive}.$$

When a flow (X,ψ) has no fixed point, it is proved that C-expansiveness is equivalent to K-expansive (see Theorem 1;(i) and (ii) in [2]). However it is unknown yet whether this is true or not when (X,ψ) has fixed points.

Definition 2. A semi-flow $\varphi = \{\varphi^t\}_{t\geqq 0}$ on a compact metric space X is a continuous map

$$\varphi : X \times [0,\infty) \longrightarrow X \; ; \; (x,t) \longrightarrow \varphi(x,t) = \varphi^t(x)$$

such that φ^0 is the identity map, $\varphi^t : X \longrightarrow X$ is surjective and $\varphi^{t+s} = \varphi^t \circ \varphi^s$ holds for every $t,s \geqq 0$. This is often denoted by (X,φ). We define a compact metric space $\tilde{X}$ and a flow $\tilde{\varphi} = \{\tilde{\varphi}^t\}_{t\in\mathbb{R}}$ on $\tilde{X}$ by

$$\tilde{X} = \{\tilde{x}=(x^t)_{t\in\mathbb{R}} : x^t \in X,\ x^t = \varphi^{t-s}(x^s) \text{ for all } s \leqq t\},$$

$$\varphi^t(\tilde{x}) = (x^{s+t})_{s\in\mathbb{R}} \qquad \text{for } \tilde{x} = (x^s)_{s\in\mathbb{R}} \text{ and } t \in \mathbb{R}.$$

The distance function on $\tilde{X}$ is defined by

$$d(\tilde{x},\tilde{y}) = \sum_{n\in\mathbb{Z}} 2^{-|n|}\, d(x^n, y^n)$$

for $\tilde{x} = (x^t)_{t\in\mathbb{R}}$, $\tilde{y} = (y^t)_{t\in\mathbb{R}}$. This distans function

satisfies $d(x^0, y^0) \leqq \tilde{d}(\tilde{x}, \tilde{y})$ for all $\tilde{x}$, $\tilde{y}$.
(We remark that this distance function $\tilde{d}$ is equivalent to the distance function $\tilde{\rho}$ defined by

$$\tilde{\rho}(\tilde{x}, \tilde{y}) = \int_{-\infty}^{\infty} e^{-|t|} d(x^t, y^t) dt$$

because each x^t varies continuously with respect to t.)
The flow $(\tilde{X}, \tilde{\psi})$ is called the inverse limit of a semi-flow (X, ψ). We denote this by $(\tilde{X}, \tilde{\psi}) = \varprojlim (X, \psi)$.

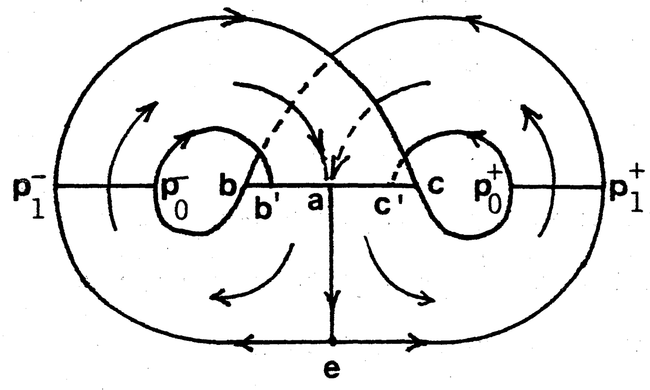
Definition 3 (geometric Lorenz attractors [13]).

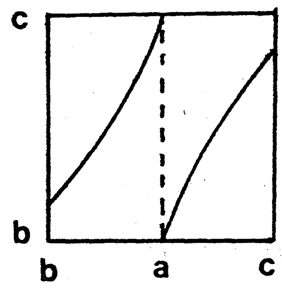
Let L be a 2-dimensional compact smooth manifold (called a Lorenz branched manifold) illustrated as in Figure 1. The set of branch points of L is the line segment $\overline{b'c'}$. The point b (resp. c) is an intersection of the boundary of L with an extention of the line $\overline{c'b'}$ (resp. $\overline{b'c'}$). We permit the case of b = b' or c = c'. The branched manifold L is embedded in $\mathbb{R}^3$ as a subset. We denote by d a distance function on L which is a usual distance function on $\mathbb{R}^3$.

We suppose that a C^1 semi-flow φ on L is defined as illustrated by some arrows in Figure 1. We call this the Lorenz semi-flow. The point e is a unique fixed point of φ. Near the point e the linearlized equation has the form

$$\dot{x} = \mu_1 x \qquad \dot{y} = \mu_2 y \qquad (0 < \mu_2 < \mu_1)$$

There is a unique point $a \in \overline{b'c'}$ such that $\varphi^t(a)$ does not return to the line segment $\overline{bc}$ but converges to the point e as $t \longrightarrow \infty$.





Graph of f

Figure 1

We identify the line segment $\overline{bc}$ with a interval I = [b,c] in $\mathbb{R}$ such that c−b = 1. Without loss of generality we may assume that $d(x,y) = |x-y|$ for all $x,y \in I$.
We denote

$$I_0^+ = (a,c], \quad I_0^- = [b,a) \quad \text{and} \quad I_0 = I_0^+ \cup I_0^-.$$

Also we set the following notation;

[a,e) = $a\cdot[0,\infty)$ the positive half orbit of a,

[a,e] = $[a,e) \cup \{e\}$,

arc(e,b] the orbit from e to b (not including e),

arc(e,c] the orbit from e to c (not including e),

arc[b,b'] the orbit from b to b',

arc[c,c'] the orbit from c to c',

[b',c'], [b',c), (b,c'] the subintervals of I=[b,c].

Define a map $T: I_0 \longrightarrow [0,\infty)$ by

$$T(x) = \inf \{s>0:\ x\cdot s \in I\} \qquad \text{for all } x \in I_0.$$

That is, T(x) is the first return time of $x \in I_0$ to I under φ. Clearly T satisfies that

$$T(x) \longrightarrow \infty \quad \text{as} \quad x \longrightarrow a\pm 0, \text{ and}$$

$$\inf \{T(x) : x \in I_0\} > 0.$$

Define a map $f: I \longrightarrow I$ by $f(a) = b$ and

$$f(x) = \varphi^{T(x)}(x) \qquad \text{if } x \neq a.$$

The map f is called the <u>return map</u> of φ. We assume that the return map f has the followeng properties;

<u>Standing assumption</u>

(1) f has a single discontinuity a, and is C^1 strictly increasing on I_0^+ and I_0^- ;

(2) $f(a-0) = c$, $f(a+0) = b$, $f(b) < a < f(c)$;

(3) $f' > 1$.

(We remark that the condition $f' > \sqrt{2}$ is assumed for the return map f in [13]. Under this condition it is proved that the return map f is locally eventually onto. In our paper, however, only the condition $f' > 1$ is enough.)

If we have a map f as above, then a Lorenz branched manifold L and a C^1 Lorenz semi-flow φ which has the return map f are uniquely determined up to the topological equivalence. Here a semi-flow (X_1, φ_1) is topologically equivalent to a semi-flow (X_2, φ_2) if there exist a homeomorphism $h: X_1 \longrightarrow X_2$ and a continuous map $\sigma: X_1 \times [0,\infty) \longrightarrow [0,\infty)$ such that $\sigma(x,\cdot): [0,\infty) \longrightarrow [0,\infty)$ is a surjective homeomorphism with $\sigma(x,0) = 0$ and

$$\varphi_2(h(x), \sigma(x,t)) = h(\varphi_1(x,t)) \text{ for all } x \in X_1 \text{ and } t \in [0,\infty).$$

Thus we denote by (L_f, φ_f) a Lorenz semi-flow on a Lorenz branched manifold with the return map f. We call the inverse limit

$$(\tilde{L}_f, \tilde{\varphi}_f) = \varprojlim (L_f, \varphi_f)$$

the <u>Lorenz attractor</u> with the return map f.

Our theorem is stated as below.

<u>Theorem</u> Let $(\tilde{L}_f, \tilde{\varphi}_f)$ be a Lorenz attractor with the return map f. Then

(I) $(\tilde{L}_f, \tilde{\varphi}_f)$ is K^*-expansive, but

(II) $(\tilde{L}_f, \tilde{\varphi}_f)$ is not K-expansive.

The following property of the return map is bacic for the proof of Theorem.

<u>Proposition 1.</u> Let $f: I \longrightarrow I$ be a return map which satisfies (1) $\sim$ (3). Then there exist a constant α_1 with $0 < \alpha_1 < 1/2$ which satisfies the following;

(4) for $x, y \in I$ with $d(x,y) \leqq \alpha_1$, $f(x) = f(y)$ implies $x = y$,

(5) for $x \in I_0^-$ and $y \in I_0^+$, $d(x,y) \leqq \alpha_1$ implies $d(f(x), f(y)) \geqq \frac{1}{2}$,

(6) for $x \in I - \bigcup_{n \geqq 0} f^{-n}(a)$, if $d(f^n x, f^n y) \leqq \alpha_1$ for all $n \geqq 0$ then $x = y$.

<u>Proof</u>. The maps $f | I_0^- : I_0^- \longrightarrow [f(b), c) = [b', c)$ and $f | I_0^+ : I_0^+ \longrightarrow (b, f(c)] = (b, c']$ are homeomorphism. Thus for each $x \in [f(b), f(c)]$ there are two points $x_1 \in I_0^-$ and $x_2 \in I_0^+$ such that $f(x_1) = f(x_2) = x$. Then

$$\alpha_1' = \frac{1}{2} \inf \{ d(x_1, x_2) : x \in [f(b), f(c)] \}$$

is positive because $d(x_1,x_2) = d(x_1,a) + d(a,x_2)$ varies continuously with respect to $x \in [f(b),f(c)]$. Then for each $x,y \in I$ with $d(x,y) \leqq \alpha_1'$, $f(x) = f(y)$ implies $x = y$. By (3) there is $\alpha_1''>0$ such that

$$f(a-s) > 3/4 \text{ and } f(a+s) < 1/4 \text{ for all } 0 < s < \alpha_1''.$$

Then for $x \in I_0^-$ and $y \in I_0^+$, $d(x,y) \leqq \alpha_1''$ implies $d(f(x),f(y)) \geqq \frac{1}{2}$. Put $\alpha_1 = \min\{\alpha_1', \alpha_1''\}$, then this clearly satisfies (4) and (5). To see (6), let $x \in I - \bigcup_{n\geqq 0} f^{-n}(a)$ and $y \in I$ satisfy $d(f^n x, f^n y) \leqq \alpha_1$ for all $n \geqq 0$. Since $\alpha_1 \leqq \alpha_1''$, for each $n \geqq 0$

$$\text{either } f^n x,\ f^n y \in I_0^-, \quad \text{or } f^n x,\ f^n y \in I_0^+$$

holds. If $x \neq y$, then $f^n x \neq f^n y$ for all $n \geqq 0$ because $\alpha_1 \leqq \alpha_1'$. Since f is continuous on I_0^+ and I_0^-, and since $f' > 1$,

$$d(f^n x, f^n y) \longrightarrow \infty \qquad \text{as } n \longrightarrow \infty.$$

This is a contradiction. Therefore $x = y$.

§2 Preliminary.

Let (L_f, φ_f) be a Lorenz semi-flow with the return map f. For each $x \in L_f$ and $t \in \mathbb{R}$ we denote

$$x \cdot t = \begin{cases} \varphi_f^t(x) & \text{if } t \geqq 0 \\ \{y \quad L_f : \varphi_f^{-t}(y) = x\} & \text{if } t < 0. \end{cases}$$

If $x \in L_f - [b',c']$, then $x\cdot(-t)$ is one point for $t>0$ small enough. If $x \in [b',c']$, then $x\cdot(-t)$ is a finite set of points having more than two points for any $t > 0$. For subsets $Y \subset L_f$ and $J \subset \mathbb{R}$ we denote

$$Y \cdot J = \{\, z : z \in x \cdot t,\ x \in Y,\ t \in J\}.$$

(P.1) Recall the map $T: I_0 \longrightarrow (0,\infty)$ as in Definition 3.

Define

$$C(x) = x\cdot[0,T(x)) \qquad \text{for all } x \in I_0,$$

$$T_n(x) = \sum_0^{n-1} T(f^i x) \quad \text{for } x \in I - \bigcup_0^{n-1} f^{-i}(a) \text{ and } n \geq 1$$

$$(\text{where } T_0(x) = 0),$$

$$L_0^\sigma = \{x\cdot t:\ t \in [0,T(x)),\ x \in I_0^\sigma\} \quad (\sigma = +,-) \text{ and}$$

$$L_0 = L_0^+ \cup L_0^- .$$

(P.2) Put

$$q_0 = \frac{1}{4} \inf \{T(x):\ x \in I_0\} > 0.$$

Define a continuous map $\zeta:\ I\cdot[-q_0,q_0] \longrightarrow I$ such that

$$\zeta(x\cdot t) = x \qquad \text{for all } x \in I \text{ and } t \in [-q_0,q_0].$$

We define the line segments J^+ and J^- as follow (Figure 1);

$$J^- = \overline{p_0^- p_1^-} \quad \text{such that } J^- \cap C(x) = \{\text{one point}\} \text{ for all } x \in I_0^-$$

$$\text{where } p_0^- \in \text{arc}[b,b'] ,\ p_0^-\cdot(2q_0) = b' \text{ and}$$

$$p_1^- \in \text{arc}(e,c] ,\ p_1^-\cdot(2q_0) = c,$$

$$J^+ = \overline{p_0^+ p_1^+} \quad \text{such that } J^+ \cap C(x) = \{\text{one point}\} \text{ for all } x \in I_0^+$$

$$\text{where } p_0^+ \in \text{arc}[c,c'],\ p_0^+\cdot(2q_0) = c' \text{ and}$$

$$p_1^+ \in \text{arc}(e,b],\ p_1^+\cdot(2q_0) = b.$$

And put $J = J^+ \cup J^-$. Define a map $\Theta: I_0 \longrightarrow (0,\infty)$ such that $\Theta(x) = \inf \{t > 0 :\ x\cdot t \in J\}$ for $x \in I_0$. Set

$L_1^\sigma = \{x\cdot t : t\in[\theta(x),T(x)),\ x\in I_0^\sigma\}$ ($\sigma = +,-$) and

$L_1 = L_1^+ \cup L_1^-$.

(P.3) For each subset $Y\subset L_f$ and $\varepsilon>0$ we denote

$B(Y,\varepsilon) = \{x\in L_f: d(x,Y) \leqq \varepsilon\}$.

Take a constant α_0 with $0<\alpha_0<\alpha_1$ which satisfies the following;

(i) $B(J^\pm,\alpha_0)\cap B(I,\alpha_0) = \emptyset$,

(ii) $B(J^\pm,\alpha_0)\cap B(L_0^\mp,\alpha_0) = \emptyset$,

(iii) $B(J^\pm,\alpha_0)\cap B([a,e],\alpha_0) = \emptyset$,

(iv) $B(I,\alpha_0)\cap B(e,\alpha_0) = \emptyset$,

(v) $B(J,\alpha_0)\cap I\cdot[-q_0,q_0] = \emptyset$ and

(vi) $B(I,\alpha_0)\subset I\cdot[-q_0,q_0]$.

(P.4) For $x\in I$ and $\varepsilon\in(0,\alpha_0)$ we denote

$Y(x,\varepsilon) = \{y\in B(I,\varepsilon): \zeta(y) = x\}$.

If $x\in[f(b),f(c)]$ then $Y(x,\varepsilon)$ is "Y", and if $x\notin[f(b),f(c)]$ then $Y(x,\varepsilon)$ is "I". For each $x\in[f(b),f(c)]$, define

$Y^+(x,\varepsilon) = Y(x,\varepsilon)\cap L_1^+$ and $Y^-(x,\varepsilon) = Y(x,\varepsilon)\cap L_1^-$.

§3 Proof of Theorem (I).

Let $\varepsilon>0$ be given. Take a $\varkappa>0$ such that

(7) $\varkappa \leqq \min\{q_0,\varepsilon\}$,

(8) $I\cdot[-2\varkappa,2\varkappa]\subset B(I,\alpha_0)$,

(9) $d(x,x\cdot t) \leqq \alpha_0/2$ for all $x\in L_f$ and $t\in[0,\varkappa]$.

There exists a $\delta>0$ with $\delta<\alpha_0/2$ such that

(10) $\quad B(I,\delta) \subset I\cdot[-\varkappa,\varkappa]$ and

(11) $\quad d(x,y) \leq \delta$ implies $d(\zeta(x),\zeta(y)) \leq \alpha_1$

for all $x,y \in I\cdot[-q_0,q_0]$.

In this section we denote simply $\tilde{L}_f$ and L_f by $\tilde{L}$ and L respectively. Let $\tilde{x} \in \tilde{L}$ be given. Suppose that $\tilde{y} \in \tilde{L}$ and $g \in K$ satisfy

$$\tilde{d}(\tilde{x}\cdot t,\tilde{y}\cdot g(t)) \leq \delta \quad \text{for all } t \in \mathbb{R}.$$

We shall show that

$$y\cdot g(t_0) \in x\cdot[t_0-\varepsilon,t_0+\varepsilon] \quad \text{for some } t_0 \in \mathbb{R}$$

by a sequence of lemmas.

<u>Lemma 1.</u> If for $y \in L$

$$d(e\cdot t,y\cdot g(t)) \leq \delta \quad \text{for all } t \geq 0,$$

then $y \in [a,e] \cap B(e,\delta)$.

<u>Proof</u>. Suppose that $y \notin [a,e]$. Then there is a $t_0 \geq 0$ such that $y\cdot g(t_0) \in J$, because g is surjective. Since $B(J,\alpha_0) \cap B(e,\alpha_0) = \emptyset$,

$$d(e\cdot t_0,y\cdot g(t_0)) = d(e,y\cdot g(t_0)) > \alpha_0 > \delta.$$

This is a contradiction, thus $y \in [a,e]$. Since $g(0) = 0$, we have $y \in [a,e] \cap B(e,\delta)$.

<u>Lemma 2.</u> If for $\tilde{y} \in \tilde{L}$

$$d(e\cdot t,y\cdot g(t)) \leq \delta \quad \text{for all } t \in \mathbb{R},$$

then $\tilde{y} = \tilde{e}$, where $\tilde{e} = (e)_{t\in\mathbb{R}} \in \tilde{L}$.

Proof. For every $t \geqq 0$ we have

$$d(e\cdot t, y^0\cdot g(t)) \leqq \tilde{d}(\tilde{e}\cdot t, \tilde{y}\cdot g(t)) \leqq \delta.$$

By Lemma 1, $y^0 \in [a,e] \cap B(e,\delta)$. Suppose that $y^0 \neq e$. Then there is an $s_0 < 0$ such that $y^{g(s_0)} = a \in I$. Thus

$$d(e,a) = d(e, y^{g(s_0)}) \leqq d(\tilde{e}\cdot s_0, \tilde{y}\cdot g(s_0)) \leqq \delta.$$

This contradicts the fact that

$$B(e,\alpha_0) \cap B(I,\alpha_0) = \emptyset \quad \text{and} \quad \delta < \alpha_0.$$

Thus $y^0 = e$, so that $\tilde{y} = \tilde{e}$.

Lemma 2 states that Theorem (I) holds for $\tilde{x} = \tilde{e}$. From now on, we will consider the case of $\tilde{x} \neq \tilde{e}$.

Lemma 3. Let $x \in I_0$ and $z \in I$ be given. If

$$d(x\cdot t, z\cdot g(t)) \leqq \alpha_0 \quad \text{for all } t \in [0, T(x)],$$

then $z\cdot g(T(x)) \in B(I,\alpha_0)$ and $\zeta(z\cdot g(T(x))) = f(z)$.

Proof. Since $x\cdot T(x) \in I$, clearly $z\cdot g(T(x))$ belongs to $B(I,\alpha_0)$. Since there is an $s \in [0,T(x)]$ such that $x\cdot s \in J$ and since $z\cdot g(s) \in B(x\cdot s,\alpha_0) \subset B(J,\alpha_0)$, we have $z\cdot[0,T(x)] \not\subset B(I,\alpha_0)$. Thus $\zeta(z\cdot g(T(x))) = f^n(z)$ for some $n \geqq 1$. Suppose $\zeta(z\cdot g(T(x))) \neq f(z)$. Then there is an $n \geqq 2$ such that $\zeta(z\cdot g(T(x))) = f^n(z)$, thus there is a $t_1 \in [0,T(x)]$ such that $z\cdot g(t_1) = f(z)$ (i.e. $g(t_1) = T(z)$). Since $d(x\cdot t_1, z\cdot g(t_1)) < \alpha_0$, $x\cdot t_1 \in B(I,\alpha_0)$. Thus

$$t_1 \in [0,q_0] \cup [T(x)-q_0, T(x)].$$

In the case of $t_1 \in [0,q_0]$, there is a $t \in [0,t_1]$ with $z\cdot g(t) \in J$ because $g(t_1) = T(z)$. It follows that

$$x\cdot t \in B(z\cdot g(t),\alpha_0) \subset B(J,\alpha_0).$$

Since $x\cdot t \in x\cdot[0,q_0] \subset I\cdot[-q_0,q_0]$, this contradicts (P.3 v). In the case of $t_1 \in [T(x)-q_0,T(x)]$, there is a $t \in [t_1,T(x)]$ with $z\cdot g(t) \in J$. Because, if $z\cdot[g(t_0),g(T(x))] \cap J = \emptyset$, then $z\cdot[g(t_0),g(T(x))] \subset B(I,\alpha_0)$, thus $\zeta(z\cdot g(T(x)) = \zeta(z\cdot g(t_0))$ $= f(z)$. This contradicts our assumption. Thus

$$x\cdot t \in B(z\cdot g(t),\alpha_0) \subset B(J,\alpha_0).$$

Since $x\cdot t \in x\cdot[T(x)-q_0,T(x)] \subset f(x)\cdot[-q_0,0] \subset I\cdot[-q_0,q_0]$, this contradicts (P.3 v). Therefore , in any case we have $\zeta(z\cdot g(T(x))) = f(z)$.

<u>Lemma 4.</u> Let $x \in I_0$, $y \in L$ and $h \in K$ be given. If $d(x\cdot t,y\cdot h(t)) \leqq \delta$ for all $t \in [0,T(x)]$, then

$$y\cdot h(T(x)) \in B(I,\delta) \text{ and } \zeta(y\cdot h(T(x))) = f(\zeta(y)).$$

<u>Proof.</u> From the fact that $x\cdot T(x) \in I$ and $d(x\cdot T(x),y\cdot h(T(x))) \leqq \delta$, it follows that $y\cdot h(T(x)) \in B(I,\delta)$. Since $y \in B(x,\delta) \subset I\cdot[-\varkappa,\varkappa]$, there is a $\tau \in [-\varkappa,\varkappa]$ such that $\zeta(y) = y\cdot\tau$. Using (9), we have

$$\begin{aligned} d(x\cdot t,\zeta(y)\cdot h(t)) &\leqq d(x\cdot t,y\cdot h(t))+d(y\cdot h(t),y\cdot(\tau+h(t)) \\ &\leqq \delta + \alpha_0/2 \leqq \alpha_0 \end{aligned}$$

for all $t \in [0,T(x)]$. Since $\zeta(y) \in I$, by Lemma 3 we have

$$\zeta(y)\cdot h(T(x)) \in B(I,\alpha_0) \text{ and } \zeta(\zeta(y)\cdot h(T(x))) = f(\zeta(y)).$$

Since $y\cdot h(T(x)) \in B(x\cdot T(x),\delta) \subset I\cdot[-\varkappa,\varkappa]$, $y\cdot(h(T(x))+\tau) \in I\cdot[-2\varkappa,2\varkappa]$, so that $\zeta(y\cdot h(T(x))) = \zeta(y\cdot(h(T(x))+\tau))$.
Therefore $\zeta(y\cdot h(T(x))) = \zeta(y\cdot(h(T(x))+\tau)) = \zeta(\zeta(y)\cdot h(T(x))) = f(\zeta(y))$.

Lemma 5. Let $n \geqq 0$, $x \in I-\bigcup_0^n f^{-j}(a)$ and $y \in L$ be given. If $d(x\cdot t, y\cdot g(t)) \leqq \delta$ for all $t\in[0,T_n(x)]$, then

$$y\cdot g(T_j(x)) \in B(I,\delta),\ \zeta(y\cdot g(T_j(x))) = f^j(\zeta(y)) \text{ and}$$
$$d(f^j x, f^j(\zeta(y))) \leqq \alpha_1 \quad \text{for all } 0 \leqq j \leqq n.$$

Proof. The induction with respect to j.
Since $d(x,y) \leqq \delta$ and $T_0(x) = 0$, we have

$$y\cdot g(T_0(x)) = y \in B(I,\delta),\ \zeta(y\cdot g(T_0(x))) = \zeta(y) \text{ and}$$
$$d(x,\zeta(y)) \leqq \alpha_1 \quad \text{(by (11))}.$$

Thus the assertion is true for $j = 0$.

Suppose that the assertion is true for $j = i$. Define $g_i \in K$ by $g_i(t) = g(t+T_i(x)) - g(T_i(x))$ for all $t \in \mathbb{R}$. Then

$$d((f^i x)\cdot t, (y\cdot g(T_i(x)))\cdot g_i(t)) =$$
$$d(x\cdot(t+T_i(x)), y\cdot g(t+T_i(x))) \leqq \delta$$

for all $t \in [0,T(f^i x)]$. By Lemma 4,

$$(y\cdot g(T_i(x)))\cdot g_i(T(f^i x)) = y\cdot g(T_{i+1}(x)) \subset B(I,\delta) \text{ and}$$
$$\zeta(y\cdot g(T_{i+1}(x))) = f(\zeta(y\cdot g(T_i(x)))) = f^{i+1}(\zeta(y)).$$

Since $d((f^i x)\cdot T(f^i x), y\cdot g(T_{i+1}(x))) \leqq \delta$, $d(f^{i+1}(x), f^{i+1}(\zeta(y))) \leqq \alpha_1$ (by (11)). Therefore the assertion is true for $j = i+1$, and so for all $0 \leqq j \leqq n$.

Lemma 6. Let $x \in I - \bigcup_{n \geq 0} f^{-n}(a)$ and $y \in L$ be given. If $d(x \cdot t, y \cdot g(t)) \leqq \delta$ for all $t \geqq 0$, then $y \in Y(x,\delta)$.

Proof. By Lemma 5, we have

$$d(f^n x, f^n(\zeta(y))) \leqq \alpha_1 \quad \text{for all } n \geqq 0.$$

By Proposition 1(6), it follows that $x = \zeta(y)$. Thus $y \in B(x,\delta) \cap x \cdot [-q_0, q_0] = Y(x,\delta)$.

Lemma 7. Let $y \in L$ and $h \in K$ be given. If $d(a \cdot t, y \cdot h(t)) \leqq \delta$ for all $t \geqq 0$, then $y \in Y(a,\delta)$.

Proof. Since $d(a,y) \leqq \delta$, $y \in B(I,\delta) \subset I \cdot [-\varkappa, \varkappa]$. There is $\tau \in [-\varkappa,\varkappa]$ such that $\zeta(y) = y \cdot \tau$. Suppose $\zeta(y) \neq a$. Then there is a $t_0 > 0$ such that $y \cdot g(t_0) \in J$, so that

$$d(a \cdot t_0, y \cdot g(t_0)) \leqq \delta < \alpha_0.$$

Since $a \cdot t_0 \in [a,e)$ and $B(J,\alpha_0) \cap B([a,e),\alpha_0) = \emptyset$, this is a contradiction. Therefore $\zeta(y) = a$, so that $y \in Y(a,\delta)$.

Lemma 8. Let $x \in \bigcup_{n \geq 0} f^{-n}(a) \subset I$ and $y \in L$ be given. If $d(x \cdot t, y \cdot g(t)) \leqq \delta$ for all $t \geqq 0$, then $y \in Y(x,\delta)$.

Proof. Let $n \geqq 0$ be the minimal integer such that $f^n(x) = a$. Since $d(x \cdot t, y \cdot g(t)) \leqq \delta$ for all $t \in [0, T_n(x)]$, by Lemma 5 it follows that $\zeta(y \cdot g(T_n(x))) = f^n(\zeta(y))$ and

$$d(f^i x, f^i(\zeta(y))) \leqq \alpha_1 \quad \text{for all } 0 \leqq i \leqq n.$$

Define $g_n \in K$ by $g_n(t) = g(T_n(x)+t) - g(T_n(x))$ for all $t \in \mathbb{R}$. Then $d(a \cdot t, (y \cdot g(T_n(x))) \cdot g_n(t)) =$

$$d(x \cdot (T_n(x)+t), y \cdot g(T_n(x)+t)) \leqq \delta \quad \text{for all } t \geqq 0.$$

Thus by Lemma 7 we have

$$a = \zeta(y \cdot g(T_n(x))),$$

that is, $f^n(x) = f^n(\zeta(y))$. Using Proposition 1 repeatedly, we have $x = \zeta(y)$. Since $y \in B(x,\delta)$, $y \in Y(x,\delta)$.

Lemma 9. Suppose $\tilde{x} \in \tilde{L}$ satisfies $\tilde{x} \neq \tilde{e}$ and $x^0 \in I$. If $\tilde{d}(\tilde{x} \cdot t, \tilde{y} \cdot g(t)) \leq \delta$ for all $t \geq 0$, then $x^0 = y^\tau$ for some $\tau \in [-\varkappa, \varkappa]$.

Proof. Notice that

$$d(x^0 \cdot t, y^0 \cdot g(t)) \leq \tilde{d}(\tilde{x} \cdot t, \tilde{y} \cdot g(t)) \leq \delta \quad \text{for all } t \geq 0.$$

Using Lemma 8 when $x^0 \in \bigcup_{n \geq 0} f^{-n}(a)$, and using Lemma 6 when $x^0 \in I - \bigcup_{n \geq 0} f^{-n}(a)$, we have $y^0 \in Y(x^0, \delta)$. Since $B(I,\delta) \subset I \cdot [-\varkappa, \varkappa]$, there is a $\tau \in [-\varkappa, \varkappa]$ such that $x^0 = y^0 \cdot \tau$, thus we have $x^0 = y^\tau$.

For $\tilde{x} \in \tilde{L}$ with $x^0 \in I$ we denote

$$T_{-1}(\tilde{x}) = \sup \{t<0 : x^t \in I\}.$$

Define recursively, if it is well defined, $S_0(\tilde{x}) = 0$ and

$$S_{-i-1}(\tilde{x}) = T_{-1}(\tilde{x} \cdot S_{-i}(\tilde{x})) \qquad \text{for each } i \geq 0,$$

and $T_{-i}(\tilde{x}) = \sum_0^i S_j(\tilde{x})$.

Lemma 10. Let $n \geq 0$ and $\tilde{x} \in \tilde{L}$ with $x^0 \in I$ be given. Suppse that $T_{-i}(\tilde{x})$ is well defined for all $0 \leq i \leq n$ and that $\tilde{d}(\tilde{x} \cdot t, \tilde{y} \cdot g(t)) \leq \delta$ for all $t \leq 0$.
If $x^0 = y^\tau$ for some $\tau \in [-\varkappa, \varkappa]$, then $x^s = y^{s+\tau}$ for all $s \in [T_{-n}(\tilde{x}), 0]$.

Proof. Suppose that there are $0 \leq i < n$ and $s \in [T_{-i-1}(\tilde{x}), T_{-i}(\tilde{x})]$ such that $x^t = y^{t+\tau}$ for all $t \in [T_{-i}(\tilde{x}), 0]$ and $x^s \neq y^{s+\tau}$. Since

$$x^s \cdot (T_{-i}(\tilde{x})-s) = x^{T_{-i}(\tilde{x})} = y^{T_{-i}(\tilde{x})+\tau} = y^{s+\tau} \cdot (T_{-i}(\tilde{x})-s),$$

it follows that

$$\text{either } x^s \in L_0^+ \text{ and } y^{s+\tau} \in L_0^- ,$$
$$\text{or } x^s \in L_0^- \text{ and } y^{s+\tau} \in L_0^+ .$$

Thus there exists the maximal $t \in [T_{-i-1}(\tilde{x}), T_{-i}(\tilde{x}))$ such that either $x^t \in J$ or $y^{g(t)} \in J$ holds.

If $x^t \in J^\pm$ we have $y^{g(t)} \in L_1^\mp$, and if $y^{g(t)} \in J^\pm$ we have $x^t \in L_1^\mp$. In any case we have $d(x^t, y^{g(t)}) > \alpha_0 > \delta$ because $B(J^\pm, \alpha_0) \cap B(L_0^\mp, \alpha_0) = \emptyset$. This contradicts the assumption $d(x^t, y^{g(t)}) \leq \tilde{d}(\tilde{x} \cdot t, \tilde{y} \cdot g(t)) \leq \delta$.

Lemma 11. Let $\tilde{x}, \tilde{y} \in \tilde{L}$ satisfies that

$\tilde{d}(\tilde{x} \cdot t, \tilde{y} \cdot g(t)) \leq \delta$ for all $t \leq 0$, and

$x^0 = b$ and $x^s \in \mathrm{arc}(e,b]$ for all $s \leq 0$

(or $x^0 = c$ and $x^s \in \mathrm{arc}(e,c]$ for all $s \leq 0$).

If $x^0 = y^\tau$ for some $\tau \in [-\varkappa, \varkappa]$, then $x^s = y^{s+\tau}$ for all $s \leq 0$.

Proof. We consider the case of $x^0 = b$. Suppose $x^s \neq y^{s+\tau}$ for some $s \leq 0$. Since $x^s \cdot (-s) = x^0 = y^\tau = y^{s+\tau} \cdot (-s)$, there is an $s' \in (s, 0)$ such that

$$x^{s'} \in Cl(L_0^+) \quad \text{and} \quad y^{s'+\tau} \in Cl(L_0^-)$$

where Cl(Y) denote the closer of Y in L. There exists the maximal $t \in (-\infty, 0]$ such that either $x^t \in J^+$ or $y^{g(t)} \in J^-$ holds. If $x^t \in J^+$ we have $y^{g(t)} \in Cl(K_1^-)$, and if $y^{g(t)} \in J^-$ we have $x^t \in Cl(L_1^+)$. In any case we have $d(x^t, y^{g(t)}) > \alpha_0 > \delta$ by the choice of α_0 ;(P.3 ii). This contradicts the assumption $d(x^t, y^{g(t)}) \leqq \tilde{d}(\tilde{x}\cdot t, \tilde{y}\cdot g(t)) \leqq \delta$, thus the assertion is true. The proof in the case of $x^0 = c$ is similer.

<u>Lemma 12.</u> Let $\tilde{x}, \tilde{y} \in \tilde{L}$ be given, and suppose $x^0 \in I$. If $\tilde{d}(\tilde{x}\cdot t, \tilde{y}\cdot g(t)) \leqq \delta$ for all $t \in \mathbb{R}$, then $\tilde{y}$ $\tilde{x}\cdot[-\varkappa, \varkappa]$.

<u>Proof.</u> Since $\tilde{d}(\tilde{x}\cdot t, \tilde{y}\cdot g(t)) \leqq \delta$ for all $t \geqq 0$, by Lemma 9, there is a $\tau \in [-\varkappa, \varkappa]$ such that $x^0 = y^\tau$. And since $\tilde{d}(\tilde{x}\cdot t, \tilde{y}\cdot g(t)) \leqq \delta$ for all $t \leqq 0$, using Lemmas 10 and 11 we have $x^s = y^{s+\tau}$ for all $s \leqq 0$. Since $x^s = x^0\cdot s = y^\tau\cdot s = y^{s+\tau}$ for all $s \geqq 0$, we get $\tilde{x} = (x^s)_{s\in\mathbb{R}} = (y^{s+\tau})_{s\in\mathbb{R}} = \tilde{y}\cdot\tau$, that is, $\tilde{y} \in \tilde{x}\cdot[-\varkappa, \varkappa]$.

<u>Lemma 13.</u> Let $\tilde{x} \in \tilde{L}$ be given, and suppose $\tilde{x} \neq \tilde{e}$. If $d(x\cdot t, y\cdot g(t)) \leqq \delta$ for all $t \in \mathbb{R}$, then $y\cdot g(t_0) \in x\cdot[t_0-\varepsilon, t_0+\varepsilon]$ for some $t_0 \in \mathbb{R}$.

<u>Proof.</u> By $\tilde{x} \neq \tilde{e}$, there is a $t_0 \in \mathbb{R}$ such that $x^{t_0} \in I$. Put $\tilde{z} = \tilde{x}\cdot t_0$, $\tilde{w} = \tilde{y}\cdot g(t_0)$ and

$$h(t) = g(t_0+t) - g(t_0) \quad \text{for all } t \in \mathbb{R}.$$

Then $\tilde{d}(\tilde{z}\cdot t, \tilde{w}\cdot h(t)) = \tilde{d}(\tilde{x}\cdot(t_0+t), \tilde{y}\cdot g(t_0+t)) \leqq \delta$ for all $t \in \mathbb{R}$

and $z^0 = x^{t_0} \in I$. Thus, by Lemma 12 we have $\tilde{w} = \tilde{z}\cdot\tau$ for some $\tau \in [-\varkappa,\varkappa]$, so that

$$\tilde{y}\cdot g(t_0) = \tilde{w} \in \tilde{z}\cdot[-\varkappa,\varkappa] \subset (\tilde{x}\cdot t_0)\cdot[-\varepsilon,\varepsilon] = \tilde{x}\cdot[t_0-\varepsilon,t_0+\varepsilon].$$

This prove Lemma 13.

Lemma 13 implies that Theorem (I) holds for $\tilde{x} \neq \tilde{e}$. By combinig Lemma 2 and Lemma 13, the proof of Theorem (I) is completed.

§4 <u>Proof of Theorem (II).</u>

We must show that there is an $\varepsilon > 0$ such that for any $\delta > 0$ there are $\tilde{x},\tilde{y} \in \tilde{L}$ and $g \in K$ such that

$$\tilde{y} \notin \tilde{x}\cdot[-\varepsilon,\varepsilon] \quad \text{and}$$
$$\tilde{d}(\tilde{x}\cdot t,\tilde{y}\cdot g(t)) \leqq \delta \quad \text{for all } t \in \mathbb{R}.$$

To do this, put $\varepsilon = 1$. Let $\delta > 0$ be given.
There is a $\tilde{z} \in \tilde{L}$ such that

$$z^0 = b \text{ and } z^s \in \operatorname{arc}(e,b] \quad \text{for all } s \leqq 0.$$

There is an integer $N > 0$ such that $\sum_{n \geqq N} 2^{-n} D \leqq \delta/4$, where D is the diameter of L. Take a $\gamma > 0$ such that

$$d(z^u,z^v) < 4\gamma \text{ implies } d(z^{u+s},z^{v+s}) < \delta/6$$

for all $s \in \mathbb{R}$ with $|s| \leqq N$. There is an $s_0 < 0$ such that

$$z^{s_0}\cdot(-\infty,0] \subset B(e,2\gamma).$$

Moreover there is an $s_1 < s_0$ such that

$$z^{s_1}\cdot(-\infty,0] \subset B(e,\gamma).$$

Take $\tilde{x},\tilde{y}\in\tilde{L}$ such that

$$x^0 = z^{s_1} \quad\text{and}\quad y^0 = z^{s_1-2\varepsilon}.$$

Since $y^0 \notin x^0\cdot[-\varepsilon,\varepsilon]$, $\tilde{y} \notin \tilde{x}\cdot[-\varepsilon,\varepsilon]$. Define $g\in K$ by

$$g(t) = \begin{cases} t & (t \leqq 0) \\ (s_0-s_1+2\varepsilon)t/(s_0-s_1) & (0 < t \leqq s_0-s_1) \\ t + 2\varepsilon & (s_0-s_1 < t\ ) \end{cases}$$

Let $t\in\mathbb{R}$ be given. In the case of $t \leqq 0$, it follows that

$$x^t = z^{s_1+t} \in z^{s_1}\cdot(-\infty,0] \subset B(e,\gamma) \text{ and}$$

$$y^{g(t)} = y^t = z^{s_1+t-2\varepsilon} \in z^{s_1}\cdot(-\infty,0] \subset B(e,\gamma).$$

Thus $d(x^t,y^{g(t)}) \leqq 2\gamma$.

In the case of $0 < t \leqq s_0-s_1$, we have

$$x^t = z^{s_0}\cdot(t-s_0+s_1) \in z^{s_0}\cdot(-\infty,0] \subset B(e,2\gamma) \text{ and}$$

$$y^{g(t)} = z^{s_1-2\varepsilon+g(t)} =$$

$$z^{s_0}\cdot(g(t)-s_0+s_1-2\varepsilon) \in z^{s_0}\cdot(-\infty,0] \subset B(e,2\gamma).$$

Thus $d(x^t,y^{g(t)}) \leqq 4\gamma$.

In the case of $s_0-s_1 \leqq t$, we have

$$x^t = z^{s_1+t} \quad\text{and}\quad y^{g(t)} = z^{s_1+g(t)-\varepsilon}.$$

Thus $d(x^t,y^{g(t)}) = 0$.

In any case we have $d(x^t, y^{g(t)}) \leqq 4\gamma$, so that

$$\tilde{d}(\tilde{x}\cdot t, \tilde{y}\cdot g(t)) \leqq \sum_{-N}^{N} 2^{-|n|}\, d(x^{n+t}, y^{n+g(t)}) + \delta/2$$

$$\leqq \sum_{-N}^{N} 2^{-|n|}(\delta/6) + \delta/2 \leqq \delta.$$

for any $t \in \mathbb{R}$. This complete s the proof of Theorem (II).

References.

[1] R.Bowen and D.Ruelle, The ergodic theory of Axiom A flows, Invent. Math., 29 (1975), 181-202.
[2] R.Bowen and P.Walters, Expansive one-parameter flows, J. Diff. Eq., 12 (1972), 180-193.
[3] J.Gukenheimer, A strange, strange attractors, in The Hopf Bifurcation Theorem and its Applications, ed. by J.E.Marsden and M.Mc Cracken, Springer-Verlag (1976), 368-381.
[4] J.Gukenheimer and P.Holmes, Nonlinear Oscillations, Dynamical Systems and Bifurcations of Vector Fields, Applied Mathematical Sciences, Vol.42, Berlin-Heidelberg-New York: Springer 1983.
[5] J.Gukenheimer and R.F.Williams, Structural stability of Lorenz attractors,IHES Publ. Math., 50 (1979), 59-72.
[6] H.B.Keynes and M.Sears, $\mathcal{F}$-expansive transformation groups, General Topology and Its Applications, 10 (1979), 65-87.
[7] H.B.Keynes and M.Sears, Modelling expansion in real flows, Pacific J. Math. 85 (1979), 111-124.
[8] H.B.Keynes and M.Sears, Real-expansive flows and topological dimension, Ergod. Th. and Dynam. Sys. 1 (1981), 179-195.
[9] M.Komuro, Lorenz attractors have not the pseudo-orbit tracing property, to appear.
[10] E.Lorenz, Deterministic nonperiodic flow, Journal of Atmosheric Sciences, 20 (1963), 130-141.
[11] D.Ruelle, The Lorenz attractor and the problem of turbulence, Lecture Notes in Math., 565 (1976), Springer, 146-158.
[12] S.Smale, Differentiable dynamical systems, Bull. Amer. Math. Soc. 73 (1967), 747-817.
[13] R.F.Williams, Structure of Lorenz attractors, IHES Publ. Math., 50 (1979), 73-99.

Department of Mathematics
Tokyo Metropolitan University
Tokyo, JAPAN