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LINKS OF EMBEDDINGS OF SURFACES AND TOPOLOGICAL ENTROPY

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ABSTRUCT

In this paper, we show that if $g:F\to F$ is an orientation preserving C^1-embedding with a g-invariant set Σ of finite points in Int F, then the dynamical complexity of g is described by the link type $L_{g,\Sigma}$ $(=\Sigma\times[0,1]/\sim)$ in the mapping torus M_g. As an application, for a class of differential equations, we give a criterion for determining whether they possess infinitely many periodic solutions by using finitely many given periodic solutions.

1. INTRODUCTION.

Let f be a self-map of a compact, orientable surface F with an f-invariant set Σ of finite points in Int F. Let M_f be the mapping torus of f i.e. M_f is obtained from $F\times[0,1]$ by identifying $(x, 0)$ to $(f(x), 1)$ $(x\in F)$. Then $\Sigma\times[0,1]$ $(\subset F\times[0,1])$ projects to a union of circles $L_{f,\Sigma}$ in M_f. In [4,5] the author showed that if f is an orientation preserving homeomorphism, then the link type $L_{f,\Sigma}$ is closely related to the dynamical complexity of f. In this paper, we prove a similar result for C^1-embeddings.

Theorem 1. Let $g:F\to F$ be an orientation preserving C^1-embedding of a compact surface such that $g(F)\subset \text{Int } F$, with a g-invariant set Σ of finite points in Int F. If the topological entropy $h(g)$ of g is zero, then $L_{g,\Sigma}$ is a graph link. Moreover, if $L_{g,\Sigma}$ is not a graph link, then g has infinitely many periodic orbits whose periods are mutually distinct.

We note that, since g is of class C^1, M_g is a C^1-manifold.

As an application of this theorem, in section 4, we will give a geometric version of the Matsuoka's theorem [7], which give a criterion for determining whether a given differential equation of certain type possesses infinitely many periodic solutions (Theorem 2).

I would like to express my gratitude to Dr. Takashi Matsuoka for helpful conversations.

2. PRELIMINARIES.

In this section, we review the results in [5]. Throughout this paper, we suppose that surfaces are connected. A general reference of topological entropy is [3, Expose 10]. A link L is a finite union of mutually disjoint circles in a 3-manifold. The exterior of L is the closure of the complement of a tubular neighborhood of L. A 3-manifold M is a graph manifold if there is a system of mutually disjoint 2-tori in Int M such that each component of M cut along these tori is (a surface) $\times S^1$. A link is a graph link if the exterior is a graph manifold.

Let $f:F \to F$ be an orientation preserving homeomorphism of a compact surface F with an f-invariant set Σ of finite points in Int F. Let M_f, $L_{f,\Sigma}$ be as in section 1. Then in [5] we proved:

Proposition 2.1. If $h(f) = 0$, then $L_{f,\Sigma}$ $(\subset M_f)$ is a graph link. Conversely, if $L_{f,\Sigma}$ is a graph link, then f is isotopic rel Σ to a homeomorphism g such that $h(g) = 0$. Moreover, if $L_{f,\Sigma}$ is not a graph link and f is differentiable at each point of Σ, then f has infinitely many periodic orbits whose periods are mutually distinct.

3. PROOF OF THEOREM 1.

In this section, we prove Theorem 1 stated in section 1. Throughout this section, let g, F, Σ, M_g, $L_{g,\Sigma}$ be as in Theorem 1.

Let $f:X \to X$ be a continuous map. A point x $(\in X)$ is a wandering point of f if it has a neighborhood U in X such that $U \cap f^n(U) = \emptyset$ for each integer n (>0); otherwise x is called nonwandering. $\Omega(f)$ denotes the set of all nonwandering points of f. It is easily seen that $\Omega(f)$ is an f-invariant closed set.

Lemma 3.1. If each component of $F - \mathrm{Int}\, g(F)$ is an annulus, then the conclusions of Theorem 1 hold.

Proof. In this case, by considering Euler characteristic, we see that each component of $\partial g(F)$ is parallel to a component of ∂F. Let $D_1, \ldots, D_n$ be 2-disks, and $\bar{F}$ be a surface obtained from F and $D_1 \cup \ldots \cup D_n$ by identifying their boundaries. Then there is a C^1-diffeomorphism $\bar{g}: \bar{F} \to \bar{F}$ such that $\bar{g}|_F = g$, each D_i contains exactly one periodic point a_i, and for each $x \in D_i - a_i$ there is a K (>0) such that if $k > K$, then $\bar{g}^{-k}(x) \notin D_i$. Let $\bar{\Sigma} = \Sigma \cup a_1 \cup \ldots \cup a_n$. $\bar{\Sigma}$ is invariant by $\bar{g}$. We note that $\Omega(\bar{g}) = \Omega(g) \cup a_1 \cup \ldots \cup a_n$. Hence, by [2, Theorem 2.4] $h(g) = h(\bar{g})$.

If $h(g) = 0$, then $h(\bar{g}) = 0$. Hence, by Proposition 2.1 $L_{\bar{g},\bar{\Sigma}}$ is a graph link. We easily see that the exterior of $L_{\bar{g},\bar{\Sigma}}$ is homeomorphic to that of $L_{g,\Sigma}$. Hence, $L_{g,\Sigma}$ is a graph link.

Suppose that $L_{g,\Sigma}$ is not a graph link. Then $L_{\bar{g},\bar{\Sigma}}$ is not a graph link. Hence, by Proposition 2.1 $\bar{g}$ has infinitely many periodic orbits whose periods are mutually distinct. We see that $\mathrm{Per}(\bar{g}) = \mathrm{Per}(g) \cup a_1 \cup \ldots \cup a_n$, where $\mathrm{Per}(f)$ denotes the set of all periodic points of f. Hence, g satisfies the last conclusion of Theorem 1.

Proof of Theorem 1. The proof is by the induction on the number of the components of ∂F. Suppose that ∂F consists of one component. By considering Euler characteristic, we see that $F - \mathrm{Int}\, g(F)$ is an annulus. Hence, by Lemma 3.1 the conclusion holds.

Suppose that ∂F consists of n (>1) components. By Lemma 3.1 we may suppose that some component of $F - \mathrm{Int}\, g(F)$ is not an annulus. Then Euler characteristic argument shows that some component of $F - \mathrm{Int}\, g(F)$ is a 2-disk D_1. Let S_1 be a component of ∂F such that $g(S_1) = \partial D_1$. Let $\bar{F}$ be a surface obtained from F and a 2-disk D_1' by identifying S_1 and $\partial D_1'$. Then there is an embedding $\bar{g}: \bar{F} \to \bar{F}$

such that $\bar{g}|_F = g$, $\bar{g}(D_1') = D_1$. Since $M_{\bar{g}}$ is obtained from M_g by attaching a 3-disk $D_1' \times [0,1]$ along a 2-disk $(S_1 \times [0,1]) \cup D_1$, $(M_{\bar{g}}, L_{\bar{g},\Sigma})$ is homeomorphic to $(M_g, L_{g,\Sigma})$ as a pair. Then each point of D_1' is wandwering. By [2], $h(\bar{g}) = h(g)$. Hence, by the assumption of induction we see that the conclusion of Theorem 1 holds for g.

This completes the proof of Theorem 1.

4. HOMEOMORPHISMS OF A DISK.

In this section, we review the results in [4]. Let f be an orientation preserving homeomorphism of a 2-disk D^2. Suppose that there is an f-invariant set $\Sigma = \{p_1,\ldots,p_n\}$ of finite points in Int D^2. In [4] we defined a link $m \cup L$ from (f,Σ), which is called a link of (f,Σ). Let us review the construction of $m \cup L$. Let α be a smooth arc properly embedded in D^2 such that $\Sigma \subset \alpha$ and $p_1,\ldots,p_n$ are on α in this order. Let Diff$(D^2,\Sigma,\text{rel } \partial)$ be the set of diffeomorphisms $\psi: D^2 \to D^2$ such that $\psi(\Sigma) = \Sigma$, $\psi|_{\partial D^2} = id_{\partial D^2}$ with smooth topology. Let D_i $(i=1,\ldots,n-1)$ be a disk in Int D^2 such that $D_i \cap \alpha$ is a proper arc in D_i which contains p_i and p_{i+1}, D_i' a small regular neighborhood of D_i. Let s_i be an element of Diff$(D^2,\Sigma,\text{rel } \partial)$ such that s_i rotates D_i in counterclockwise direction in π radian, $s_i(p_i) = p_{i+1}$, $s_i(p_{i+1}) = p_i$, and $s_i|_{D^2-\text{Int } D_i'} = id_{D^2-\text{Int } D_i'}$. Let B_n be a braid group ([1]);

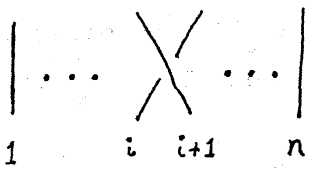
$$< \sigma_1,\ldots,\sigma_{n-1} \mid \sigma_i\sigma_j = \sigma_j\sigma_i \text{ if } |i-j| > 1,\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \text{ if } 0 < i < n-1 >.$$

Then in [3,Expose 2] the following is shown.

Lemma 4.1. B_n *is isomorphic to* $\pi_0(\text{Diff}(D^2,\Sigma,\text{rel } \partial))$, *where* σ_i *corresponds to* s_i.

There is an isomorphic correspondence between an element σ of B_n and a geometric braid ([1]) b_σ such that:

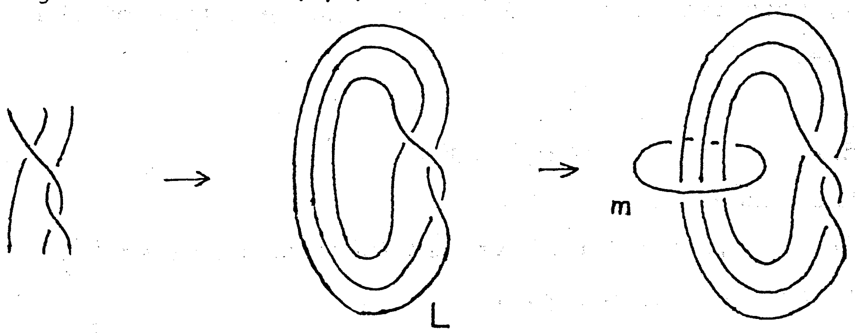
(i) σ_i corresponds to the geometric braid b_{σ_i}



(ii) the product $\sigma\sigma'$ corresponds to connecting the bottom endpoints of b_σ to the top endpoints of $b_{\sigma'}$.

For example, $\sigma_1\sigma_2{}^2 \in B_3$ corresponds to the following geometric braid.

Then isotope f rel Σ to t ($\in \mathrm{Diff}(D^2,\Sigma,\mathrm{rel}\ \partial)$). By Lemma 4.1 t defines an element σ of B_n. We obtain a link L in the 3-sphere S^3 by connecting the top and bottom of the geometric braid b_σ. Then the link obtained from L by adding a special component m as in the following is a link of (f,Σ)



We note that the exterior of $m \cup L$ is homeomorphic to the exterior of $L_{f,\Sigma}$ ($\subset M_f$). Hence, $L_{f,\Sigma}$ <u>is a graph link if and only if</u> $m \cup L$ <u>is a graph link</u>.

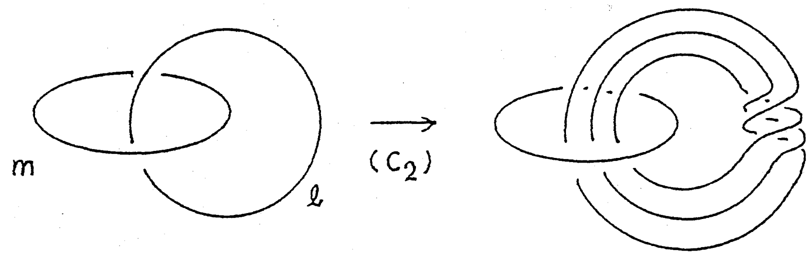
In [4,Theorem 2], we gave a characterization of graph links which are kind of links of homeomorphisms of a disk. For the statement of this result, we define two operations (C_1), (C_2) which give a new link L' from a link L.

(C_1) (adding a cable link of a component) let ℓ be a component of L, N a tubular neighborhood of ℓ, and $\bar{L} \subset \partial N$ be a link each component of which is not contractible in N then $L' = L \cup \bar{L}$.

(C_2) (replecing a component to its cable link) let ℓ, $\bar{L}$ be as above then $L' = (L - \ell) \cup \bar{L}$.

Then we have:

Proposition 4.2. Let $m \cup L$ be a graph link, which is a link of a homeomorphism of a disk. Then L is obtained from a Hopf link $m \cup \ell$ by performing a finite sequence of (C_1) or (C_2) operations on components each of which is not m.



5. APPLICATIONS.

In [7], Matsuoka considered differential equation of the following type.

(*) $dx/dt = f(t,x) \quad t \in \mathbb{R},\ x \in \mathbb{R}^2$

with the assumptions

(**) (i) $f(t,x)$ is an $\mathbb{R}^2$-valued function of class C^1,

(ii) $f(t+1,x) = f(t,x)$,

(iii) for any initial condition $(t_0,x_0) \in \mathbb{R}\times\mathbb{R}^2$, there exists a solution $x = \Phi(t;t_0,x_0)$ of the equation defined on $-\infty < t < \infty$,

(iv) there is a 2-disk K embedded in $\mathbb{R}^2$ such that $T(K) \subset K$, where $T: \mathbb{R}^2 \to \mathbb{R}^2$ denotes the Poincaré transformation; $x \to T(x) = \Phi(1;0,x)$.

Let $x_1(t),\dots,x_n(t)$ be a system of periodic solutions of (*) such that $\{x_1(0),\dots,x_n(0)\} = \{x_1(1),\dots,x_n(1)\} \subset K$. Then Matsuoka gave a method to estimate the number of p-periodic solitions of (*) for each $p \geq 1$, by seeing the "linking" of $x_1(t),\dots,x_n(t)$. He used the Burau representation of a braid group to describe the "linking". In this paper, we will see the "linking" directly by using the link theory.

Let $\Sigma = \{x_1(0),\ldots,x_n(0)\}$, $t:K\to K$ be an element of $\mathrm{Diff}(K,\Sigma,\mathrm{rel}\ \partial)$ which is isotopic to $T|_K$ rel Σ. Let $m\cup L$ be a link of (t,Σ) defined in section 4. Then it can be shown that the exterior of $m\cup L$ is homeomorphic to the exterior of $L_{T|_K,\Sigma}$ $(\subset M_{T|_K})$. Hence, by Theorem 1 we have:

Theorem 2. _If $m\cup L$ is not a graph link, then (*) has infinitely many periodic solutions whose periods are mutually distinct._

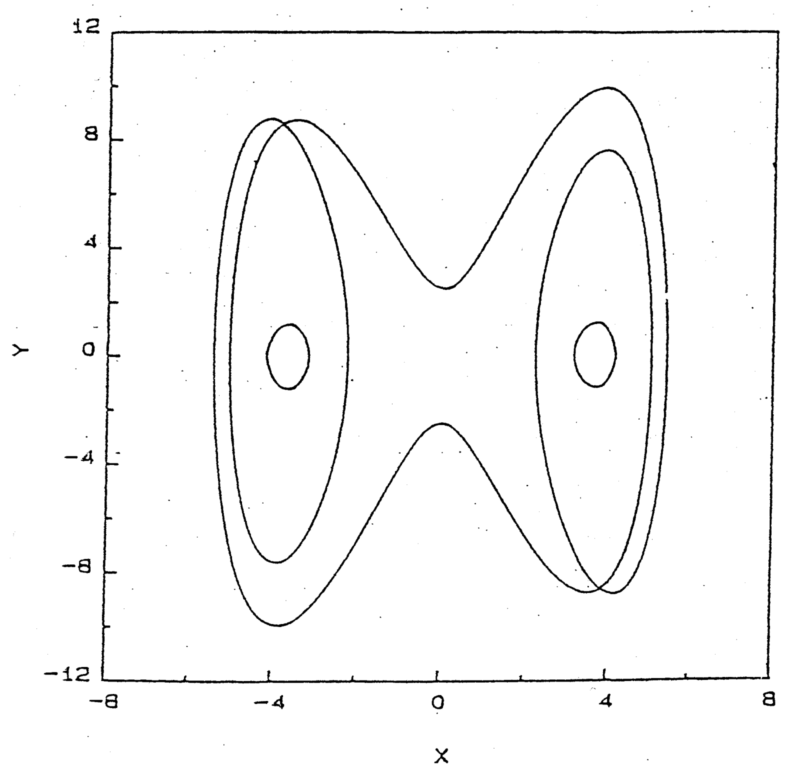
I would like to thank Professor Hiroshi Kawakami for introducing me the following example.

Example(Kawakami) Let us consider the differential equation:

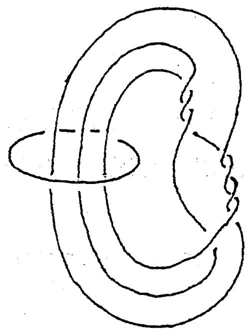
$$dy/dt = -0.1\,y - (-14 + 3.3\cos(2t))\,x - x^3$$

$$dx/dt = y.$$

It has three periodic solutions such that the trace of them in x - y plane is as in the following.



Then these periodic solutions defines the following link. We can show that this is not a graph link by using Proposition 4.2. But we must note that it is not known whether this equation satisfies the assumption (**) (iv). If it is shown, then by Theorem 2, we see that this equation possesses infinitely many periodic solutions.



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