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逆ガウス型分布における一様最小分散不偏推定

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２母数逆ガウス型分布における任意次数の半不変係数を含む母数の関数の或るクラスに対する一様最小分散不偏推定量を与える．　任意の標本の大きさに対する母分散の推定量の分散式を具体的に求め，その上界および下界を導く．
（研究会での報告は岩瀬が行なった.）

UNIFORMLY MINIMUM VARIANCE UNBIASED ESTIMATION FOR THE INVERSE GAUSSIAN DISTRIBUTION

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Uniformly minimum variance unbiased estimators of the cumulants of arbitrary order and others are derived for the inverse Gaussian distribution. The variance of the second order cumulant is given explicitly.
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1. Introduction

In wide variety of fields, the (two-parameter) inverse Gaussian distribution is considered as the population model. The distribution is known by name as the first passage time distribution of Brownian process. Because of the inverse relationship between the cumulant generating function of the first passage time distribution and that of the Gaussian distribution, Tweedie (1957a) proposed the name inverse Gaussian for the distribution. When the population mean is equal to unity, the distribution is often referred to as the standard Wald distribution. The inverse Gaussian distribution shares with the gamma and lognormal, and other skewed distributions, asymptotic convergence to normality.

Tweedie (1957a, 1957b) discussed the statistical properties of the distribution. He gave, among others, the maximum likelihood estimators of some parameters for a general case and derived a remarkably simple relation between the positive and negative moments of the distribution. He also studied the problem of estimating the reciprocal of inverse Gaussian means in a more general form and discussed these estimates in greater detail. This type of parameter estimation problem commonly arises whenever actual observations are made in inverted scale. A traffic engineer, for example, deals with it while monitoring the speed of a car where time is recorded for every distance interval.

Chhikara and Folks (1975) have proposed the inverse Gaussian distribution as a life time model. They investigated the properties of the inverse Gaussian failure rate function and considered maximum likelihood estimation for both the reliability function and failure rate function. They also showed that the inverse Gaussian distribution was a viable alternative to the lognormal distribution as a lifetime model and pointed out several advantages of the inverse Gaussian model over the lognormal one. Padgett and Wei (1979) introduced a threshold parameter to the inverse Gaussian distribution and investigated the esimation of parameters based on the method of moments and the maximum likelihood estimation for this three-parameter distribution.

Recently, Korwar (1980) derived the uniformly minimum variance unbiased estimator (UMVU estimator) for the variance and the reciprocal of the variance of the (two-parameter) inverse Gaussian distribution. His expression, however, is extremely complicated. A compact expression for the UMVU estimator was given in terms of the hypergeometric function by Iwase (1981). The main purpose of the present paper is to construct the UMVU estimator for the cumulant of arbitrary order. The variance of the estimator of the second cumulant is also given in a closed form, together with some numerical examples.

In the next section, a main theorem is proved and, as a consequence of the theorem, the UMVU estimator of the r-th cumulant is given. Other estimators, useful in practice, are listed in Table I. Technical detailes of the proof of the main theorem are described in Appendix. In Sec. 3, the variance of the UMVU estimator for the second cumulant is given in an arbitrary sample size. Some numerical computations for the variances are performed. The asymptotic variance of the UMVU estimator is derived and a connection of this estimator with the consistent and asymptotic unbiased estimator obtained by Iwase (1981) is displayed in Table II. The final section is devoted to some discussions.

2. UMVU Estimators

A random variable X is said to be distributed as Inverse Gaussian with mean parameter μ and a shape parameter λ, denoted as $X \sim I(\mu,\lambda)$, if its probability density function is given by

$$f(x) = \begin{cases} \sqrt{\dfrac{\lambda}{2\pi x^3}} \exp[-\dfrac{\lambda(x-\mu)^2}{2\mu^2 x}], & x > 0 \\ 0, & \text{otherwise} \end{cases} \tag{2.1}$$

where μ and λ are assumed to be positive. Let $X_1, X_2, \cdots, X_n$ be a random sample from (2.1), $\overline{X}$ $(=\Sigma_{i=1}^{n} X_i/n)$ be the sample mean and $V = \Sigma_{i=1}^{n}(X_i^{-1} - \overline{X}^{-1})$. We have the following main

Theorem 1. *For any real numbers* α, β *and* τ *and any integer* n *such that* $(n-1)/2 + \tau > 0$ *and* $n \geq 2$, *it holds that*

$$E[\overline{X}^{\alpha+\beta+1/2} V^{\tau} \cdot F(\alpha, \beta; \frac{n-1}{2}+\tau; -\frac{\overline{X}V}{n})]$$

$$= \frac{\Gamma((n-1)/2+\tau)}{\Gamma((n-1)/2)} \mu^{\alpha+\beta+1/2} \left(\frac{2}{\lambda}\right)^{\tau} \cdot \sqrt{\frac{2n\lambda}{\pi\mu}} e^{n\lambda/\mu} \cdot K_{\beta-\alpha}(\frac{n\lambda}{\mu}), \tag{2.2}$$

where F *is a hypergeometric function and* K *is a modified Bessel function.*

Tweedie (1957a) proved that $\overline{X} \sim I(\mu, n\lambda)$ and that $\lambda V \sim \chi^2_{n-1}$ and that they are independent. A jointly sufficient statistics for (μ,λ) is $(\overline{X},V)$ and the family of density functions (2.1) is complete. Because of these, the statistics $\overline{X}$ and V are the basic ones used in making inferences about the inverse Gaussian distribution. Using the Lehmann-Scheffé theorem and the equality $\sqrt{2n\lambda/\pi\mu}\exp[n\lambda/\mu] \cdot K_{1/2}(n\lambda/\mu) = 1$, we have

Corollary 1. *The UMVU estimator of the r-th cumulant*

$$\kappa_r = \frac{2^{r-1}}{\sqrt{\pi}} \Gamma(r-\tfrac{1}{2}) \cdot \mu^{2r-1} \lambda^{1-r}, \quad r \geq 1$$

is given by

$$\hat{\kappa}_r = \frac{\Gamma(r-1/2)}{\sqrt{\pi}} \frac{\Gamma((n-1)/2)}{\Gamma((n-1)/2+r-1)} \overline{X}^{2r-1} V^{r-1} \cdot F(r-1, r-\tfrac{1}{2}; \frac{n-1}{2}+r-1; -\frac{\overline{X}V}{n}), \quad n \geq 2. \tag{2.3}$$

For $r=2$, this reduces to the result due to Korwar (1980)† and Iwase (1981). Table I gives some other UMVU estimators of parametric functions of μ and λ.

Table I

Proof of Theorem 1. Since $\overline{X}$ and V are independent, the left hand side of (2.2) is expressed as

$$\frac{\Gamma((n-1)/2+\tau)}{\lambda^{\tau}\Gamma((n-1)/2+\tau-\beta)\Gamma(\beta)}\int_0^1 t^{\beta-1}(1-t)^{(n-1)/2+\tau-\beta-1} \times E_{\overline{X}}[\overline{X}^{\alpha+\beta+1/2}\cdot E_V[\frac{(\lambda V)^{\tau}}{(1+t\overline{X}V/n)^{\alpha}}]]dt,$$

where $E_{\overline{X}}$ and E_V mean the expectations with respect to $\overline{X}$ and V, respectively. It is clear that we can safely interchange the order of integration, since we are dealing only with the absolutely convergent integrals. It is to be noted that the above formula is valid only for $(n-1)/2+\tau>\beta>0$, for we have used the integral representation for the hypergeometric function. From the fact that $\overline{X}\sim I(\mu,n\lambda)$ and $\lambda V\sim\chi^2_{n-1}$, the above expression is transformed, with the aid of (A.3) and (A.2), into

$$\frac{\Gamma((n-1)/2+\tau)}{\Gamma((n-1)/2)}\left(\frac{2}{\lambda}\right)^{\tau}\frac{(n\lambda)^{\alpha}}{\Gamma(\alpha)}\int_0^{\infty}s^{\alpha-1}(1+2s)^{-\beta}\cdot E_{\overline{X}}[\overline{X}^{\beta+1/2}e^{-n\lambda s/\overline{X}}]ds$$

$$=\frac{\Gamma((n-1)/2+\tau)}{\Gamma((n-1)/2)}\cdot\mu^{\alpha+\beta+1/2}\left(\frac{2}{\lambda}\right)^{\tau}\cdot\left(\frac{n\lambda}{2\mu}\right)^{\alpha}\frac{1}{\Gamma(\alpha)}\sqrt{\frac{2n\lambda}{\pi\mu}}e^{n\lambda/\mu} \times\int_1^{\infty}(x-1)^{\alpha-1}x^{-\beta/2}\cdot K_{\beta}(\frac{n\lambda}{\mu}\sqrt{x})dx.$$

† The expression for Y in the formula (2.1) given by Korwar (1980) should read $\tan^{-1}(V\overline{X})^{1/2}$ instead of $2\tan^{-1}(V\overline{X})^{1/2}$ for the case n is even.

The last integral can be evaluated with the help of the formula (6.592) in Gradshteyn and Ryzhik (1971) (abbreviated as G-R hereafter) to give the required result for the case $(n-1)/2+\tau>\beta>0$ and $\alpha>0$. To extend the domain of validity of (2.2) in parameter space of α, β and τ to that specified in Theorem 1, we may use the recurrence relation for the hypergeometric function or make the analytic continuation of both sides of (2.2) with respect to the (complex) parameters α, β and τ. This completes the proof.

Theorem 1 has much wide contents compared with those of Corollary 1, since the latter corresponds to the case $\beta-\alpha=1/2$ (and $\alpha=\tau=r-1$) in the former. All the estimators listed in Table I, on the other hand, are obtained by putting $\beta-\alpha=1/2$ and choosing α and τ suitably. For the case $\beta-\alpha$ is equal to a half integer, the parametric function in the right hand side of (2.2) reduces to the finite linear combination of the functions for $\beta-\alpha=1/2$. Theorem 1 also gives the estimators corresponding to the other case ($\beta-\alpha\neq$ half integer).

3. Variance of $\hat{\kappa}_2$

In this section, the variance of UMVU estimator of the second cumulant $\kappa_2 = \mu^3/\lambda$ is considered. The estimator in question corresponds to the case $r=2$ in (2.3), so that its explicit form is

$$\hat{\kappa}_2 = \frac{1}{n-1}\overline{X}^3 V \cdot F(1,\tfrac{3}{2};\tfrac{n+1}{2};-\tfrac{\overline{X}V}{n}), \quad n \geqq 2. \tag{3.1}$$

Theorem 2. *The variance of the UMVU estimator of the variance of* (3.1) *is given by*

$$\mathrm{Var}[\hat{\kappa}_2] = n^2\mu^4 \left\{\frac{2n}{n-1}\cdot\frac{\mu}{n\lambda}e^{n\lambda/\mu}\int_1^\infty e^{-n\lambda t/\mu}F(3,\tfrac{n-2}{2};n;1-t^2)dt \right.$$

$$\left. -2(\mu/n\lambda)^2 + 15(\mu/n\lambda)^3 \right\}, \quad n \geqq 2. \tag{3.2}$$

Proof. From (9.135) in G-R, (3.1) is rewritten as

$$\hat{\kappa}_2 = n\overline{X}^2\left\{1 - F(\tfrac{1}{2},1;\tfrac{n-1}{2};-\tfrac{\overline{X}V}{n})\right\}.$$

To obtain the variance of $\hat{\kappa}_2$, we must calculate the following expectations;

$$E[\hat{\kappa}_2^2] = E[n^2\overline{X}^4] - 2E[n^2\overline{X}^4F(\tfrac{1}{2},1;\tfrac{n-1}{2};-\tfrac{\overline{X}V}{n})] + E[n^2\overline{X}^4F^2(\tfrac{1}{2},1;\tfrac{n-1}{2};-\tfrac{\overline{X}V}{n})]. \tag{3.3}$$

Setting $\alpha=0$, $\beta=7/2$ and $\tau=0$ in (2.2), we find that the first term in the right hand side of (3.3) is equal to

$$n^2\mu^4(1+6(\mu/n\lambda)+15(\mu/n\lambda)^2+15(\mu/n\lambda)^3). \tag{3.4}$$

The second term is reduced, with the help of (A.6) and the formula (6.592) in G-R, to

$$-2n^2\mu^4(1+5(\mu/n\lambda)+8(\mu/n\lambda)^2). \tag{3.5}$$

Through the Euler's transformation for the hypergeometric function and the expression for the square of ${}_2F_1$ in terms of ${}_3F_2$ (p.64 and p.185 in Bateman manuscript project, Vol.1 (1953) respectively), the third term is expressed as

$$E[n^2\overline{X}^4(1+\frac{\overline{X}V}{n})^{-1}\cdot {}_3F_2(n-3,1,\frac{n-2}{2};n-2,\frac{n-1}{2};\frac{\overline{X}V}{n}/(1+\frac{\overline{X}V}{n}))] .$$

This reduces from the formula (7.512) in G-R to

$$\frac{n^2\Gamma(n-2)\Gamma((n-1)/2)}{\Gamma(3/2)\Gamma(n-3)\Gamma((n-2)/2)}\int_0^1 t^{1/2}(1-t)^{n-4}E[\overline{X}^4(1+\frac{\overline{X}V}{n}t)^{-1}]\cdot F(\frac{1}{2},\frac{n-2}{2};\frac{3}{2};t)dt .$$

From (A.3) and (A.2), the above formula is calculated to be

$$\lambda n^3\int_0^\infty E_{\overline{X}}[\overline{X}^3e^{-\lambda ns/\overline{X}}]\cdot(1+2s)^{-1/2}F(1,\frac{n-2}{2};n-2;-2s)ds$$

$$=\frac{\lambda n^3\mu^3}{2}e^{n\lambda/\mu}\sqrt{\frac{2n\lambda}{\pi\mu}}\int_1^\infty x^{-5/4}K_{5/2}(\frac{n\lambda}{\mu}\sqrt{x})x^2F(1,\frac{n-2}{2};n-2;1-x)dx .$$

Since

$$x^{-5/4}K_{5/2}(a\sqrt{x})=-\frac{2}{a}\cdot\frac{d}{dx}(x^{-3/4}K_{3/2}(a\sqrt{x}))=e^{-a}\sqrt{\frac{\pi}{2a}}\left(\frac{2}{a}\right)^2\frac{d^2}{dx^2}(e^{-a\sqrt{x}}/\sqrt{x})$$

and

$$\frac{d^2}{dx^2}(x^2F(1,\frac{n-2}{2};n-2;1-x))=\frac{n}{2(n-1)}F(3,\frac{n-2}{2};n;1-x) ,$$

the expression is transformed through integration by parts into

$$n^2\mu^4\left\{1+\frac{4\mu}{n\lambda}+\frac{2n}{n-1}\cdot\frac{\mu}{n\lambda}e^{n\lambda/\mu}\int_1^\infty e^{-n\lambda t/\mu}F(3,\frac{n-2}{2};n;1-t^2)dt\right\} . \qquad (3.6)$$

Summing the results (3.4), (3.5) and (3.6), we arrive at (3.2).

From the representation (3.2), we can immediately obtain the lower and upper bounds for $\mathrm{Var}[\hat{\kappa}_2]$:

Theorem 3. *For any sample size* $n \geqq 2$, *it holds that*

$$\sigma_m^2 \leqq \mathrm{Var}[\hat{\kappa}_2] \leqq \sigma_M^2 , \tag{3.7}$$

where σ_m^2 *and* σ_M^2 *are defined by*

$$\sigma_m^2 = \mu^4\Big\{\frac{2}{n-1}\Big(\frac{\mu}{\lambda}\Big)^2 + \frac{1}{n}\Big(\frac{\mu}{\lambda}\Big)^3 \cdot [15 - \frac{6(n-2)}{n-1}\Big(1+\frac{\mu}{n\lambda}\Big)]$$

$$+ \frac{4!}{n^2}\Big(\frac{\mu}{\lambda}\Big)^4 \cdot \frac{(n-2)n}{(n-1)(n+1)}\Big(1+\frac{3\mu}{n\lambda}+\frac{3\mu^2}{n^2\lambda^2}\Big)$$

$$- \frac{5!}{n^3}\Big(\frac{\mu}{\lambda}\Big)^5 \cdot \frac{(n-2)n}{(n-1)(n+1)}\Big(1+\frac{6\mu}{n\lambda}+\frac{15\mu^2}{n^2\lambda^2}+\frac{15\mu^3}{n^3\lambda^3}\Big)\Big\} ,$$

$$\sigma_M^2 = \sigma_m^2 + \mu^4\frac{6!}{n}\Big(\frac{\mu}{\lambda}\Big)^6 \cdot \frac{(n-2)n(n+4)}{(n-1)(n+1)(n+3)}\Big(1+\frac{10\mu}{n\lambda}+\frac{45\mu^2}{n^2\lambda^2}+\frac{105\mu^3}{n^3\lambda^3}+\frac{105\mu^4}{n^4\lambda^4}\Big) .$$

Proof. From the formula (3.2) in Theorem 2, (8.432) in G-R and the inequality due to Alam (1980)

$$\sum_{r=0}^{2n+1} \frac{(a)_r (b)_r}{(c)_r} \cdot \frac{(-x)^r}{r!} \leqq F(a,b;c;-x) \leqq \sum_{r=0}^{2n} \frac{(a)_r (b)_r}{(c)_r} \cdot \frac{(-x)^r}{r!}$$

with $(a)_r = \Gamma(a+r)/\Gamma(a)$ and $c \geqq min(a,b) > 0$, $x \geqq 0$, (3.7) can be easily obtained and the proof may be omitted.

Corollary 2. *The asymptotic variance of* $\hat{\kappa}_2$ *is given by*

$$\mathrm{Var}[\hat{\kappa}_2] = \frac{\mu^4}{n} \cdot \frac{1}{\phi^2}\Big(2+\frac{9}{\phi}\Big) + 0(n^{-2}), \quad n \to \infty. \ (\phi = \lambda/\mu) \tag{3.8}$$

Numerical computations of (3.2), (3.7) and (3.8), assuming ϕ to have 0.5, 1.0 and 2.0, are shown in Figs. 1, 2 and 3 respectively.

Fig. 1 Fig. 2 Fig. 3

Using a property of a hypergeometric function, we may obtain a consistent and asymptotic unbiased estimator of κ_2 such that

$$\hat{\kappa}_2^* = \frac{1}{n-1}\overline{X}^3 V = \frac{n}{n-1}\overline{X}^2 (\overline{X}/\overline{X}_H - 1)$$

where $\overline{X}_H$ is the harmonic sample mean. A property of the above estimator was considered by Iwase (1981). From Corollary 2, it may be easily shown

Corollary 3.

$$\lim_{n\to\infty} n\mathrm{Var}[\hat{\kappa}_2] = \lim_{n\to\infty} n\mathrm{MSE}[\hat{\kappa}_2^*]$$

where MSE[·] *denotes a mean square error.*

Numerical computations of MSE[$\hat{\kappa}_2^*$] is also given in Figs. 1, 2 and 3. In order to facilitate the use of the unbiased estimator $\hat{\kappa}_2$, Table II gives

$$\hat{\kappa}_2/\hat{\kappa}_2^* = F(1,\frac{3}{2};\frac{n+1}{2};-\frac{\overline{X}V}{n})$$

for $\overline{X}V/n = 0.25(0.25)2.00$ and $n = 2(1)10(10)100$.

Table II

4. Discussion

It may be convenient to use the reciprocal of a variate X having an inverse Gaussian distribution. The distribution of X^{-1} is called the random walk distribution. The UMVU estimators of the r-th cumulant of X and X^{-1} are given in Sec. 2. The UMVU estimators of k-th power of the r-th cumulant of X and X^{-1} can be derived in the same manner. For the mode of X and X^{-1}, however, the UMVU estimators have not been obtained in a simple form.

Corollary 3 implies that the asymptotic efficiency of $\hat{\kappa}_2^*$, relative to $\hat{\kappa}_2$, as an estimator for μ^3/λ is equal to 1. As is seen in Figs. 1, 2 and 3, however, when the sample size is not so large, it is dangerous to use $\hat{\kappa}_2^*$ instead of $\hat{\kappa}_2$.

The variance of the UMVU estimator for the second order cumulant was obtained in Sec. 3. By the similar method to that employed there, the variance of $\hat{\kappa}_r$ may be expressed in a closed form. The resulting formula, however, will not have a simple form compared with that for $\hat{\kappa}_2$. This deserves further investigations.

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Appendix

In this Appendix, we shall derive the formulas used in the proof of Theorems 1 and 2. Our starting points are

$$E[(\lambda V)^{\tau} e^{-\lambda s V}] = \frac{\Gamma((n-1)/2+\tau)}{\Gamma((n-1)/2)} 2^{\tau}\cdot(1+2s)^{-(n-1)/2-\tau}, \qquad \text{(A.1)}$$

for $(n-1)/2+\tau>0$ and $s>-1/2$

and

$$E[\bar{X}^{\kappa} e^{-n\lambda s/\bar{X}}] = \mu^{\kappa}(1+2s)^{\kappa/2-1/4}\sqrt{\frac{2n\lambda}{\pi\mu}}e^{n\lambda/\mu}\cdot K_{\kappa-1/2}\left(\frac{n\lambda}{\mu}\sqrt{1+2s}\right), \qquad \text{(A.2)}$$

for $s>-1/2$,

which follow immediately from $\lambda V \sim \chi^2_{n-1}$ and $\bar{X} \sim I(\mu, n\lambda)$. By noting that $\bar{X}$ and V are independent and by using the integral representation

$$\frac{1}{(1+t\bar{X}V/n)^{\alpha}} = (n\lambda/\bar{X})^{\alpha}\frac{1}{\Gamma(\alpha)}\int_0^{\infty} s^{\alpha-1} e^{-n\lambda s/\bar{X}}\cdot e^{-\lambda t s V} ds, \text{ for } \alpha>0 \text{ and } t>0,$$

it readily follows from (A.1) that

$$E_V\left[\frac{(\lambda V)^{\tau}}{(1+t\bar{X}V/n)^{\alpha}}\right] = \frac{\Gamma((n-1)/2+\tau)}{\Gamma((n-1)/2)}\cdot 2^{\tau}\left(\frac{n\lambda}{\bar{X}}\right)^{\alpha}\frac{1}{\Gamma(\alpha)}$$

$$\times\int_0^{\infty} s^{\alpha-1} e^{-n\lambda s/\bar{X}}\cdot(1+2ts)^{-(n-1)/2-\tau} ds. \qquad \text{(A.3)}$$

For $(n-1)/2+\tau>\beta>0$ and $\alpha>0$, the integral representation

$$F\left(\alpha,\beta;\frac{n-1}{2}+\tau;-\frac{\bar{X}V}{n}\right) = \frac{\Gamma((n-1)/2+\tau)}{\Gamma((n-1)/2+\tau-\beta)\Gamma(\beta)}$$

$$\times\int_0^1 t^{\beta-1}(1-t)^{(n-1)/2+\tau-\beta-1}\frac{1}{(1+t\bar{X}V/n)^{\alpha}}dt$$

together with (A.3) implies that

$$E_V[(\lambda V)^{\tau} F(\alpha,\beta\,;\frac{n-1}{2}+\tau\,;-\frac{\overline{X}V}{n})] = \frac{\Gamma((n-1)/2+\tau)}{\Gamma((n-1)/2)}\cdot 2^{\tau}\left(\frac{n\lambda}{\overline{X}}\right)^{\alpha}\frac{1}{\Gamma(\alpha)}$$

$$\times\int_0^{\infty} s^{\alpha-1} e^{-n\lambda s/\overline{X}}\cdot(1+2s)^{-\beta} ds. \qquad \text{(A.4)}$$

A moment reflection reveals that (A.4) is valid for $(n-1)/2+\tau > 0$ and $\alpha > 0$. Combining (A.2) and (A.4) and making the change of the integration variable s to $x = 2s+1$, we arrive at

$$E[\overline{X}^{\kappa} V^{\tau} F(\alpha,\beta\,;\frac{n-1}{2}+\tau\,;-\frac{\overline{X}V}{n})] = \frac{\Gamma((n-1)/2+\tau)}{\Gamma((n-1)/2)}\cdot\mu^{\kappa}\left(\frac{2}{\lambda}\right)^{\tau}\cdot\left(\frac{n\lambda}{2\mu}\right)^{\alpha}\frac{1}{\Gamma(\alpha)}\cdot\sqrt{\frac{2n\lambda}{\pi\mu}}e^{n\lambda/\mu}$$

$$\times\int_1^{\infty}(x-1)^{\alpha-1}x^{(\kappa-\alpha)/2-1/4-\beta}\cdot K_{\kappa-\alpha-1/2}\left(\frac{n\lambda}{\mu}\sqrt{x}\right)dx, \qquad \text{(A.5)}$$

for $(n-1)/2+\tau > 0$ and $\alpha > 0$.

For $\kappa = 4$, $\tau = 0$, $\alpha = 1/2$ and $\beta = 1$, (A.5) reduces to

$$E[\overline{X}^4 F(\tfrac{1}{2},1\,;\frac{n-1}{2}\,;-\frac{\overline{X}V}{n})] = \mu^4\cdot\sqrt{\frac{n\lambda}{2\mu}}\,\frac{1}{\sqrt{\pi}}\sqrt{\frac{2n\lambda}{\pi\mu}}e^{n\lambda/\mu}$$

$$\times\int_1^{\infty}(x-1)^{-1/2}x^{1/2}\cdot K_3\left(\frac{n\lambda}{\mu}\sqrt{x}\right)dx. \qquad \text{(A.6)}$$

The above integration is easily performed with the aid of the relation $x^{1/2} = x^{-3/2}((x-1)^2+2(x-1)+1)$ and the formula (6.592) in G-R.

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Table I

parameters in (2.2) α	β	τ	population parameter	sample size	uniformly minimum variance unbiased estimator
-1/2	0	-1	λ (a scale parameter)	$n \geq 4$	$(n-3)/V$
-1/2	0	1	λ^{-1} (reciprocal of λ)	$n \geq 2$	$V/(n-1)$
1/4	3/4	1/2	$\sqrt{\mu^3/\lambda}$ (standard deviation)	$n \geq 2$	$\frac{\Gamma((n-1)/2)}{\sqrt{2}\Gamma(n/2)}\sqrt{\bar{X}^3 V}\cdot F(\frac{1}{4},\frac{3}{4};\frac{n}{2};-\frac{\bar{X}V}{n})$
-1	-1/2	-1	$\phi = \lambda/\mu$ (a dispersion parameter)	$n \geq 4$	$(n-3)/\bar{X}V - 1/n$
0	1/2	0	μ (mean)	$n \geq 2$	$\bar{X}$
1	3/2	1	μ^3/λ (variance)	$n \geq 2$	the formula (3.1)
-1	-1/2	0	μ^{-1} (reciprocal of mean)	$n \geq 2$	$1/\bar{X} - V/n(n-1)$
-2	-3/2	-1	λ/μ^3 (reciprocal of variance)	$n \geq 4$	$(n-3)/\bar{X}^3 V - 6/n\bar{X}^2 + 3V/n^2(n-1)\bar{X}$
-1/4	1/4	1/2	$3\phi^{-1/2}$ (skewness)	$n \geq 2$	$\frac{3\Gamma((n-1)/2)}{\sqrt{2}\Gamma(n/2)}\sqrt{\bar{X}V}\cdot F(-\frac{1}{4},\frac{1}{4};\frac{n}{2};-\frac{\bar{X}V}{n})$
0	1/2	1	$15\phi^{-1}$ (kurtosis)	$n \geq 2$	$15\bar{X}V/(n-1)$
-1/4	1/4	1/2	$\phi^{-1/2}$ (coefficient of variation)	$n \geq 2$	$\frac{\Gamma((n-1)/2)}{\sqrt{2}\Gamma(n/2)}\sqrt{\bar{X}V}\cdot F(-\frac{1}{4},\frac{1}{4};\frac{n}{2};-\frac{\bar{X}V}{n})$
$r-1$	$r-\frac{1}{2}$	$r-1$	κ_r (r-th cumulant of X)	$n \geq 2$	the formula (2.3)
$\frac{r-2}{4}$	$\frac{r}{4}$	$\frac{r}{2}$	γ_r (standardized r-th cumulant of X)	$n \geq 2$	$\frac{2^{r/2+1}}{\sqrt{\pi}}\cdot\frac{\Gamma(r+3/2)\Gamma((n-1)/2)}{\Gamma((n-1)/2+r/2)}(\bar{X}V)^{r/2}\cdot F(\frac{r-2}{4},\frac{r}{4};\frac{n-1}{2}+\frac{r}{2};-\frac{\bar{X}V}{n})$
$\frac{r+s-1}{2}$	$\frac{r+s}{2}$	s	μ'_r (r-th moment of X)	$n \geq 2$	$\sum_{s=0}^{r-1}\frac{(r-1+s)!}{s!(r-1-s)!}\cdot\frac{2^{-2s}\Gamma((n-1)/2)}{\Gamma((n-1)/2+s)}\bar{X}^{s+r}V^{s}\cdot F(\frac{s+r-1}{2},\frac{s+r}{2};\frac{n-1}{2}+s;-\frac{\bar{X}V}{n})$
$\{\begin{matrix}-1\\-1/2\end{matrix}$	$\begin{matrix}-1/2\\0\end{matrix}$	$\begin{matrix}-1\\r\end{matrix}$	κ_{-r} (r-th cumulant of X^{-1})	$n \geq 2$	$\frac{\Gamma((n-1)/2)}{\Gamma((n-1)/2+r)}\bar{X}^{-1}V^{r-1}\left\{\frac{\Gamma(r-1/2)}{\sqrt{\pi}}(\frac{n-1}{2}+r)+(\frac{(r-1)!}{2}-\frac{(r-1/2)}{2n\sqrt{\pi}})\bar{X}V\right\}$
$\frac{s-r-1}{2}$	$\frac{s-r}{2}$	s	μ'_{-r} (r-th moment of X^{-1})	$n \geq 2$	$\sum_{s=0}^{r}\frac{(r+s)!}{s!(r-s)!}\frac{2^{-2s}\Gamma((n-1)/2)}{\Gamma((n-1)/2+s)}\bar{X}^{s-r}V^{s}\cdot F(\frac{s-r-1}{2},\frac{s-r}{2};\frac{n-1}{2}+s;-\frac{\bar{X}V}{n})$

Table II

The ratio $\hat{\kappa}_2/\hat{\kappa}_2^*$ for several values of n and $\bar{X}V/n$

$\bar{X}V/n$ =		0.25	0.50	0.75	1.00	1.25	1.50	1.75	2.00
n =	2	.300	.667	.571	.500	.444	.400	.364	.333
	3	.345	.734	.651	.586	.533	.490	.456	.423
	4	.873	.778	.703	.644	.595	.553	.518	.487
	5	.392	.308	.741	.686	.640	.600	.566	.536
	6	.906	.331	.770	.719	.675	.638	.605	.575
	7	.917	.349	.793	.745	.704	.668	.636	.608
	8	.925	.363	.311	.767	.728	.693	.663	.635
	9	.932	.375	.327	.785	.748	.715	.685	.659
	10	.938	.385	.340	.300	.765	.733	.705	.679
	20	.966	.935	.907	.382	.858	.336	.315	.796
	30	.977	.955	.935	.916	.398	.381	.364	.349
	40	.982	.965	.949	.934	.920	.906	.893	.380
	50	.986	.972	.959	.946	.934	.922	.911	.900
	60	.988	.976	.965	.954	.944	.934	.924	.915
	70	.990	.980	.970	.960	.951	.942	.934	.925
	80	.991	.982	.973	.965	.957	.949	.941	.934
	90	.992	.984	.976	.969	.961	.954	.947	.940
	100	.993	.986	.979	.972	.965	.958	.952	.946

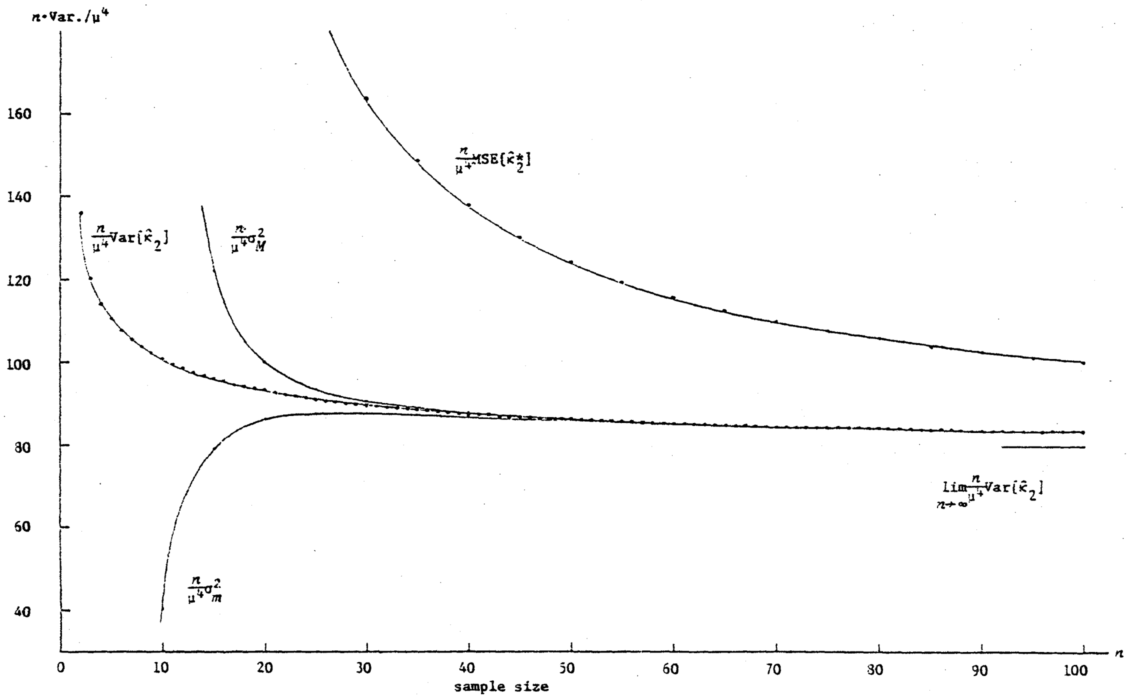

Fig. 1. Numerical results of variances for the case $\phi = 0.5$.

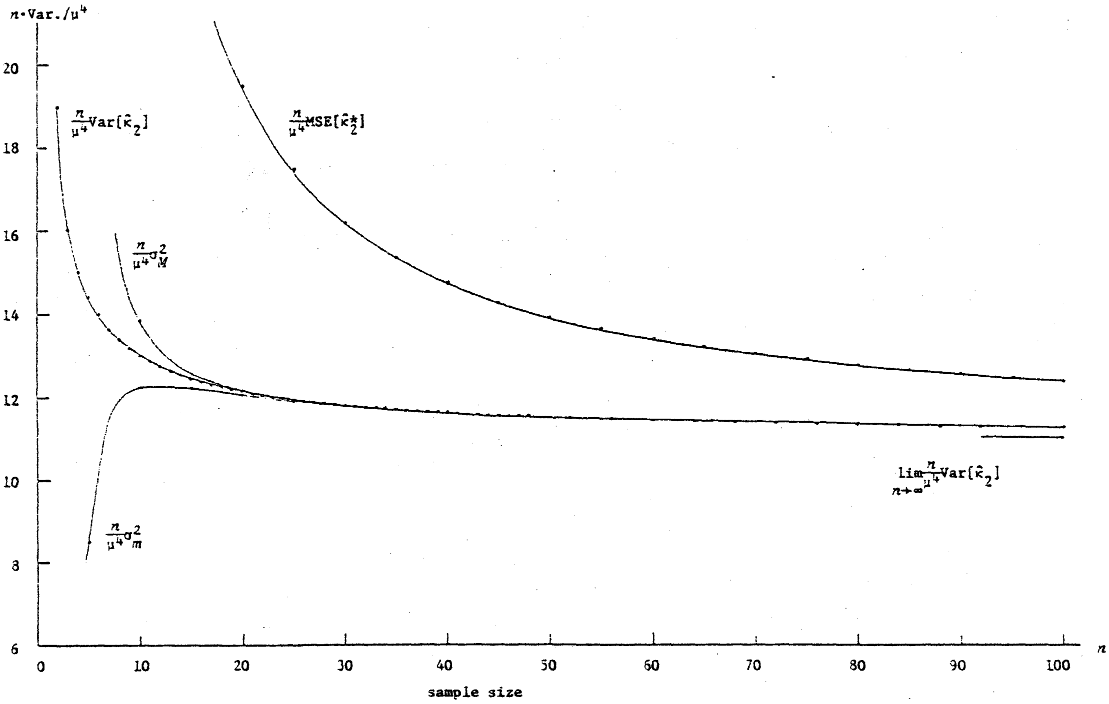

Fig. 2. Numerical results of variances for the case $\phi = 1.0$.

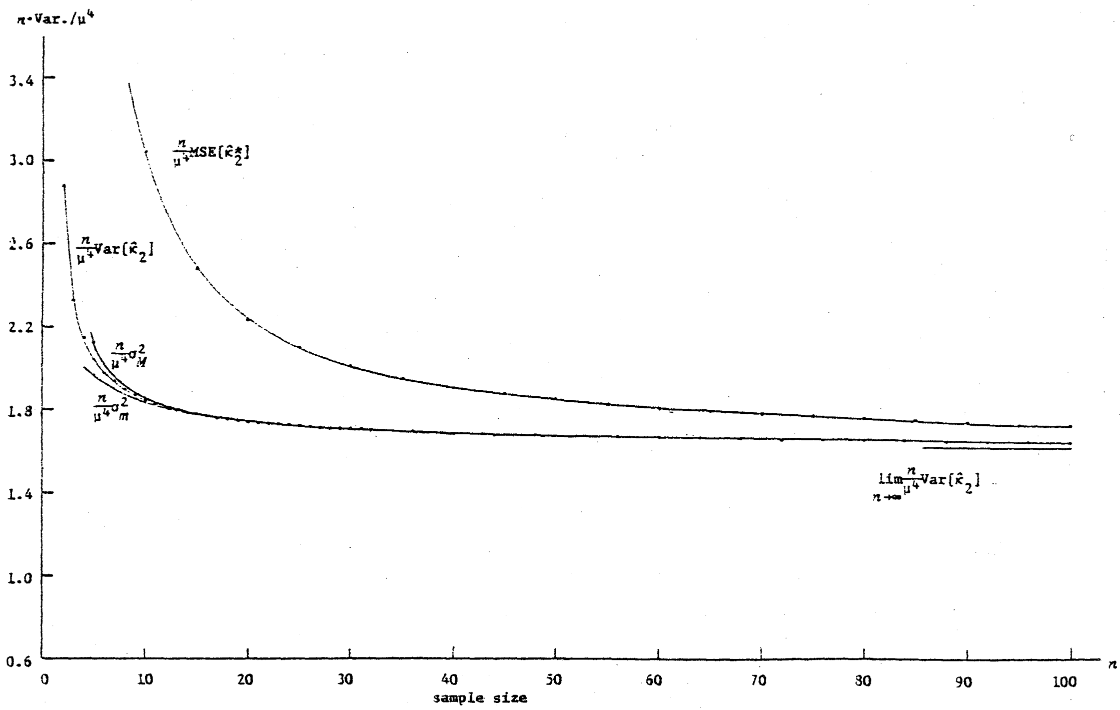

Fig. 3. Numerical results of variances for the case $\phi = 2.0$.