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Structure theory for $\underset{\sim}{\Pi}^1_n$ sets in the plain with countable sections

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Abstract

We develop structure theory for $\underset{\sim}{\Pi}^1_n$ sets in ${}^\omega\omega \times {}^\omega\omega$ with countable sections under the assumption of the projective determinacy when it is needed. It is shown that several theorems of Luzin about Borel and analytic sets cannot be extended to $\underset{\sim}{\Pi}^1_n$ sets. We also generalize a Friedman's theorem to Π^1_{2n+1} sets.

§0. Introduction.

Tanaka [39] extent several theorems of Luzin [24] about Borel and analytic sets in ${}^{\omega}\omega \times {}^{\omega}\omega$ with countable sections to $\utilde{\Delta}^1_{2n+1}$ and $\utilde{\Sigma}^1_{2n+1}$ sets respectively under the assumption of projective determinacy. He also shows that these theorems fail for Π^1_{2n+1} sets and announced these also fail for $\utilde{\Pi}^1_{2n+1}$ sets. Our main theme is to complete structure theory for projective sets in ${}^{\omega}\omega \times {}^{\omega}\omega$ with countable sections under the assumptions of projective determinacy. To this we prove that, among other things, every $\utilde{\Sigma}^1_{2n+1}$ set in ${}^{\omega}\omega \times {}^{\omega}\omega$ with countable sections can be uniformized by the difference of two $\utilde{\Sigma}^1_{2n+1}$ sets, and there is a Π^1_{2n+1} set in ${}^{\omega}\omega \times {}^{\omega}\omega$ with countable sections which cannot covered by either countably many $\utilde{\Sigma}^1_{2n+2}$ or $\utilde{\Pi}^1_{2n+2}$ curves. We also generalize a Friedman's theorem as follows : There is an infinitely countable Π^1_{2n+1} set of reals every member of which except one is Δ^1_{2n+2} real. We present several applications of these results. Our proof methods are parametrization of Δ^1_{2n+1} reals by integers, uniformization theorem and higher-level analoges of Gödel's L which are all consequences of projective determinacy.

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§1. Preliminaries.

We use in this paper standard terminology and notation in descripive set theory, following in most instances that of Moschovakis' monograph [27]. Our basic spaces will be ω, ${}^{\omega}\omega$ (the reals, denoted by $\mathcal{N}$ in [27]) and ${}^{\omega}2$. (Product) spaces are of the form

$$\mathcal{X} = X_1 \times \ldots \times X_k,$$

where X_i, $1 \leq i \leq k$, is a basic space, members of these spaces are called points and subsets of them pointsets or simply sets. Some-times we think of them as predicates on the space $\mathcal{X}$ and write interchangeably for each $x \in \mathcal{X}$

$$x \in P \Leftrightarrow P(x).$$

A pointclass is a collection of pointsets, usually in all product spaces.

We will adhere to the following notational conventions throughout this paper. Letters e, i, j, k, l, m, n denote always members of ω, $\alpha, \beta, \gamma, \delta$ members of ${}^{\omega}\omega$.

If Γ is a pointclass, we put

$$\check{\Gamma} = \{\mathcal{X} - P : \text{for some } \mathcal{X} \text{ and } P \in \Gamma,\ P \subseteq \mathcal{X}\},$$

call it the dual of Γ,

$$(\Gamma)_p = \{P \cap Q : \text{for some } \mathcal{X},\ P, Q, P \in \Gamma,\ Q \in \check{\Gamma},\ P, Q \subseteq \mathcal{X}\},$$

call it the difference of two Γ pointsets, for each α

$$\Gamma(\alpha) = \{ P : \text{for some } \mathcal{X} \text{ and } Q \in \Gamma, \ P(x) \Leftrightarrow Q(\alpha, x)\},$$

call it the relativization of Γ to α, and for each product space $\mathcal{X}$

$$\Gamma \upharpoonright \mathcal{X} = \{ P \subseteq \mathcal{X} : P \in \Gamma\}.$$

After Kondô [20] for each $x \in \mathcal{X}$ and $P \subseteq \mathcal{X} \times \mathcal{Y}$

$$P^{\langle x \rangle} = \{ y \in \mathcal{Y} : P(x, y)\},$$

and call it the section of P at x. We also call a partial function $\Phi : \mathcal{X} \to \mathcal{Y}$ a curve and identify Φ with its graph $\{(x, y) : \Phi(x) = y\}$.

We shall be concerned in this paper the projective pointclasses :

$$\underset{\sim}{\Sigma}{}^0_1 = \text{all open pointsets},$$

$$\underset{\sim}{\Pi}{}^0_1 = \text{all closed pointsets},$$

$$\underset{\sim}{\Sigma}{}^1_1 = \{ P \subseteq \mathcal{X} : \text{for some product space } \mathcal{X} \text{ and some } \underset{\sim}{\Pi}{}^0_1 \text{ pointset } Q \subseteq {}^{\omega}\omega \times \mathcal{X}, \ P(x) \Leftrightarrow \exists \alpha\, Q(\alpha, x)\}.$$

In classical terminology these are the analytic sets or A-sets. Then we let $\underset{\sim}{\Pi}{}^1_1$ be the pointclass of all complements of $\underset{\sim}{\Sigma}{}^1_1$ sets, i.e.

$$\underset{\sim}{\Pi}{}^1_1 = \check{\underset{\sim}{\Sigma}}{}^1_1 .$$

Classicaly again these are known as the coanalytic sets or CA-sets. Then we let

$$\underset{\sim}{\Sigma}{}^1_2 = \{ P : \text{for some product space } \mathcal{X} \text{ and some } \underset{\sim}{\Pi}{}^1_1 \text{ pointset } Q \subseteq {}^{\omega}\omega \times \mathcal{X}, \ P(x) \Leftrightarrow \exists \alpha\, Q(\alpha, x) \}$$

(the classical PCA-set), and

$$\underset{\sim}{\Pi}{}^1_2 = \check{\underset{\sim}{\Sigma}}{}^1_2 \text{ (the classical CPCA-set)},$$

and in general inductively

$$\underset{\sim}{\Sigma}^1_{n+1} = \{P : \text{for some product space } \mathcal{X} \text{ and some } \underset{\sim}{\Pi}^1_n \text{ pointset } Q \subseteq {}^{\omega}\omega \times \mathcal{X} \ , \ P(x) \Leftrightarrow \exists \alpha\, Q(\alpha, x)\},$$

$$\underset{\sim}{\Pi}^1_{n+1} = \check{\underset{\sim}{\Sigma}}{}^1_{n+1}.$$

We also define the <u>ambiguous</u> pointclass

$$\underset{\sim}{\Delta}^1_n = \underset{\sim}{\Sigma}^1_n \cap \underset{\sim}{\Pi}^1_n.$$

The class of <u>projective sets</u> is the union

$$\bigcup_{n=0}^{\infty} \underset{\sim}{\Sigma}^1_n.$$

It only remain to clarify the relationship between the Borel set and $\underset{\sim}{\Delta}^1_1$. According to a excellent theorem of Souslin [34]

$$\text{The Borel sets} = \underset{\sim}{\Delta}^1_1,$$

i.e. the Borel sets coincide with the analytic-coanalytic ones.

We shall also be concerned the <u>analytical</u> pointclasses :

$$\Sigma^0_1 = \text{all recursively enumerable pointsets},$$

$$\Pi^0_1 = \check{\Sigma}^0_1,$$

$$\Sigma^1_1 = \{P : \text{for some } \mathcal{X} \text{ and } \Sigma^0_1 \text{ set } Q \subseteq {}^{\omega}\omega \times \mathcal{X} \ P(x) \Leftrightarrow \exists \alpha\, Q(\alpha, x)\},$$

$$\Pi^1_1 = \check{\Sigma}^1_1,$$

$$\Sigma^1_{n+1} = \{P : \text{for some } \mathcal{X} \text{ and } \Pi^1_n \text{ set } Q \subseteq {}^{\omega}\omega \times \mathcal{X} \ P(x) \Leftrightarrow \exists \alpha\, Q(\alpha, x)\},$$

$$\Pi^1_{n+1} = \check{\Sigma}^1_{n+1},$$

$$\Delta^1_n = \Sigma^1_n \cap \Pi^1_n.$$

The class of <u>analytical</u> <u>sets</u> is the union

$$\bigcup_{n=0}^{\infty} \Sigma^1_n.$$

One can also define the effective analogs of the Borel sets, which are known as hyperarthmetic (HYP) sets. Kleene's theorem [18] (the effective analog of Souslin's theorem) assert that

$$\mathrm{HYP} = \Delta^1_1.$$

What is the interrelationship between the classical and the effective notions ? The key lies in the concept of relativization, introduced in Kleene [16].

We proceed exactly as before to define, for each real α, $\Sigma^0_1(\alpha)$, $\Pi^0_1(\alpha)$, $\Sigma^1_1(\alpha)$, $\Pi^1_1(\alpha)$, ...

Here is now the precise relationship between the classical and the (relativized) effective notions :

$$\underset{\sim}{\Sigma}{}^0_1 = \bigcup_{\alpha\in{}^\omega\omega} \Sigma^0_1(\alpha),$$

$$\underset{\sim}{\Pi}{}^0_1 = \bigcup_{\alpha\in{}^\omega\omega} \Pi^0_1(\alpha),$$

$$\underset{\sim}{\Sigma}{}^1_1 = \bigcup_{\alpha\in{}^\omega\omega} \Sigma^1_1(\alpha),$$

$$\underset{\sim}{\Pi}{}^1_1 = \bigcup_{\alpha\in{}^\omega\omega} \Pi^1_1(\alpha), \text{ etc.}$$

Also for a function $f : \mathcal{X} \to \mathcal{Y}$, f is continnuous if and only if f is recursive in some real α. Thus the effective notions are refinment of the classical ones. Note also the following facts. For each $n \geq 1$ there is a Σ^1_n set G in ${}^\omega\omega \times \mathcal{X}$ which is universal for the $\underset{\sim}{\Sigma}{}^1_n$ sets in $\mathcal{X}$, i.e. a set $P \subseteq \mathcal{X}$ is $\underset{\sim}{\Sigma}{}^1_n$ if and only if for some real α,

$$P = G^{\langle\alpha\rangle}.$$

Similarly for Σ^0_1, Π^0_1, Σ^1_n. Thus e.g. the $\underset{\sim}{\Sigma}{}^1_1$ sets are just the sections of the Σ^1_1 set G (Addison [1] and Tugué).

Almost any proof in effective descriptive set theory involving the absolute notions : recursiv, Σ^0_1, Π^0_1, etc, relativizes immediately to an arbitrary parameter α, by just "plugging-in" the parameter at all appropriate places in the proof and so yields its relativized version. In view of the previous explained relationship between the relativized concepts and the classical ones, this maks clear that the effective result immediately implies its classical version. In that sense effective descriptive set theory is a strengthening of the classical one. And usually the methods of the effective theory allow for much simpler and elegant proofs.

Many times now the formulation and proof of an effective result involve concepts which are only meningful in the effective theory, but nevertheless throw a lot of light to a related classical result.

For example, the classical perfect set theorem for $\underset{\sim}{\Sigma}^1_1$ sets asserts that a $\underset{\sim}{\Sigma}^1_1$ set which contains no nonempty perfect subset must be countable. Let us say that a real β is $\Delta^1_1(\alpha)$ if and only if its graph $\{(i, j) : \beta(i) = j\}$ is in $\Delta^1_1(\alpha)$. Denote by $\mathcal{D}^1_1(\alpha)$ the set of all $\Delta^1_1(\alpha)$ reals. Clearly $\mathcal{D}^1_1(\alpha)$ is countable. We have now the following basic result of effective descriptive set theory.

<u>Theorem</u> (Harrison [11]). Let $A \subseteq {}^{\omega}\omega$ be $\Sigma^1_1(\alpha)$. If A contains no nonempty perfect set, then $A \subseteq \mathcal{D}^1_1(\alpha)$.

Thus not only we know that every $\Sigma^1_1(\alpha)$ set with no nonempty perfect subset is countable, but we know what kind of members it contains, namely only $\mathcal{D}^1_1(\alpha)$ ones. Put in another way, we have a fixed countable set $\wp_1^1(\alpha)$ such that a necessary and sufficient condition for $A \in \Sigma_1^1(\alpha)$ to contain a nonempty perfect subset is to contain at least one element not in it. So the fact that effective descriptive set theory can develop a concept of classification of "complexity" of individual reals (as for example being Δ_1^0, Δ_1^1, Δ_2^1, ...) serves to clarify consderably a classical situation. (Note that the concept of a real being $\underset{\sim}{\Delta}{}_1^0$, $\underset{\sim}{\Delta}{}_1^1$, $\underset{\sim}{\Delta}{}_2^1$, ... is trivial ; every real is such.)

Finally, and very importantly, effective set theory provides powerful methods for the solusion of problems of unddobtedly classical character and contents, for which no classical type proofs are known at present. A such example is Steel and Martin negative solution of one of Luzin's uniformization problems [24] : That is to say there is a $\underset{\sim}{\Sigma}{}_1^1$ set in ${}^\omega\omega \times {}^\omega\omega$ which cannot be uniformized by the difference of two $\underset{\sim}{\Sigma}{}_1^1$ sets (see Moschovakis [27; 4F.21]). Also this is an example of strongest negative solution for classical type problems, since the counter example is a light face Σ_1^1 set.

As Luzin [25] predcted, the classical methods of descriptive set theory are not successful in solving non-trivial problems concerning projective sets for levels begining with the third, and sometimes for the second and even the first level. Powerful as they are, the methods from logic and recursion theory cannot solve this "difficulties of the theory of projective sets" , since they too are restricted by the limitations of Zermelo-Fraenfel set teory. (see Cohen [4] , Gödel [7] , Harrington [9, 10] ,Lévy [23]).

Since properties of definable sets can usually be established effectively, without use of the full axiom of choice AC, we shall work in set theory <u>without the full axiom of choice</u>. However, we shall assume a weak form of the axiom of choice. The reson is that in descriptive set theory one frequently considers unions and intersections of "countably many countable sets is countable". Thus we shall work, throughout this paper, in set theory ZF + DC, where DC is the <u>axiom of dependent choices</u> :

Axiom of dependent choices (DC). For every set of pairs $P \subseteq A \times A$ from nonempty set A,

$$\forall x \in A\ \exists y \in A\ P(x, y) \Rightarrow \exists f : \omega \rightarrow A\ \forall n\ \ P(f(n), f(n+1)).$$

Recall some consequences of DC :

(1) The countable axiom of choice.

(2) Every infinite set has a countable subset.

(3) The union of countably many countable sets is countable.

(4) A binary relation without infinite descending chains is wellfounded.

The full axiom of choice implies DC easily and Kechris [16] has shown that DC is consistent with PD.

For any pointclass Γ ,

$$\mathrm{Det}(\Gamma)$$

is an abbreviation for the assertion that all games in Γ are determined. In particular,

$$\mathrm{PD} \Leftrightarrow \bigvee_{n=0}^{\infty} \mathrm{Det}(\underset{\sim}{\Sigma}{}^1_n)$$

is the projective determinacy.

As stated in introduction, this paper is a sequel to well-known classic book Luzin [25]. Its main purpose is to show how game theoretic hypotheses, definable determinacy, can be used to natural extension of Luzin's theory of structual properties about Borel and analytic sets in ${}^{\omega}\omega \times {}^{\omega}\omega$ with countable sections to higher level projective sets.

We assume from now on for the rest of this section $\mathrm{Det}(\underset{\sim}{\Delta}{}^1_{2n})$. (For $n = 0$, $\underset{\sim}{\Sigma}{}^1_0 = \underset{\sim}{\Sigma}{}^0_1$, $\underset{\sim}{\Pi}{}^1_0 = \underset{\sim}{\Pi}{}^0_1$, $\underset{\sim}{\Delta}{}^1_0 = \underset{\sim}{\Delta}{}^0_1$ so $\mathrm{Det}(\underset{\sim}{\Delta}{}^1_0)$ is just clopen determinacy which is provable in ZF + DC, thus no strong hypothesis is being made in this case.)

Let Γ be a pointclass, A a pointset. A norm on A is a map $\varphi : A \twoheadrightarrow \kappa$ from A onto an ordinal κ. We call φ a Γ-norm if the two relations below

$$x \leq^*_{\varphi} y \Leftrightarrow x \in A \ \& \ (\varphi(x) \leq \varphi(y)),$$

$$x <^*_{\varphi} y \Leftrightarrow x \in A \ \& \ (\varphi(x) < \varphi(y))$$

are in Γ where we put $\varphi(y)$ = an ordinal bigger than $\sup\{\varphi(x) : x \in A\}$, for all $y \notin A$. Finally we say that Γ is normed if and only if every pointsets in Γ admts a Γ-norm.

The pointclasses Π^1_{2n+1}, Σ^1_{2n+2} are normed (First Periodcity Theorem ; Moschovakis [27 ; 6B. 1]). Some corollaries of this fact are the following :

(1) Π^1_{2n+1} (Σ^1_{2n+2}) satisfy reduction and Σ^1_{2n+1} (Π^1_{2n+2}) satisfy separetion (Moschovakis [29; 4B.10-11]).

(2) The <u>number uniformization theorem</u> for Π^1_{2n+1}, i.e. if $P(x, n)$ is Π^1_{2n+1} there is $P^*(x, n)$ in Π^1_{2n+1} such that

$$P^* \subseteq P$$

and

$$\exists n\, P(x, n) \Leftrightarrow \exists!\, n\, P^*(x, n)$$

(Moschovakis [29; 4B.4]).

(3) Let $\mathcal{D}^1_{2n+1}$ denote the set of Δ^1_{2n+1} reals and $\mathcal{D}^1_{2n+1}(\alpha)$ the relativized notion, i.e.

$$\beta \in \mathcal{D}^1_{2n+1}(\alpha) \Leftrightarrow \beta \in \Delta^1_{2n+1}(\alpha)\ .$$

Then there are partial functions $\underset{\sim}{d} : \omega \times {}^{\omega}\omega \times \omega \longrightarrow \omega$ and $\underset{\sim}{c} : {}^{\omega}\omega \times {}^{\omega}\omega \longrightarrow \omega$ with Π^1_{2n+1} graphs such that

$$\beta \in \mathcal{D}^1_{2n+1}(\alpha) \Leftrightarrow \exists e\ \forall i\ (\beta(i) = \underset{\sim}{d}(e, \alpha, i)),$$

$$(\beta, \alpha) \in \mathrm{dom}(\underset{\sim}{c}) \Leftrightarrow \beta \in \mathcal{D}^1_{2n+1}(\alpha),$$

for $\beta \in \mathcal{D}^1_{2n+1}(\alpha)$

$$\forall i\ (\underset{\sim}{d}(\underset{\sim}{c}(\beta, \alpha), \alpha, i) = \beta(i)),$$

and the relatons "$\beta(i) = \underset{\sim}{d}(e, \alpha, i)$" and "$e = \underset{\sim}{c}(\beta, \alpha)$" are Δ^1_{2n+1}, uniformly on $(e, \alpha, i) \in \mathrm{dom}(\underset{\sim}{d})$ and $(\beta, \alpha) \in \mathrm{dom}(\underset{\sim}{c})$ respectively, i.e. , for example, there are Q, R in Σ^1_{2n+1}, Π^1_{2n+1} respectively , such that for $(e, \alpha, i) \in \mathrm{dom}(\underset{\sim}{d})$

$$\beta(i) = \underset{\sim}{d}(e, \alpha, i) \Leftrightarrow Q(e, \alpha, i, \beta) \Leftrightarrow R(e, \alpha, i, \beta)$$

(Moschovakis [29; 4D.2, 4D.5, 4D.15 and 6B.2]).

(4) The <u>bounded quantification theorem</u> for Π^1_{2n+1}, i.e. for each $P(\alpha, \beta, x)$ in Π^1_{2n+1} the pointset

$$R(\beta, x) \Leftrightarrow \exists \alpha \in \mathfrak{D}^1_{2n+1}(\beta)\ P(\alpha, \beta, x)$$

is also in Π^1_{2n+1} (Moschovakis [29; 4D.3 and 6B.2]). In particular the relation

$$\beta \in \Delta^1_{2n+1}(\alpha)$$

is Π^1_{2n+1}.

Harrington [10] has shown that "First Periodicity Theorem" is consistent with ZF + DC + I, so its all consequences, e.g. (1)-(4), are consistent with ZF + DC + I, where I is the statment which say there is an incessible cardinal.

Again let Γ be a pointclass and A a pointset. A <u>scale</u> on A is a sequence $\bar{\varphi} = \{\varphi_n\}$ of norms on A such that

(i) If $x_i \in A$, $i = 0, 1, \ldots$ and $x_i \to x$

and

(ii) For each n, and for all large enough i

$$\varphi_n(x_i) = \text{constant} = \lambda_n ,$$

then $x \in A$ and $\varphi_n(x) \leq \lambda_n$. We call $\{\varphi_n\}$ a Γ-<u>scale</u> if the pointsets

$$R(n, x, y) \Leftrightarrow x \leq^*_{\varphi_n} y ,$$

$$S(n, x, y) \Leftrightarrow x <^*_{\varphi_n} y$$

are in Γ. We say that Γ is <u>scaled</u> if every $A \in \Gamma$ admits a Γ-scale.

The pointclasses Π^1_{2n+1}, Σ^1_{2n+2} are scaled (Second Periodicity Theorem ; Moschovakis [29; 6C.3]). Some corollaries of this fact are the following :

(1) The uniformization theorem for Π^1_{2n+1}, i.e. if $P(x, y)$ is Π^1_{2n+1} there is $P^*(x, y)$ in Π^1_{2n+1} such that

$$P^* \subseteq P$$

and

$$\exists y\, P(x, y) \Leftrightarrow \exists !\, y\, P^*(x, y)$$

(Moschovakis [29; 6C.5]).

(2) The basis theorem for Σ^1_{2n+2}, i.e. every nonempty Σ^1_{2n+2} set contains a Δ^1_{2n+2} real (Moschovakis [29; 6C.6]).

We turn now to definability estimates for winning strategies. The basic theorem here is the Third Periodicity Theorem (Moschovakis [29; 6E.1]), which asserts that in every Σ^1_{2n} game in which Player I has a winning strategy, he actually has a Δ^1_{2n+1} winning strategy. We shall also use the following consequences of this result :

(1) The Spector-Gandy Theorem for Π^1_{2n+1}, which asserts that every Π^1_{2n+1} pointset $P(x)$ can be written as

$$P(x) \Leftrightarrow \exists \alpha \in \Delta^1_{2n+1}(x)\; R(\alpha, x) ,$$

for some R in Π^1_{2n} (Moschovakis [29; 6E.7]).

(2) Every thin (i.e. containing no nonempty perfect subset) Σ^1_{2n+1} set contains only Δ^1_{2n+1} reals (so in particular is countable). Also every nonempty Δ^1_{2n+1} thin set A can be written as $\{(\varepsilon)_n : n \in \omega\}$ for some Δ^1_{2n+1} real ε (Moschovakis [29; 6E.5]).

§2. The uniformization of $\underset{\sim}{\Sigma}{}^1_{2n+1}$ sets with countable sections.

We assume in this section $\mathrm{Det}(\underset{\sim}{\Delta}{}^1_{2n})$.

Theorem 2.1. (Yasuda [43, 45]). Every Σ^1_{2n+1} set in ${}^\omega\omega \times {}^\omega\omega$ with countable sections can be uniformized by a $(\Sigma^1_{2n+1})_\rho$ set.

Proof. Let P be a Σ^1_{2n+1} set in ${}^\omega\omega \times {}^\omega\omega$ with countable sections. Since for each α $P^{\langle\alpha\rangle}$ is a ~~thin~~ $\Sigma^1_{2n+1}(\alpha)$ set,

$$P^{\langle\alpha\rangle} \subseteq \mathcal{D}^1_{2n+1}(\alpha).$$

Let P* define by

$$P^*(\alpha, \beta) \Leftrightarrow P(\alpha,\beta) \;\; \& \;\; \forall\gamma\,(P(\alpha,\gamma) \Rightarrow \underset{\sim}{\rho}(\beta,\alpha) \leq \underset{\sim}{\rho}(\gamma, \alpha)),$$

where $\leq$ is the usual wellordering on ω . Then we have

$$P^* \subseteq P,$$

and

$$\exists\beta\, P(\alpha, \beta) \Leftrightarrow \exists!\beta\, P^*(\alpha, \beta).$$

Thus P* uniformizes P, and from our definiton of P* it is clealy $(\Sigma^1_{2n+1})_\rho$ set. □

From this theorem, using the remarks in preliminaries, we have

Corollary 2.2. Every $\underset{\sim}{\Sigma}{}^1_{2n+1}$ set in ${}^\omega\omega \times {}^\omega\omega$ with countable sections can be uniformized by a $(\underset{\sim}{\Sigma}{}^1_{2n+1})_\rho$ set. □

<u>Theorem 2.3.</u> (Yasuda [45]). There is a Σ^1_{2n+1} set P in ${}^{\omega}\omega \times {}^{\omega}\omega$ with the properties :

(i) For each α $P^{\langle\alpha\rangle}$ is nonempty and has at most two elements,

(ii) P cannot be uniformized by either a $\underset{\sim}{\Sigma}^1_{2n+1}$ or a $\underset{\sim}{\Pi}^1_{2n+1}$ set.

<u>Proof.</u> Let G be a Σ^1_{2n+1} set in ${}^{\omega}\omega \times {}^{\omega}\omega \times {}^{\omega}\omega$ which is universal for all $\underset{\sim}{\Sigma}^1_{2n+1}$ sets in ${}^{\omega}\omega \times {}^{\omega}\omega$ and Q define by

$$Q(\alpha, e) \Longleftrightarrow \forall\beta(G(\alpha,\alpha,\beta) \Rightarrow \forall i\ (\beta(i) = \underset{\sim}{d}(e, \alpha, i))).$$

Since Q is a Π^1_{2n+1} set in ${}^{\omega}\omega \times \omega$, using the number uniformization theorem we can find a Π^1_{2n+1} set Q* which uniformizes Q. Now let R define by

$$R(\alpha, \beta) \Longleftrightarrow \exists e\ (Q^*(\alpha, e)\ \&\ \forall i\ (\beta(i) = \underset{\sim}{d}(e, \alpha, i))).$$

Then R is a Π^1_{2n+1} set whose each section has at most one $\Delta^1_{2n+1}(\alpha)$ real as an elements. The following is clear from the properties of R.

<u>Fact.</u> For each α , if $G^{\langle\alpha,\alpha\rangle}$ contains just one element then

$$G^{\langle\alpha,\alpha\rangle} = R^{\langle\alpha\rangle}.$$

Finally, put

$$P(\alpha, \beta) \Longleftrightarrow (\ \forall i\ (\beta(i) = 0)\ \vee\ \forall i\ (\beta(i) = 1))\ \&\ \neg R(\alpha, \beta)\ .$$

Then P is Σ^1_{2n+1} and for each α $P^{\langle\alpha\rangle}$ is nonempty and at most two elements. We will show that P satisfies (ii). Let A be a $\underset{\sim}{\Sigma}^1_{2n+1}$ subset of P which is the graph of a partial function from ${}^{\omega}\omega$ into ${}^{\omega}\omega$. Then there is a real α_0 such that

$$A = G^{\langle\alpha_0\rangle}.$$

Since $A^{\langle\alpha_o\rangle}$ has at most one element, if $A^{\langle\alpha_o\rangle}$ is nonempty then by the fact

$$A^{\langle\alpha_o\rangle} = R^{\langle\alpha_o\rangle}.$$

But

$$A^{\langle\alpha_o\rangle} \subseteq P^{\langle\alpha_o\rangle} \subseteq {}^{\omega}2 - R^{\langle\alpha_o\rangle} .$$

We must have $A^{\langle\alpha_o\rangle}$ is empty. Thus $A^{\langle\alpha_o\rangle}$ is the empty set. This means P cannot be uniformized by a $\underset{\sim}{\Sigma}{}^1_{2n+1}$ set. Now suppose that P can be uniformized by a $\underset{\sim}{\Pi}{}^1_{2n+1}$ set C. Then for some real δ C is in $\Pi^1_{2n+1}(\delta)$ and we have

$$\neg C(\alpha, \beta) \Leftrightarrow \exists \gamma \in \Delta^1_{2n+1}(\alpha, \delta)(C(\alpha, \gamma) \;\&\; \gamma \neq \beta).$$

By the bounded quantification theorem, this equivalence shows that the complement of C is also $\Pi^1_{2n+1}(\delta)$, i.e. C is $\underset{\sim}{\Delta}{}^1_{2n+1}$. Thus it is a $\underset{\sim}{\Sigma}{}^1_{2n+1}$ uniformizator for P , so we have a contradiction. Therefor P also cannot be uniformized by a $\underset{\sim}{\Pi}{}^1_{2n+1}$ set. $\square$

<u>Corollary</u> 2.4. There is a Σ^1_{2n+1} set in ${}^{\omega}\omega \times {}^{\omega}\omega$ with countable sections which cannot be uniformized by either a $\underset{\sim}{\Sigma}{}^1_{2n+1}$ or a $\underset{\sim}{\Pi}{}^1_{2n+1}$ set. $\square$

<u>Corollary</u> 2.5. There is a $\underset{\sim}{\Sigma}{}^1_{2n+1}$ set in ${}^{\omega}\omega \times {}^{\omega}\omega$ which cannot uniformized by either a $\underset{\sim}{\Sigma}{}^1_{2n+1}$ or a $\underset{\sim}{\Pi}{}^1_{2n+1}$ set. $\square$

Let D define by

$$D = \{\alpha : R(\alpha, \lambda n [0]) \vee R(\alpha, \lambda n [1])\},$$

and C* by

$$C^*(\alpha, \beta) \Leftrightarrow \alpha \in D \ \& \ ((R(\alpha, \lambda n[0]) \ \& \ \beta = \lambda n[1]) \vee (R(\alpha, \lambda n[1]) \ \& \ \beta = \lambda n[0])).$$

Then D and C^* are Π^1_{2n+1} and C^* is a partial function which is contained in P. Now put

$$P^*(\alpha, \beta) \Leftrightarrow (\alpha \in D \ \& \ C^*(\alpha, \beta)) \vee (\alpha \notin D \ \& \ \beta = \lambda n[0]).$$

Then P^* uniformizes P, and it is the sum of a Σ^1_{2n+1} and a Π^1_{2n+1} sets.

__Problem.__ Is there a $\underset{\sim}{\Sigma}^1_{2n+1}$ set in ${}^\omega\omega \times {}^\omega\omega$ with countable sections which cannot be uniformized by the sum of a $\underset{\sim}{\Sigma}^1_{2n+1}$ and a $\underset{\sim}{\Pi}^1_{2n+1}$ sets ?

For this problem we have no answer at present, but a related result for $n = 0$.

__Theorem 2.6.__ (Tanaka [39] for $n = 0$). There is an uncountable Σ^1_{2n+1} set in ${}^\omega 2$ with no nonempty Π^1_{2n+1} subsets.

__Proof.__ Let G be a Π^1_{2n+1} set in $\omega \times {}^\omega 2$ which is universal for all Π^1_{2n+1} sets in ${}^\omega 2$. By the uniformization theorem for Π^1_{2n+1}, we can find a Π^1_{2n+1} set G^* in $\omega \times {}^\omega 2$ which uniformizes G. Put

$$Q(\alpha) \Leftrightarrow \exists e \ G^*(e, \alpha).$$

Then Q is a Π^1_{2n+1} set and intersects with every nonempty Π^1_{2n+1} set in ${}^\omega 2$. therefor the complement $A = {}^\omega 2 - Q$ is a Σ^1_{2n+1} set and it contains no nonempty Π^1_{2n+1} subsets. $\square$

(Cenzer and Mauldin [4]).

Theorem 2.7. For each uncountable Σ^1_1 set A in ${}^{\omega}2$, we can find a nonempty perfect subset A^* of A which is also Σ^1_1. $\square$

Corollary 2.10. There is a perfect Polish space D which is a Σ^1_1 set in ${}^{\omega}2$ with no nonempty Π^1_1 subsets.

Proof. Let A be an uncountable Σ^1_1 set with the properties as in theorem 2.6. Then ,using theorem 2.7 , we can find a nonempty perfect Σ^1_1 subset D of A. Clealy D is a perfect Polish space, since D has a countable base

$$\{N_s \cap D : s \in 2^{<\omega}\},$$

where

$$N_s = \{\alpha \in {}^{\omega}2 : \alpha \restriction \text{the length of } s = s\}. \qquad \square$$

Theorem 2.11. There is a $\underset{\sim}{\Sigma}^1_1$ set P in ${}^{\omega}2 \times {}^{\omega}2$ with the following properties :

(i) For each real α, $P^{\langle\alpha\rangle}$ has at most two reals as elements.

(ii) P cannot be uniformized by a $\underset{\sim}{\Sigma}^1_1$ set.

(iii) P contains no nonempty Π^1_1 curves.

Proof. Let D be a perfect Polish space as in corollary 2.10, and $f : D \longrightarrow {}^{\omega}2$ a Borel isomorphism (we cannot find such f which is Δ^1_1 isomorphism, since if such f exists then D must contains nonempty Π^1_1 sets), and G a Σ^1_1 set in ${}^{\omega}2 \times {}^{\omega}2 \times {}^{\omega}2$ which is universal for all $\underset{\sim}{\Sigma}^1_1$ sets in ${}^{\omega}2 \times {}^{\omega}2$. We can define a new univasal set G^+ for all $\underset{\sim}{\Sigma}^1_1$ sets in ${}^{\omega}2 \times {}^{\omega}2$ by

$$G^+(\alpha, \beta, \gamma) \Leftrightarrow \alpha \in D \ \& \ G(f(\alpha), \beta, \gamma).$$

Since D is Σ^1_1 and f is a Borel function, G^+ is a $\underset{\sim}{\Sigma}^1_1$ in ${}^{\omega}2 \times {}^{\omega}2 \times {}^{\omega}2$, then we use this G^+ as G in the proof of theorem 2.3. □

From this theorem, using a Δ^1_1 isomorphism between two spaces ${}^{\omega}2$ and ${}^{\omega}\omega$, we have

<u>Corollary</u> 2.12. There is a $\underset{\sim}{\Sigma}^1_1$ set in ${}^{\omega}\omega \times {}^{\omega}\omega$ with countable sections which cannot be uniformized by the sum of a $\underset{\sim}{\Sigma}^1_1$ and a Π^1_1 sets. □

The same reason as befor the set P in theorem 2.11 can be uniformized by the sum of a $\underset{\sim}{\Sigma}^1_1$ and a $\underset{\sim}{\Pi}^1_1$ sets. We do not know at present whether corollary 2.12 can be extended to $\underset{\sim}{\Sigma}^1_{2n+1}$ set , where $n > 0$.

Note in the proof of theorem 2.7 we really used the fact that C is Π^0_1 only to prove that the game G^* is determined. Thus we have

<u>Theorem</u> 2.13. If $\mathrm{Det}(\underset{\sim}{\Pi}^1_{2n+1})$ then every uncountable Σ^1_{2n+2} set in ${}^{\omega}2$ has a nonempty Σ^1_{2n+3} perfect subset. □

§3. On the partial $\underset{\sim}{\Pi}^1_n$ functions from ${}^{\omega}\omega$ into ${}^{\omega}\omega$.

Luzin [26] proved that every analytic curve can be extended to a Borel curve. We assume that $\mathrm{Det}(\underset{\sim}{\Delta}^1_{2n})$ holds in this section. Under this assumpution, Tanaka [42] extend this Luzin's result as follows.

Theorem 3.1. (Tanaka [26]). Every Σ^1_{2n+1} partial function from ${}^{\omega}\omega$ into ${}^{\omega}\omega$ can be extended to a Δ^1_{2n+1} function. □

From this, using the notion of relativization, we have

Corollary 3.2. Every $\underset{\sim}{\Sigma}^1_{2n+1}$ partial function from ${}^{\omega}\omega$ into ${}^{\omega}\omega$ can be extended to a $\underset{\sim}{\Delta}^1_{2n+1}$ function. □

Now we show that these results do not hold for $\underset{\sim}{\Pi}^1_{2n+1}$ and $\underset{\sim}{\Sigma}^1_{2n+2}$ partial functions in the strongest form.

Lemma 3.3. Let Γ be an analytical pointclass, and G a Γ pointset in ${}^{\omega}\omega \times {}^{\omega}\omega \times {}^{\omega}\omega$ which is universal for all $\underset{\sim}{\Gamma}$ sets in ${}^{\omega}\omega \times {}^{\omega}\omega$, and put

$$D = \{(\alpha, \beta) : G(\alpha, \alpha, \beta)\} .$$

Then D cannot be embedded in a $\check{\underset{\sim}{\Gamma}}$ set P in ${}^{\omega}\omega \times {}^{\omega}\omega$ with the property : for each α

$$P^{\langle\alpha\rangle} \neq {}^{\omega}\omega .$$

Proof. Assume, in order to obtain a contradiction, that there is a $\check{\underset{\sim}{\Gamma}}$ set P in ${}^{\omega}\omega \times {}^{\omega}\omega$ such that

$$D \subseteq P$$

and for each α

$$P^{\langle\alpha\rangle} \neq {}^{\omega}\omega.$$

Now consider the $\underset{\sim}{\Gamma}$ set $Q = ({}^{\omega}\omega \times {}^{\omega}\omega) - P$. Since Q is $\underset{\sim}{\Gamma}$ there is a real α_0 suh that

$$Q = G^{\langle\alpha_0\rangle}.$$

Then

$$D^{\langle\alpha_0\rangle} = \{\beta : G(\alpha_0, \alpha_0, \beta)\} = Q^{\langle\alpha_0\rangle},$$

so we have

$$Q^{\langle\alpha_0\rangle} = \emptyset .$$

But $Q^{\langle\alpha_0\rangle}$ is not empty since $P^{\langle\alpha_0\rangle}$ is not equal to ${}^{\omega}\omega$. Thus we have a contradiction, so the lemma is proved. $\square$

Theorem 3.4. (i) There is a partial Π^1_{2n+1} function from ${}^{\omega}\omega$ into ${}^{\omega}\omega$ which cannot be extended to a $\underset{\sim}{\Delta}^1_{2n+1}$ function.
(ii) There is a partial Σ^1_{2n+} function from ${}^{\omega}\omega$ into ${}^{\omega}\omega$ which cannot be extended to a $\underset{\sim}{\Delta}^1_{2n+2}$ function.

Proof. Let Γ be $\Pi^1_{2n+1}(\Sigma^1_{2n+2})$ and Φ a partial Π^1_{2n+1} (Σ^1_{2n+2}) function from ${}^{\omega}\omega$ into ${}^{\omega}\omega$ which uniformizes the Π^1_{2n+1} (Σ^1_{2n+2}) set D in the lemma 3.3, we can find such function using the uniformization theorem for $\Pi^1_{2n+1}(\Sigma^1_{2n+2})$. By lemma 3.3, Φ cannot be extended to $\underset{\sim}{\Delta}^1_{2n+1}(\underset{\sim}{\Delta}^1_{2n+2})$ function. $\square$

We have also

Theorem 3.5. There is a total Π^1_{2n+1} function from ${}^\omega\omega$ into ${}^\omega\omega$ which is not $\underset{\sim}{\Delta}^1_{2n+1}$.

To prove this we need first

Lemma 3.6. There is a Π^1_{2n} set P in ${}^\omega\omega \times {}^\omega\omega$ such that

(i) $$\forall\alpha\exists\beta P(\alpha, \beta),$$

(ii) $$\forall\alpha\neg\exists\beta\in\mathcal{D}^1_{2n+1}(\alpha)\ P(\alpha, \beta).$$

Proof. Since $\mathcal{D}^1_{2n+1}(\alpha)$ is Π^1_{2n+1}, there is a Π^1_{2n} set R in ${}^\omega\omega\times{}^\omega\omega\times{}^\omega\omega$ such that

$$\beta\notin\mathcal{D}^1_{2n+1}(\alpha) \Leftrightarrow \exists\gamma R(\alpha, \beta, \gamma).$$

Put

$$P(\alpha, \beta) \Leftrightarrow R(\alpha, (\beta)_0, (\beta)_1).$$

Then P is Π^1_{2n} and since

$$\forall\alpha\exists\beta\exists\gamma R(\alpha, \beta, \gamma),$$

$$\forall\alpha\exists\beta P(\alpha, \beta).$$

Since if

$$\neg\forall\alpha\neg\exists\beta\in\mathcal{D}^1_{2n+1}(\alpha)\ P(\alpha, \beta)$$

then for some real α and $\Delta^1_{2n+1}(\alpha)$ real β

$$R(\alpha, (\beta)_0, (\beta)_1),$$

so $(\beta)_0$ is not $\Delta^1_{2n+1}(\alpha)$ real. This contradiction shows

$$\forall\alpha\neg\exists\beta\in\mathcal{D}^1_{2n+1}(\alpha)\ P(\alpha, \beta). \qquad \square$$

<u>Proof</u> of theorem 3.5. Let P be as in lemma 3.6. By the uniformization theorem for Π^1_{2n+1} we can find a Π^1_{2n+1} set P* which uniformizes P. By lemma 3.6 (i), P* is a total function. Assume, in order to obtain a contradiction, that P* is $\underset{\sim}{\Delta}{}^1_{2n+1}$. Then for some real α_0 P* is in $\Delta^1_{2n+1}(\alpha_0)$. Thus there is just one $\Delta^1_{2n+1}(\alpha_0)$ real β_0 such that

$$P^*(\alpha_0, \beta_0).$$

This implies

$$\neg \forall \alpha \neg \exists \beta \in \mathcal{D}^1_{2n+1}(\alpha)\ P(\alpha, \beta).$$

But this formula contradicts with the lemma 3.6 (ii). $\square$

Theorem 3.5 can be extended to even levels using Moschovakis' models $M^{2n+2}(\alpha)$ which is the smallest Σ^1_{2n+2}-correct standard model of ZFC containing all ordinals and real α. We also need the sharp operation for these models (for detailed theory of these models and its sharps , the reader consalts Becker [3]).

<u>Lemma</u> 3.7.(Becker [3]). Assuming $\overbrace{{}^{\omega}\omega \cap M^{2n+2}(\alpha)}^{\text{for all } \alpha,}$ is countable, the real $\underbrace{\alpha^{\#}_{2n+2}}_{\notin M^{2n+2}(\alpha)}$ exists, the relation

$$P(\alpha, \beta) \Leftrightarrow \beta = \alpha^{\#}_{2n+2}$$

is Π^1_{2n+2}, and

$$\forall \alpha\, \exists !\, \beta P(\alpha, \beta). \qquad \square$$

Lemma 3.7. If β is $\Delta^1_{2n+2}(\alpha)$ real, then β is in $M^{2n+2}(\alpha)$.

Proof. Let β be a $\Delta^1_{2n+2}(\alpha)$ real. Then

$$\beta(i) = j \Leftrightarrow P(i, j), \text{ where } P \text{ is } \Sigma^1_{2n+2}(\alpha)$$

$$\Leftrightarrow M^{2n+2}(\alpha) \models P(i, j).$$

Since

$$ZF + V = M^{2n+2}(\alpha) \vdash \exists X\ (X \subseteq \omega \times \omega\ \&\ \forall i\ \ \forall j\ ((i, j) \in X \Leftrightarrow P(i,j))),$$

$$M^{2n+2}(\alpha) \models \exists X\ (X \subseteq \omega \times \omega\ \&\ \forall i\ \ \forall j\ ((i, j) \in X \Leftrightarrow P(i, j))).$$

This implies that there is a set X in $M^{2n+2}(\alpha)$ such that

$$X \subseteq \omega \times \omega,$$

and for each i, j

$$M^{2n+2}(\alpha) \models (i, j) \in X \Leftrightarrow P(i, j),$$

so by the Σ^1_{2n+2} correctness of $M^{2n+2}(\alpha)$

$$(i, j) \in X \Leftrightarrow M^{2n+2}(\alpha) \models P(i, j)$$

$$\Leftrightarrow P(i, j)$$

$$\Leftrightarrow \beta(i) = j.$$

Thus $X = \beta$ and hence $\beta \in M^{2n+2}(\alpha)$. $\square$

Corollary 3.8. Assuming ${}^{\omega}\omega \cap M^{2n+2}(\alpha)$ is countable, $\alpha^{\#}_{2n+2}$ is not in $\Delta^1_{2n+2}(\alpha)$. $\square$

Theorem 3.9. Assume that for all α $M^{2n+2}(\alpha) \cap {}^{\omega}\omega$ is countable. Then there is a total Π^1_{2n+2} function form ${}^{\omega}\omega$ into ${}^{\omega}\omega$ which is not in $\undertilde{\Delta}^1_{2n+2}$.

Proof. Consider the P in the lemma 3.7 which is a total Π^1_{2n+2} function from ${}^\omega\omega$ into ${}^\omega\omega$. Suppose that P is in $\underset{\sim}{\Delta}^1_{2n+..}$, so for some real α_0 P is in $\Delta^1_{2n+..}(\alpha_0)$. Then $\alpha_0{}^{\#}_{2n+1}$ is in $\Delta^1_{2n+..}(\alpha_0)$. This contradicts with corollary 3.8. □

Remark. Det($\underset{\sim}{\Sigma}^1_{2n+2}$) implies ${}^\omega\omega \cap M^{2n+2}(\alpha)$ is countable (see Becker [3]).

Luzin [26] proved the so-called "Théorème sur la projection d'ensemble d'unicité" . This says that let $\mathcal{E}$ be a Borel set in ${}^\omega\omega \times {}^\omega\omega$ and $\mathcal{E}_1$ denote the set of all points (α, β) of $\mathcal{E}$ such that the section $\mathcal{E}^{\langle\alpha\rangle}$ consists of a single point . Further, let E_1 be the projection of $\mathcal{E}_1$ on the first axis :

$$E_1 = \text{Proj } \mathcal{E}_1.$$

Luzin called $\mathcal{E}_1$ "l'ensemble d'unicité" of $\mathcal{E}$, and showed that both $\mathcal{E}_1$ and E_1 are $\underset{\sim}{\Pi}^1_1$ sets. Tugué and Tanaka [43] obtained the effective version of this classical fact and also proved this classical fact from its eff ctive version.

We will extend these facts to higher levels of projective sets.

Theorem 3.10. (Tugué and Tanaka [43] for n = 0). For each Δ^1_{2n+1} set B in ${}^\omega\omega \times {}^\omega\omega$ the pointset

$$P(\alpha) \Leftrightarrow \exists !\, \beta\, B(\alpha, \beta)$$

is also in Π^1_{2n+1}.

<u>Proof.</u> As is well known

(*) $$\exists!\ \beta B(\alpha, \beta) \Rightarrow B^{<\alpha>} \text{ is a } \Sigma^1_{2n+1}(\alpha) \text{ singlton}$$
$$\Rightarrow \exists\beta\in\mathcal{D}^1_{2n+1}(\alpha)\ B(\alpha, \beta).$$

Since, by the baunded quantification theorem, $\exists\beta\in\mathcal{D}^1_{2n+1}(\alpha)\ B(\alpha, \beta)$ is Π^1_{2n+1}, from (*) we have

$$\exists!\ \beta B(\alpha, \beta) \Leftrightarrow \exists\beta\in\mathcal{D}^1_{2n+1}(\alpha)\ B(\alpha, \beta)\ \&\ \forall\beta, \beta'\ (B(\alpha, \beta)$$
$$\&\ B(\alpha, \beta') \Rightarrow \beta = \beta').$$

This shows that P is Π^1_{2n+1}. $\square$

<u>Corollary</u> 3.11. (Tugué and Tanaka [43] for $n = 0$). For each $\underset{\sim}{\Delta}^1_{2n+1}$ set B in ${}^{\omega}\omega \times {}^{\omega}\omega$ the pointset

$$R(\alpha) \Leftrightarrow \exists!\ \beta B(\alpha, \beta)$$

is also in $\underset{\sim}{\Pi}^1_{2n+1}$. $\square$

Let $\mathcal{E}$ and $\mathcal{E}_1$ be the sets in ${}^{\omega}\omega \times {}^{\omega}\omega$ defined as follows.

$$(\alpha, \beta)\in\mathcal{E} \Leftrightarrow B(\alpha, \beta)$$

and

$$(\alpha, \beta)\in\mathcal{E}_1 \Leftrightarrow (\alpha, \beta)\in\mathcal{E}\ \&\ \forall\beta'\ ((\alpha, \beta')\in\mathcal{E} \Rightarrow \beta = \beta').$$

Since $\mathcal{E}$ is $\underset{\sim}{\Delta}^1_{2n+1}$, $\mathcal{E}_1$ is $\underset{\sim}{\Pi}^1_{2n+1}$. Since

$$\alpha\in E_1 \Leftrightarrow \exists!\ \beta((\alpha, \beta)\in\mathcal{E}),$$

we have

$$\alpha\in E_1 \Leftrightarrow R(\alpha),$$

so E_1 is also $\underset{\sim}{\Pi}^1_{2n+1}$.

$$(\alpha, \beta) \in \Phi \Leftrightarrow \alpha \in D \ \& \ \beta = \beta_0 .$$

Then Φ is a Σ^1_{2n+1} partial function from ${}^\omega\omega$ into ${}^\omega\omega$, but whose domain D is not in $\underset{\sim}{\Delta}{}^1_{2n+1}$.

Finally, we notice that every partial Δ^1_{2n+1} function from ${}^\omega\omega$ into ${}^\omega\omega$ can be extended to a total Δ^1_{2n+1} function.

Corollary 3.12. (Luzin [26], Tugué and Tanaka [43]). Let B be a $\underset{\sim}{\Delta}^1_{2n+1}$ set in ${}^{\omega}\omega \times {}^{\omega}\omega$ such that

$$\exists \beta B(\alpha, \beta) \Rightarrow \exists! \beta B(\alpha, \beta).$$

Then $\exists \beta B(\alpha, \beta)$ is also in $\underset{\sim}{\Delta}^1_{2n+1}$. $\square$

Corollary 3.13. (Tanaka [42]). The domain of a Δ^1_{2n+1} partial function $\Phi : {}^{\omega}\omega \to {}^{\omega}\omega$ is also Δ^1_{2n+1}.

Proof. In theorem 3.11 or corollary 3.12 set $B = \Phi$. Then $\mathrm{Dom}(\Phi) = R$ is in Δ^1_{2n+1}. $\square$

This corollary 3.13 does not hold for Δ^1_{2n+2} partial functions.

Theorem 3.14. There is a partial Π^1_{2n+1} function $\Phi : {}^{\omega}\omega \to {}^{\omega}\omega$ whose domain is not in $\underset{\sim}{\Delta}^1_{2n+2}$.

Proof. Let D be a set in ${}^{\omega}\omega$ such that

$$D \in \Sigma^1_{2n+2} - \underset{\sim}{\Delta}^1_{2n+2}$$

By the uniformization theorem, we can find a Π^1_{2n+1} partial function $\Phi : {}^{\omega}\omega \to {}^{\omega}\omega$ such that

$$\mathrm{dom}(\Phi) = D. \qquad \square$$

Cleary this theorem 3.14 holds for Σ^1_{2n+1} partial functions. Let D be a set in ${}^{\omega}\omega$ as follows

$$D \in \Sigma^1_{2n+I} - \underset{\sim}{\Delta}^1_{2n+1}.$$

Let β_0 be a fixed recursive real, and put

§4. On the $\underset{\sim}{\Pi}^1_n$ pointsets in ${}^\omega\omega \times {}^\omega\omega$ with countable sections.

One of well known theorems of Luzin is

Theorem 4.1. (Luzin [26]). Every $\underset{\sim}{\Delta}^1_1$ set in ${}^\omega\omega \times {}^\omega\omega$ with countable sections is the union of countably many $\underset{\sim}{\Delta}^1_1$ curves. $\square$

From this he also obtained

Corollary 4.2. (Luzin [26]). Every $\underset{\sim}{\Sigma}^1_1$ set in ${}^\omega\omega \times {}^\omega\omega$ with countable sections is the union of countably many $\underset{\sim}{\Sigma}^1_1$ curves. $\square$

Effective versions and extentions of these results are obtained by Kondô [23] and Tanaka [42] . We assume from now on Det($\underset{\sim}{\Delta}^1_{2n}$) holds.

Theorem 4.3. (Kondô [23] for $n = 0$, Tanaka [42]). For every Δ^1_{2n+1} set P in ${}^\omega\omega \times {}^\omega\omega$ with countable sections, we can find a Δ^1_{2n+1} set P^* in $\omega \times {}^\omega\omega \times {}^\omega\omega$ such that for each n and α $P^{*\langle n,\alpha\rangle}$ has at most ome element and

$$P(\alpha, \beta) \Leftrightarrow \exists n\, P^*(n, \alpha, \beta). \qquad \square$$

Corollary 4.4. (Kondô [23] for $n = 0$, Tanaka [42]). For every Σ^1_{2n+1} set P in ${}^\omega\omega \times {}^\omega\omega$ with countable sections, we can find a Σ^1_{2n+1} set P^* in $\omega \times {}^\omega\omega \times {}^\omega\omega$ such that for each n and α $P^{*\langle n,\alpha\rangle}$ has at most one element and

$$P(\alpha, \beta) \Leftrightarrow \exists n\, P^*(n, \alpha, \beta). \qquad \square$$

Then Luzin [26] proposed the question :

Does every $\underset{\sim}{\Pi}_1^1$ set in ${}^{\omega}\omega \times {}^{\omega}\omega$ can be covered by countably many $\underset{\sim}{\Pi}_1^1$ curves ?

A negative answer of this and its effective analoge are obtained by Tanaka [42] . Here we shall show

Theorem 4.5. Assume that Det($\underset{\sim}{\Sigma}_{2n+1}^1$). Then there is a Π_{2n+1}^1 set in ${}^{\omega}\omega \times {}^{\omega}\omega$ with countable sections which cannot be covered by either countably many $\underset{\sim}{\Sigma}_{2n+2}^1$ or $\underset{\sim}{\Pi}_{2n+2}^1$ curves.

Proof. Let G be a Σ_{2n+1}^1 set in $\omega \times {}^{\omega}\omega \times \omega$ whic is universal for all $\Sigma_{2n+1}^1(\alpha)$ sets in ω for each real α , and for each e and α $\beta_{e,\alpha}$ the characteristic function of the set $G^{\langle e,\alpha\rangle}$ in ω. Since for each e and α $G^{\langle e,\alpha\rangle}$ is $\Sigma_{2n+2}^1(\alpha)$, the real $\beta_{e,\alpha}$ is in $(\Sigma_{2n+2}^1(\alpha))_{\rho}$. Put

$$P = \{(\alpha, \beta) : \exists e\ (\beta = \beta_{e,\alpha})\}.$$

We want to prove that the set P is really Σ_{2n+2}^1. To this we need the following results of Kechris and Moschovakis (for details see Kechris [12]).

Theorem 4.6. (Solovay [36] for n = 0, Kechris and Moschovakis [13]). Assume that Det($\underset{\sim}{\Sigma}_{2n+1}^1$) holds. Then for each real α

(i) There is a lagest thin (not containing nonempty perfect subsets) $\Pi_{2n+1}^1(\alpha)$ set of reals $C_{2n+1}(\alpha)$. (For n = 0, this is also due independently to Sacks [30]).

(ii) If

$$C_{2n+2}(\alpha) = \{ \beta : \exists \gamma \in C_{2n+1}(\alpha)(\beta \text{ is recursive in } \gamma)\},$$

then $C_{2n+2}(\alpha)$ is a $\Sigma^1_{2n+2}(\alpha)$ set containing all thin $\Sigma^1_{2n+2}(\alpha)$ sets.

(iii) $C_{2n+2}(\alpha)$ is the largest countable $\Sigma^1_{2n+2}(\alpha)$ set. $\square$

We say that for each real α a wellordering $<_\alpha$ on a pointset A is $\Delta(\alpha)$-good, where $\Gamma(\alpha)$ is a pointclass and

$$\Delta(\alpha) = \Gamma(\alpha) \cap \check{\Gamma}(\alpha),$$

if and only if for each real $\beta \in A$

$$\{\gamma : \gamma <_\alpha \beta\} \text{ is countable}$$

and the relation

$$\mathrm{InSeq}_{<_\alpha}(\gamma, \beta) = \{(\gamma)_n : n \in \omega\} = \{\delta : \delta <_\alpha \beta\}$$

is in $\Delta(\alpha)$ for $\beta \in A$, i.e. for Q, R in $\Gamma(\alpha)$, $\check{\Gamma}(\alpha)$ respectively

$$\beta \in A \Rightarrow (\mathrm{InSeq}_{<_\alpha}(\gamma, \beta) \Leftrightarrow Q(\gamma, \beta) \Leftrightarrow R(\gamma, \beta)).$$

Theorem 4.7. (Kechris [12, 15]). Assume that $\mathrm{Det}(\underset{\sim}{\Sigma}^1_{2n+1})$ for $n \geq 1$. $C_{2n+2}(\alpha)$ admits a $\Sigma^1_{2n+2}(\alpha)$-good wellordering $<^{2n+2}_\alpha$ which has the property :

$$\beta <^{2n+2}_\alpha \gamma \Rightarrow \beta \in \Delta^1_{2n+2}(\alpha, \gamma). \qquad \square$$

Lemma 4.9. The set $\{(\alpha,\gamma) : \gamma \in D^{2n+2}(\alpha)\}$ is in Σ^1_{2n+2}.

Proof. This is clear from $\Sigma^1_{2n+2}(\alpha)$-goodness of wellordering $<^{2n+2}_\alpha$ and the definition of $D^{2n+2}(\alpha)$. $\square$

Lemma 4.10. If γ is in $D^{2n+2}(\alpha)$ and for each $\Delta^1_{2n+2}(\alpha)$ real β

$$\beta <^{2n+2}_\alpha \gamma,$$

then for each e

$$\beta_{e,\alpha} = \beta^{\gamma}_{e,\alpha}.$$

Proof. Since γ is in $C_{2n+2}(\alpha)$, it is sufficient to show that

$$(*) \qquad \exists\beta\phi(\beta, e, \alpha, i) \Leftrightarrow \exists\beta <^{2n+2}_\alpha \gamma\, \phi^{<^{2n+2}_\alpha \gamma}(\beta, e, \alpha, i).$$

Suppose that $\exists\beta\phi(\beta, e, \alpha, i)$. Then $\Sigma^1_{2n+1}(\alpha)$ set $\{\beta : \phi(\beta, e, \alpha, i)\}$ is nonempty. By the basis theorem for Σ^1_{2n+2}, there is a $\Delta^1_{2n+2}(\alpha)$ real β_0 such that

$$(**) \qquad \phi(\beta_0, e, \alpha, i).$$

Since γ is in $D^{2n+2}(\alpha)$ and $\beta_0 <^{2n+2}_\alpha \gamma$, by (**)

$$\beta_0 <^{2n+2}_\alpha \gamma\, \phi^{<^{2n+2}_\alpha \gamma}(\beta_0, e, \alpha, i).$$

Thus we have

$$\exists\beta <^{2n+2}_\alpha \gamma\, \phi^{<^{2n+2}_\alpha \gamma}(\beta, e, \alpha, i).$$

To prove right to left implication of (*), suppose that $\exists\beta <^{2n+2}_\alpha \gamma\, \phi(\beta, e, \alpha, i)$. Since γ is in $D^{2n+2}(\alpha)$, we have

$$\exists\beta\phi(\beta, e, \alpha, i).$$

So we have proved the formula (*). $\square$

Lemma 4.11. There is a real γ in $C_{2n+2}(\alpha)$ such that

(i) $L^{2n+2}(\alpha) \models$ "For each $(\Sigma^1_{2n+2}(\alpha))_\rho$ real β, $\beta <^{2n+2}_\alpha \gamma$ ".

(ii) $L^{2n+2}(\alpha) \models$ " $\gamma \in D^{2n+2}(\alpha)$".

Proof. Now we work in the model $L^{2n+2}(\alpha)$. Let γ be the $<^{2n+2}_\alpha$-least real such that for all real β

$$\beta <^{2n+2}_\alpha \gamma \Rightarrow \beta \in \mathcal{D}^1_{2n+3}(\alpha).$$

(i) Suppose that there is a $(\Sigma^1_{2n+2}(\alpha))_\rho$ real β such that

$$\neg \beta <^{2n+2}_\alpha \gamma.$$

Then

$$\gamma <^{2n+2}_\alpha \beta \vee \gamma = \beta.$$

So γ is a $\Delta^1_{2n+3}(\alpha)$ real. This contradicts with our choice of γ.

(ii) This can be proved by the induction on the construction of the Π^1_{2n+1} formula ϕ. Let ψ be a Σ^1_{2n} formula such that

$$\forall\delta\psi(\delta, \beta, e, \alpha, i) \Leftrightarrow \phi(\beta, e, \alpha, i).$$

Suppose that

(*) $$\forall \beta <^{2n+2}_\alpha \gamma\, (\psi^{<^{2n+2}_\alpha \gamma} (\delta, \beta, e, \alpha, i) \Leftrightarrow \psi(\delta, \beta, e, \alpha, i)).$$

We must show that for all $\beta <^{2n+2}_\alpha \gamma$,

(**) $$\forall \delta <^{2n+2}_\alpha \gamma\, \psi^{<^{2n+2}_\alpha \gamma}(\delta, \beta, e, \alpha, i) \Leftrightarrow \forall\delta\psi(\delta, \beta, e, \alpha, i).$$

To prove from left to right of formula (**), suppose that $\neg\forall\delta\psi(\delta, \beta, e, \alpha, i)$. Then $\exists\delta\neg\psi(\delta, \beta, e, \alpha, i)$. Since the Σ^1_{2n+2} set $\{\delta : \neg\psi(\delta, \beta, e, \alpha, i)\}$ is nonempty, by the basis theorem for Σ^1_{2n+2} there is a $\Delta^1_{2n+2}(\alpha)$ real δ_0 such that

$$\neg\psi(\delta_0, \beta, e, \alpha, i),$$

I.e.

$$\exists \delta <_{\alpha}^{2n+2} \gamma \ \neg\psi(\delta, \beta, e, \alpha, i).$$

Thus we have a contradiction. This prove that the left to right implication of (**). Using (*) from the right to the left of (**) is clear. Therefor (ii) is proved. □

Lemma 4.12. For each α there is a γ such that

(i) $\gamma \in D^{2n+2}(\alpha)$.

(ii) If β is a $(\Sigma^1_{2n+2}(\alpha))_P$ real, then $\beta <_{\alpha}^{2n+2} \gamma$.

Proof. Since the model $L^{2n+2}(\alpha)$ is Σ^1_{2n+2}-absolute, by lemmas 4.9 and 4.11 (i) and (ii) are clear. □

Lemma 4.13. The following equality holds.

$$P = \{(\alpha, \beta) : \exists e \ \exists \gamma (\gamma \in D^{2n+2}(\alpha) \ \& \ \beta <_{\alpha}^{2n+2} \gamma \ \& \ \beta = \beta^{\gamma}_{e,\alpha})\}.$$

Proof. To prove the inclusion from the left to the right, let $P(\alpha, \beta)$. Then there is an e such that

$$\beta = \beta_{e,\alpha}.$$

By lemma 4.12 there is a γ such that

$$\gamma \in D^{2n+2}(\alpha)$$

and

$$\beta <_{\alpha}^{2n+2} \gamma.$$

By lemma 4.10

$$\beta = \beta^{\gamma}_{e,\alpha}.$$

Conversely let (α, β) be such that

$$\exists e \quad \exists \gamma (\gamma \in D^{2n+2}(\alpha) \ \&\ \beta <_{\alpha}^{2n+2} \gamma \ \&\ \beta = \beta_{e,\alpha}^{\gamma}).$$

I for each $\Delta^1_{2n+2}(\alpha)$ real δ,

$$\delta <_{\alpha}^{2n+2} \gamma .$$

Then by lemma 4.10

$$P(\alpha, \beta).$$

If there is a $\Delta^1_{2n+2}(\alpha)$ real δ such that

$$\neg \delta <_{\alpha}^{2n+2} \gamma ,$$

then, since the absoluteness of the $\Delta^1_{2n+2}(\alpha)$ reals $\delta \in C_{2n+2}(\alpha)$,

$$\gamma <_{\alpha}^{2n+2} \delta \ \vee \ \gamma = \delta .$$

Thus

$$\beta <_{\alpha}^{2n+2} \delta .$$

Therefore β is a $\Delta^1_{2n+2}(\alpha)$ real. Since $\beta (= \beta_{e,\alpha}^{\gamma})$ is the characteristic function of the $\Sigma^1_{2n+2}(\alpha)$ set $\{ i : \beta(i) = 1 \}$, we have

$$P(\alpha, \beta). \qquad \Box$$

<u>Lemma</u> 4.14. The set P has the following properties :

(i) For each α $P^{\langle\alpha\rangle}$ is countable.

(ii) There is no countable family of $\underset{\sim}{\Sigma}^1_{2n+2}$ curves whose union contains P.

(iii) There is no countable family of $\underset{\sim}{\Pi}^1_{2n+2}$ curves whose union contains P.

Proof. (i) Since for each α there are only countably nany $\Sigma^1_{2n+2}(\alpha)$ sets in ω, $P^{\langle\alpha\rangle}$ is countable.

(ii) Assume, in order to obtain a contradiction, that there is a countable family $\{P_m\}$ of $\underset{\sim}{\Sigma}^1_{2n+2}$ curves such that

$$P \subseteq \bigcup_{m=0} P_m.$$

Take a real α_0 and a set S in ω as follows :

$$\forall m\ (P_m \in \Sigma^1_{2n+2}(\alpha_0)),$$

and

$$S \in \Sigma^1_{2n+2}(\alpha_0) - \Delta^1_{2n+2}(\alpha_0).$$

Let β_0 be the characteristic function of the set S. Then

$$P(\alpha_0, \beta_0),$$

so there is a m_0 such that

$$P_{m_0}(\alpha_0, \beta_0).$$

Since P_{m_0} is a curve, β_0 is in $\Sigma^1_{2n+2}(\alpha_0)$, so β_0 is a $\Delta^1_{2n+2}(\alpha_0)$ real. The set S is written as

$$S = \{ i : \beta_0(i) = 1\},$$

S is in $\Delta^1_{2n+2}(\alpha_0)$. This contradicts with our choice of S.

To prove lemma 4.14, (iii) we need the following lemma.

Lemma 4.16. There is a β in $P^{\langle\alpha\rangle}$ which is not $\Pi^1_{2n+2}(\alpha)$ singleton, i.e. $\{\beta\}$ is in $\Pi^1_{2n+2}(\alpha)$.

Proof. Since $P^{\langle\alpha\rangle}$ is $\Sigma^1_{2n+2}(\alpha)$ and $\{\beta : \{\beta\} \in \Pi^1_{2n+2}(\alpha)\}$ is in $\Pi^1_{2n+3}(\alpha) - \Sigma^1_{2n+3}(\alpha)$,

$$\emptyset \neq \{\beta : \{\beta\} \in \Pi^1_{2n+2}(\alpha) \ \& \ P(\alpha, \beta)\} \subsetneq P^{<\alpha>}. \qquad \square$$

Proof of lemma 4.14, (iii). Suppose that there is a countable family $\{P_m\}$ of $\underset{\sim}{\Pi}^1_{2n+2}$ curves such that

$$P \subseteq \bigcup_{m=0} P_m.$$

Then there is a α_0 such that

$$\forall m \ (P_m \in \Pi^1_{2n+2}(\alpha_0)).$$

By lemma 4.16, there is a β_0 in $P^{<\alpha_0>}$ which is not $\Pi^1_{2n+2}(\alpha_0)$ singleton. Since

$$P(\alpha_0, \beta_0),$$

there is a m_0 such thst

$$P_{m_0}(\alpha_0, \beta_0).$$

Since P_{m_0} is a $\Pi^1_{2n+2}(\alpha_0)$ curve, β_0 is a $\Pi^1_{2n+2}(\alpha_0)$ singleton. Thus we have a contradiction. $\square$

Let $h : {}^\omega\omega \times {}^\omega\omega \longrightarrow {}^\omega\omega$ be a recursive homeomorpism and for $i=0,1$ $h_i : {}^\omega\omega \longrightarrow {}^\omega\omega$ recursive functions such that for each α,

$$h(h_0(\alpha), h_1(\alpha)) = \alpha.$$

By the uniformization theorem there is a Π^1_{2n+1} set P^* in ${}^\omega\omega \times {}^\omega\omega \times {}^\omega\omega$ such that

$$\mathrm{dom}(P^*) = P$$

and

$$\forall \gamma, \gamma' \ (P^*(\alpha, \beta, \gamma) \ \& \ P^*(\alpha, \beta, \gamma') \Rightarrow \gamma = \gamma').$$

Put

$$P^{**} = \{(\alpha, h(\beta, \gamma)) : P^*(\alpha, \beta, \gamma)\}.$$

We will show that the Π^1_{2n+1} set P^{**} cannot covered by either countably many $\underset{\sim}{\Sigma}^1_{2n+2}$ or $\underset{\sim}{\Pi}^1_{2n+2}$ curves.

Suppose that there is countably many $\underset{\sim}{\Sigma}^1_{2n+2}$ curves $\{P_m\}$ such that

$$P^{**} \subseteq \bigcup_{m=0} P_m .$$

Take α_0 such that

$$\forall m\, (P_m \in \Sigma^1_{2n+2}(\alpha_0))$$

and, take β_0 in ω_2 such that

$$\beta_0 \in (\Sigma^1_{2n+2}(\alpha_0))_\rho - \Delta^1_{2n+2}(\alpha_0)$$

and

$$P(\alpha_0, \beta_0).$$

Let γ_0 be a real such that

$$P^{**}(\alpha_0, h(\beta_0, \gamma_0)).$$

Then there is a m_0 such that

$$P_{m_0}(\alpha_0, h(\beta_0, \gamma_0)),$$

so $h(\beta_0, \gamma_0)$ is a $\Delta^1_{2n+2}(\alpha_0)$ real. By the substitusion property (see Moschovakis [29]) $\beta_0 = h_0(h(\beta_0, \gamma_0))$ is a $\Delta^1_{2n+2}(\alpha_0)$ real. This contradicts with our choice of β_0.

Now suppose that there is countably many Π^1_{2n+2} curves $\{P_m\}$ such that

$$P^{**} \subseteq \bigcup_{m=0} P_m$$

Take α_0 such that

$$\forall m \, (P_m \in \Pi^1_{2n+2}(\alpha_0)) \ ,$$

and take β_0 such that

$$P(\alpha_0, \beta_0)$$

but β_0 is not $\Pi^1_{2n+2}(\alpha_0)$ singleton (by lemma 4.16 such a β_0 exists). Let γ_0 be a real such that

$$P^{**}(\alpha_0, h(\beta_0, \gamma_0)).$$

Then there is a m_0 such that

$$P_{m_0}(\alpha_0, h(\beta_0, \gamma_0)),$$

so $h(\beta_0, \gamma_0)$ is a $\Pi^1_{2n+2}(\alpha_0)$ singleton. By the subustitusion property $\beta_0 = h_1(h(\beta_0, \gamma_0))$ is also $\Pi^1_{2n+2}(\alpha_0)$ singleton. This contradicts with our choice of β_0. Therefor the proof of theorem 4.5 is completed. $\square$

Since by theorem 4.1 every Π^1_{2n+1}, Σ^1_{2n+2} and Π^1_{2n+2} set in ${}^\omega\omega \times {}^\omega\omega$ with countable sections can be covered by countably many Δ^1_{2n+3} curves, theorem 4.5 is the best possible extension of theorem II.7 of Tanaka [42] .

§5. A generalization of a Friedman's theorem.

Friedman proved the following theorem.

Theorem 5.1. (See Mathais [28 ; T3210]). There is an infinitely countable Π^1_1 set of reals every member of which except one is Δ^1_2 real .

Using the method developed in §4, we shall prove the following generalization of theorem 5.1.

Theorem 5.2. Assume that $\mathrm{Det}(\underset{\sim}{\Sigma}^1_{2n+1})$ for $n > 0$. There is an infinitely countable Π^1_{2n+1} set of reals every member of which except one is Δ^1_{2n+2} real.

Proof. Let G be a Σ^1_{2n+2} set in $\omega\times\omega$ which is universal for all Σ^1_{2n+2} sets in ω, ϕ a Π^1_{2n+1} formula such that

$$\exists\beta\phi(\beta, i) \Leftrightarrow G(i, i).$$

Let $\phi^{<^{2n+2}\gamma}(\beta, i)$ be the Σ^1_{2n+2} formula which is obtaind from the formula $\phi(\beta, i)$ by replacing quantifiers $\forall\delta$, $\exists\delta$ in ϕ by $\forall\delta<^{2n+2}\gamma$, $\exists\delta<^{2n+2}\gamma$ respectively, where $<^{2n+2}$ is a Σ^1_{2n+2}-good wellordering on C_{2n+2} such that

$$\beta<^{2n+2}\alpha \Rightarrow \beta\in\Delta^1_{2n+2}(\alpha).$$

Lemma 5.3. There is a real γ such that if β is a Δ^1_{2n+2} real then

$$\beta<^{2n+2}\gamma.$$

Proof is similar one of lemma 4.12. $\square$

Lemma 5.4. Let γ be such that if β is a Δ^1_{2n+2} real then $\beta <^{2n+2} \gamma$. Then following formula holds.

$$\exists\beta\phi(\beta, i) \Leftrightarrow \exists\beta <^{2n+2}\gamma\ \phi^{<^{2n+2}\gamma}(\beta, i).$$

Proof is similar one of lemma 4.11of lemma 4.10. $\square$

Put

$$A = \{\varepsilon \in {}^{\omega}2 : \forall i(\varepsilon(i) = 1 \Leftrightarrow \exists\beta <^{2n+2}\gamma\ \phi^{<^{2n+2}\gamma}(\beta, i))\}.$$

Then A is Σ^1_{2n+2}. Since the set $\{i : G(i, i)\}$ is in $\Sigma^1_{2n+2} - \Delta^1_{2n+2}$, its characteristic function ε^* is in $(\Sigma^1_{2n+2})_\rho - \Delta^1_{2n+2}$.

By lemmas 5.3, 5.4, there is a real γ such that

$$\begin{aligned}\varepsilon^*(i) = 1 &\Leftrightarrow G(i, i)\\ &\Leftrightarrow \exists\beta\phi(\beta, i)\\ &\Leftrightarrow \exists\beta <^{2n+2}\gamma\ \phi^{<^{2n+2}\gamma}(\beta, i).\end{aligned}$$

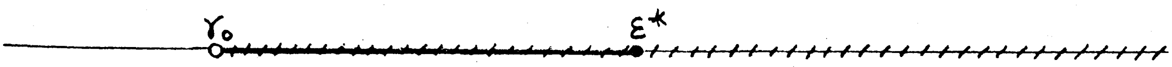
Thus ε^* is in A. By the basis theorem for Σ^1_{2n+2}, A must have infinitely many Δ^1_{2n+2} reals. Let γ_0 be the smallest real such that

$$\beta \in \Delta^1_{2n+2} \Rightarrow \beta <^{2n+2} \gamma_0$$

(such γ_0 exsists by lemma 5.3). Then, by lemma 5.4, if $\gamma_0 \leq^{2n+2} \gamma$

$$\exists\beta <^{2n+2}\gamma\ \phi^{<^{2n+2}\gamma}(\beta, i) \Leftrightarrow \exists\beta\phi(\beta, i).$$

Since ε^* is in Δ^1_{2n+3}, $\gamma_0 \leq^{2n+2} \varepsilon^*$. Clearly between γ_0 and ε^* there is no elements of A.



In this interval, there is no element of A

In this interval, there is only one element of A, i.e. the real ε^*

Let A^* be a Π^1_{2n+1} set in ${}^\omega\omega \times {}^\omega\omega$ such that

$$\mathrm{dom}(A^*) = A$$

and

$$\forall \beta, \beta' \, (A^*(\alpha, \beta) \;\&\; A^*(\alpha, \beta') \Rightarrow \beta = \beta')$$

by the uniformization theorem such A^* can be finded). Now put

$$A^{**} = h(A^*),$$

where h is a recursive homeomorphism from ${}^\omega\omega \times {}^\omega\omega$ onto ${}^\omega\omega$. Let β^* be the unique real such that

$$A^*(\varepsilon^*, \beta^*).$$

Since ε^* is $(\Sigma^1_{2n+2})_P - \Delta^1_{2n+2}$, β^* is a $\Delta^1_{2n+3} - \Delta^1_{2n+2}$ real, so is $h(\varepsilon^*, \beta^*)$ which is in A^{**}.

$$\begin{aligned} A^{**}(\alpha) \;\&\; \alpha \neq h(\varepsilon^*, \beta^*) &\Rightarrow \exists \varepsilon \neq \varepsilon^* \; \exists \beta \, (\alpha = h(\varepsilon, \beta) \;\& \\ &\qquad A^*(\varepsilon, \beta) \;\&\; A(\varepsilon)) \\ &\Rightarrow \alpha \in \Delta^1_{2n+2}. \end{aligned}$$

Thus the Π^1_{2n+1} set A^{**} has just one non-Δ^1_{2n+2} real $h(\varepsilon^*, \beta^*)$ and other members of A^{**} are all Δ^1_{2n+2} reals. Therefor theorem 5.2 is proved. $\square$

From this theorem we have

Corollary 5.5. Effective perfect set theorem for Σ^1_{2n+2} fail. $\square$

Finally, we state one more theorem which is essentially included in the theorem 4.5, but it is interrested itself.

<u>Theorem</u> 5.6. Assume that $\mathrm{Det}(\underset{\sim}{\Sigma}^1_{2n+1})$. Then there is a Π^1_{2n+1} set in ${}^\omega\omega$ which contains at least one non Π^1_{2n+2} singleton real.

<u>Proof</u>. Let G be a Σ^1_{2n+2} set in $\omega\times\omega$ which is univarsal for all Σ^1_{2n+2} sets in ω. Now put

$$A = \{\xi\in{}^\omega 2 : \exists e \ \ \exists i\ (\xi(i) = 1 \Leftrightarrow G(e,\ i)\}.$$

Then appling the proof method of theorem 4.5 to the set A using the fact

$$\emptyset \neq \{\xi\in{}^\omega 2 : \{\xi\}\in\Pi^1_{2n+2} \ \& \ \xi\in A\} \subsetneq A. \qquad \square$$

§6. Enumerability.

We begin an application of theorem 5.2 to the problem of effective enumerability of countable projective sets of reals.

Since there is non-$\underset{\sim}{\Delta}^1_{2n+1}$ infinitely countable Σ^1_{2n+1} set in ${}^{\omega}\omega$, the elements of a Σ^1_{2n+1} set of reals are not necessarily enumerated by a $\underset{\sim}{\Delta}^1_{2n+1}$ function, but

Theorem 6.1. Assume that Det($\underset{\sim}{\Delta}^1_{2n}$). Then the elements of a countable Δ^1_{2n+1} set in ${}^{\omega}\omega$ can be enumerated by a Δ^1_{2n+1} function. $\square$

Theorem 6.2. (Tanaka [40] for $n = 0$). Assume that Det($\underset{\sim}{\Delta}^1_{2n}$) An infinitely countable Σ^1_{2n+1} set P in ${}^{\omega}\omega$ cannot be contain Δ^1_{2n+1} reals of arbitrarily high degrees : that is, there is a Δ^1_{2n+1} real ε such that

(*) $$\forall \alpha\,(P(\alpha) \Rightarrow \alpha \text{ is recursive in } \varepsilon).$$

Proof. By Moschovakis [29; 4F.5], there is Δ^1_{2n+1} real ε such that

$$P \subseteq \{(\varepsilon)_0, (\varepsilon)_1, (\varepsilon)_2, \ldots\}.$$

Using this real ε we have (*). $\square$

Is was a difficult work the one performs any enumeration of a countable $\underset{\sim}{\Pi}^1_1$ set in ${}^{\omega}\omega$. In fact it is undecidable in ZFC. But under the projective determinacy, we can prove , using theorem 5.2,

Theorem 6.3. Assume that Det($\underset{\sim}{\Sigma}^1_{2n+1}$). Then there is a Π^1_{2n+1} set in ${}^{\omega}\omega$ which cannot be enumerated by a Δ^1_{2n+2} function. $\square$

Therefor the following theorem is the best possible one.

<u>Theorem</u> 6.4. Assume that $\mathrm{Det}(\underset{\sim}{\Sigma}^1_{2n+1})$. Then every infinitely countable Π^1_{2n+1} set in ${}^{\omega}\omega$ can be enumerated without repetition by a Δ^1_{2n+3} function.

<u>Proof.</u> Let P be a Π^1_{2n+1} set in ${}^{\omega}\omega$. Put

$$P_0(\beta) \Leftrightarrow \forall i, j\ (i \neq j \Rightarrow (\beta)_i \neq (\beta)_j)\ \&\ \forall i\ (P((\beta)_i)\ \&$$
$$\forall \alpha (P(\alpha) \Rightarrow \exists i\ (\alpha = (\beta)_i)).$$

Then P_0 is a countable Δ^1_{2n+3} set, by the Δ-uniformization criterion (Moschovakis [29; 4D.4]), we can find a Δ^1_{2n+3} set P_0^* such that

$$\exists \beta\ P_0^*(\beta)$$

and

$$\forall \beta, \beta'\ (P_0^*(\beta)\ \&\ P_0^*(\beta') \Rightarrow \beta = \beta').$$

Now we can define the function $\varphi : \omega \longrightarrow {}^{\omega}\omega$ by

$$\varphi(i) = \alpha \Leftrightarrow \exists \beta (P_0^*(\beta)\ \&\ (\beta)_i = \alpha)$$
$$\Leftrightarrow \forall \beta (P_0^*(\beta) \Rightarrow (\beta)_i = \alpha).$$

Thus the function φ is in Δ^1_{2n+3} and enumerats without repetition the elements of P. $\square$

Closing this section, we state an extention of Sampei [33] and Tanaka [39] theorem.

Theorem 6.5. (sampei [33] and Tanaka [39] for $n = 0$). Assume that $\mathrm{Det}(\underset{\sim}{\Delta}{}^1_{2n})$. Then every Σ^1_{2n+1} set in ${}^{\omega}\omega$ can be enumerated by a Δ^1_{2n+2} function without repitition.

Proof is similar one of theorem 6.4, using the uniformization theorem for Δ^1_{2n+2} (see Kondô [21]) instead of the Δ-uniformization criterion. $\square$

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