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A decision method for a set of first order classical formulas and its application to decision problems for non-classical propositional logics.

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I. Main Theorem

Let LN be the first order classical predicate logic without equality which has a fixed binary predicate symbol R , unary predicate symbols P1,..,PN and no other non-logical constant symbols. Suppose that X is a set of sentences in LN. Then *a decision method for X* is a method by which, given a sentence A in X , we can decide in a finite number of steps whether or not it has a model. X is said to be decidable if there is a decision method for X. It is well-known that the set of all the R-free sentences (sentences is LN which have no occurrences of R) is decidable , but the set of all the sentences in LN is not. R-formulas are formulas belonging to the least set X such that; (i) All R-free formulas belong to X, (ii) X is closed under $\neg, \wedge, \vee, \supset$, (iii) If $A(x)$ belongs to X, then $\exists v A(v)$, $\exists v(R(x,v) \wedge A(v))$, $\exists v(R(v,x) \wedge A(v))$ belong to X. R-positive formulas are formulas which have no negative occurrences of R. Also, Tr is the sentence $\forall u \forall v \forall w(R(u,v) \wedge R(v,w). \supset R(u,w))$ and Sym is the sentence $\forall u \forall v(R(u,v) \supset R(v,u))$. Let FN be the set of finite conjunctions of sentences: R-sentences, R-positive sentences, Tr and Sym. Then, our main theorem is:

MAIN THEOREM. FN is decidable.

In fact, we show that for each sentence A in FN, we can calculate a natural number $n(A)$ such that if A has a model, then A has a model whose cardinality is at most $n(A)$. This fact clearly implies our main theorem.

II. Applications.

Suppose that L is a formal logic. Then a decision method for L is a method by which, given a formula of L, we can decide in a finite number of steps whether or not it is provable in L.

1) Intuitionistic propositional logic.

Let IPL be the intuitionistic propositional logic whose propositional variables are $p1,\dots,pN$. For each formula A in IPL, and each free variable x in LN, let (A,x) be the formula in LN defined by: (pi,x) is $Pi(x)$, $(\neg A,x)$ is $\forall v(R(x,v) \supset \neg (A,v))$, $(A \wedge B,x)$ is $(A,x) \wedge (B,x)$, $(A \vee B,x)$ is $(A,X) \vee (B,x)$, and, $(A \supset B,x)$ is $\forall v(R(x,v) \supset ((A,v) \supset (B,v)))$.

Then, by Kripke's completeness theorem for IPL, we have:

Completeness Theorem for IPL. For each formula A in IPL, A is provable in IPL iff the sentence $Tr \wedge \bigwedge_{i=1}^{N} Tr(Pi) \wedge \exists v \neg (A,v)$ has no models, where $Tr(Pi)$ is the R-sentence $\forall u(Pi(u) \supset \forall v(R(u,v) \supset P_i(v))$.

Since Tr, $\wedge \bigwedge_{i=1}^{N} \mathrm{Tr}(Pi) \wedge \exists v \neg (A,v)$ belongs to FN, our main theorem clearly implies that the logic IPL is decidable.

2) Modal propositonal logics.

Let MPL be the modal propositional language whose logical constants are $\neg, \wedge, \vee, \supset$ and $\Box$, and whose propositional variables are $p1, \ldots, pN$. For each formula A in MPL and each free variable x in LN, let $\langle A,x\rangle$ be the formula in LN defined by; $\langle pi,x\rangle$ is $Pi(x)$, $\langle \neg A,x\rangle$ is $\neg\langle A,x\rangle$, $\langle A \wedge B,x\rangle$ is $\langle A,x\rangle \wedge \langle B,x\rangle$, $\langle A \vee B,x\rangle$ is $\langle A,x\rangle \vee \langle B,x\rangle$, $\langle A \supset B,x\rangle$ is $\langle A,x\rangle \supset \langle B,x\rangle$ and $\langle \Box A,x\rangle$ is $\forall v(R(x,v) \supset \langle A,v\rangle)$.

Let M, S4, B, S5 be modal propositional logics in Kripke [ ], whose language is MPL. Then, by Kripke's completeness theorem for modal logics, we have:

Completeness Theorem for modal logics. For any formula A in MPL,

(i) A is provable in M iff $\forall u R(u,u) \wedge \exists v \langle \neg A,v\rangle$ has no models,

(ii) A is provable in S4 iff $\forall u R(u,u) \wedge \mathrm{Tr} \wedge \exists v \langle \neg A,v\rangle$ has no models,

(iii) A is provable in B iff $\forall u R(u,u) \wedge \mathrm{Sym} \wedge \exists v \langle \neg A,v\rangle$ has no models,

(iv) A is provable in S5 iff $\forall u R(u,u) \wedge \mathrm{Tr} \wedge \mathrm{Sym} \wedge \exists v \langle \neg A,v\rangle$ has no models.

Since finite conjunctions of sentences $\forall u R(u,u)$, Tr, Sym, and $\exists v \langle \neg A,v\rangle$ belong to FN, our main theorem clearly implies that four logics M, S4, B, S5 are all decidable.

III. A proof.

1) R-degree. For each R-formula A, let R-deg(A) be the non-negative integer, called the R-degree of A, defined by: R-deg(A) = 0 if A is R-free, $\text{R-deg}(\neg A) = \text{R-deg}(A)$, $\text{R-deg}(A \wedge B) = \text{R-deg}(A \vee B) = \text{R-deg}(A \supset B) = \max \{ \text{R-deg}(A), \text{R-deg}(B) \}$, $\text{R-deg}(\exists v A(v)) = \text{R-deg}(A(x))$, and $\text{R-deg}(\exists v(R(x,v) \wedge A(v))) = \text{R-deg}(\exists v(R(v,x) \wedge A(v)) = \text{R-deg}(A(x)) + 1$.

2) R-basic sentences.

Define $\Sigma n (n = 0,1,2,\ldots)$ and Σ by: $\Sigma 0 = \text{Pow}(\{1,\ldots,N\})$
$\Sigma(n+1) = \Sigma n \times \text{Pow}(\Sigma n) \times \text{Pow}(\Sigma n), (n = 0,1,2,\ldots)$ and $\Sigma = \bigcup_{n<\omega} \Sigma n$,
where Pow(Z) is the power set of Z.

For each σ in Σ, let $A(\sigma,x)$ be the unary formula defined by:

$A(\sigma,x)$ is $\bigwedge_{i \in \sigma} Pi(x) \wedge \bigwedge_{i \notin \sigma} Pi(x)$ if $\sigma \in \Sigma 0$ and

$$A(\sigma,x) \text{ is } A(\nu,x) \wedge \bigwedge_{\alpha \in 1} \exists v(R(v,x) \wedge A(\alpha,v)) \wedge \bigwedge_{\alpha \notin 1} \neg \exists v(R(v,x) \wedge A(\alpha,v)) \wedge$$
$$\bigwedge_{\alpha \in r} \exists v(R(x,v) \wedge A(\alpha,v)) \wedge \bigwedge_{\alpha \notin r} \neg \exists v(R(x,v) \wedge A(\alpha,v))$$
$$\text{if } \sigma = \langle \nu, 1, r \rangle \in \Sigma(n+1).$$

Then, $A(\sigma,x)$ is an R-formula whose R-degree is n if $\sigma \in \Sigma n$.

For each subset X of Σn, let AX be the sentence;

$$\bigwedge_{\sigma \in X} \exists v A(\sigma,v) \wedge \bigwedge_{\sigma \notin X} \neg \exists v A(\sigma,v)$$

AX $(X \subseteq \Sigma n)$ are called R-basic sentences of R-degree n.

3) Representation theorem.

(1) For each R-formula $A(x,\ldots,y)$ of R-degree n, whose free variables are among $x,\ldots,y$, we can concretely construct a Boolean combination $B(x,\ldots,y)$ of formulas of the forms: $\exists vA(\sigma,v), A(\sigma,x), \ldots, A(\sigma,y)$, where $\sigma \in \Sigma n$ such that A and B are equivalent in LN.

(2) For each R-sentence A of R-degree n, we can concretely obtain finite subsets $X1,\ldots,Xn$ of Σn such that $AX1 \vee \cdots \vee AXn$ and A are equivalent in LN.

4) Reduction lemmas.

Let GN (HN) be the set of sentences in FN which are finite conjunctions of the sentences: R-basic sentence (R-basic sentences of R-degree 1), R-positive sentences, Tr and Sym. Then $HN \subsetneq GN \subsetneq FN$.

Reduction Lemma 1. If GN is decidable, then FN is decidable.

Reduction Lemma 2. If HN (N = 1,2,...) are all decidable, then GN (N = 1,2,....) are all decidable.

5) Main Lemma. For each sentence A in HN, if A has a model, then A has a model of cardinality no more than $2^{\nu} \times 2^{2^{\nu}} \times 2^{2^{\nu}}$.

Clearly, Reduction Lemma 1, Reduction lemma 2 and Main Lemma imply our main theorem.