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Completeness of A[B]

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0. Introduction.

In [4], Dwinger considered the completeness of Boolean powers of complete Boolean algebras. Dwinger obtained a necessary and sufficient condition in algebraic form:

Theorem (Dwinger[4]). Let A and B be complete Boolean algebras. A[B] is complete if and only if $\bigvee_{x\varepsilon A}\Big(\bigwedge_{y\nleq x}\sim f(y)\cdot\bigwedge_{x\nleq z}\bigvee_{u\nleq z}f(u)\Big)=1$ for each $f:A\longrightarrow B$.

In this paper, we consider some relationship which exists between the completeness of A[B] and the distributivity of B.

In the notation of Boolean valued models of set theory, the Boolean power A[B] is isomorphic to

$$\hat{A}=\{f\varepsilon V^{(B)}\mid [\![f\varepsilon\check{A}]\!]^{(B)}=1\}$$

where A is an element of $V^{(B)}$ such that $\check{A}=\{\check{a}\mid a\varepsilon A\}\times\{1\}$. We can have a better perspective, if we deal with $\check{A}$ in $V^{(B)}$ instead of A[B]. By virtue of 5.5 of Solovay and Tennenbaum[15], A[B] is complete if and only if $[\![\check{A}$ is complete $]\!]^{(B)}=1$. Since

$$[\![\check{A}\text{ is complete}]\!]^{(B)}$$
$$=[\![\forall X\subset\check{A}\,\exists x\varepsilon\check{A}[\forall y\varepsilon X[y\leq x]\wedge\forall z\varepsilon A[\forall u\varepsilon X[u\leq z]\longrightarrow x\leq z]]]\!]^{(B)}$$
$$=\bigwedge_{f:A\longrightarrow B}\Big(\bigvee_{x\varepsilon A}\Big(\bigwedge_{y\nleq x}\sim f(y)\cdot\bigwedge_{x\nleq z}\bigvee_{u\nleq z}f(u)\Big)\Big),$$

we can obtain a proof of Dwinger's theorem which uses Boolean valued models of set theory. This suggests why we are going to work in $V^{(B)}$. With respect to basicfacts on $V^{(B)}$ we refer the reader to [8,9,15]. We assume that $V^{(B)}$ is separated,

i.e., $[\![x=y]\!]^{(B)}=1$ is equivalent to $x=y$ for every $x,y \in V^{(B)}$.
We write $[\![\Phi]\!]=1$ instead of $[\![\Phi]\!]^{(B)}=1$, if there is no confusion.

Our main results are as follows:

Theorem. Let A and B be complete Boolean algebras, κ be a infinite cardinal and $\{a(\alpha) \in A^{+} | \alpha < \kappa\} \in \mathrm{Part}(A)$. Suppose that $[\![\check{\kappa} < \delta$ and δ is a cardinal$]\!]^{(B)}=1$. The following conditions are equivalent.

(i) $[\![\check{A}$ is δ-complete$]\!]^{(B)}=1$.

(ii) $[\![(A \restriction a(\alpha))^{\vee}$ is δ-complete$]\!]^{(B)}=1$ for every $\alpha < \kappa$ and B satisfies the (κ,σ)-DL where $\sigma(\alpha)=|A \restriction a(\alpha)|$ for every $\alpha<\kappa$.

Corollary 4. Let A be a.c.c.c. complete Boolean algebra and B be a complete Boolean algebra.

(1) $[\![\check{A}$ is countably complete$]\!]^{(B)}=1$ if and only if B satisfies the $(\omega,|A|)$-DL.

(2) In particular, $[\![\check{A}$ is complete$]\!]^{(B)}=1$ implies that B satisfies the $(\omega,|A|)$-DL.

(3) If $[\![\check{A}$ is c.c.c.$]\!]^{(B)}=1$, then $[\![\check{A}$ is complete$]\!]^{(B)}=1$ if and only if B satisfies the $(\omega,|A|)$-DL.

(For precise definitions see §1.)

Let F be a free Boolean algebra with κ generators and A be its completion. A satisfies the c.c.c.. Since ' F is a free Boolean algebra' is absolute, $[\![\check{F}$ is free$]\!]^{(B)}=1$ for every complete Boolean algebra B. $[\![\check{F} \subset \check{A} \subset \overline{\check{F}}]\!]=1$ where $\overline{\check{F}}$ is the completion of $\check{F}$ in $V^{(B)}$. Hence

$$[\![A \text{ satisfies the c.c.c.}]\!] \geq [\![F \text{ satisfies the c.c.c.}]\!] = 1 .$$

There is an (ω,∞)-distributive complete Boolean algebra which is not $(\omega_1,2)$-distributive (see [13]). Hence Corollary 4.(3) shows the negative answer to the question in [4] whether $|A|=\kappa$ and completeness of A[B] imply that B satisfies the (κ,κ)-DL.

The converse of Corollary 4.(2) is not a theorem of Zermelo-Fraenkel set theory(ZFC). It is proved from the negation of Suslin's Hypothesis that there is a c.c.c. (ω,∞)-distributive atomless complete Boolean algebra (see [6]). Such a complete Boolean algebra C satisfies $[\![\check{C} \text{ is complete}]\!]^{(C)} = 0$. It remains open, however, whether the converse of Corollary 4.(2) is consistent with ZFC. Corollary 4.(3) shows that it is closely related to the nonexistence of c.c.c. Boolean algebra which is not a.c.c.c.[1] (see 1). Since $[\![\check{C} \text{ is c.c.c.}]\!]^{(C)} = 0$, it also remains open whether it is a theorem of ZFC that '$[\![\check{A} \text{ is complete}]\!]^{(B)} = 1$ if and only if $[\![\check{A} \text{ is c.c.c.}]\!]^{(B)} = 1$ and B satisfies the $(\omega,|A|)$-DL for every c.c.c. complete Boolean algebra A and complete Boolean algebra B'.

In §1 we give basic notations and definitions. In §2 we prove the theorem and its corollaries. In §3 we investigate sufficient conditions for $[\![\check{A} \text{ is c.c.c.}]\!]^{(B)} = 1$ where B is $(\omega,|A|)$-distributive.

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1) Professor Jech asked me whether we can construct such a Boolean algebra in ZFC. I cannot construct one yet.

§1. Preliminaries.

We denote the first infinite cardinal by ω and the first uncountable cardinal by ω_1. We use letters α, β for ordinals and δ, κ, λ for infinite cardinals. And we use letters A,B,C for infinite Boolean algebras. We denote the finite Boolean operations by $+_B$, $\cdot_B$, $\sim_B$, the least element by 0_B and the greatest element by 1_B. $\leqq_B$ is the canonical ordering of B. We shall omit the subscripts if ther is no confusion. For every $b \varepsilon B^+ = B - \{0\}$, $B \restriction b$ is the Boolean algebra $\{a \mid a \leqq b\}$. An element b of B^+ is an atom of B if there is no element a such that $0<a<b$. B is atomless if it has no atom. B is atomic if for every b B^+ there is an atom a such that $a \leqq b$. The cardinality of a set S is denoted by $|S|$. B is κ-complete if the supremum $\bigvee S$ exists for every subset S of B such that $|S|<\kappa$. B is countably complete if it is ω_1-complete. B is complete if it is λ-complete for every λ. A partition of B is a maximal pairwise disjoint family. The set of all partitions of B is denoted by Part(B). B satisfies the κ-chain condition or is κ-c.c. if there is no partition P of B such that $|P|=\kappa$. Sat(B) is the least cardinal κ such that B is κ-c.c.. The ω_1-chain condition is called the countable chain condition (c.c.c.). A is absolutely c.c.c. (a.c.c.c.) if $[\![\check{A} \text{ is c.c.c.}]\!]^{(B)} = 1$ for every complete B. Let σ be a function from κ to λ. Complete B satisfies the (κ,σ)-distributive law ((κ,σ)-DL) or is (κ,σ)-distributive if

$$\bigwedge_{\alpha<\kappa} \bigvee_{\beta<\sigma(\alpha)} b_{\alpha,\beta} = \bigvee_{f \varepsilon \prod_{\alpha<\kappa} \sigma(\alpha)} \bigwedge_{\alpha<\kappa} b_{\alpha,f(\alpha)}$$

for every $\{\{b_{\alpha,\beta} \mid \beta<\sigma(\alpha)\} \mid \alpha<\kappa\}$ Part(B).

If σ is a constant function such that $\sigma(\alpha)=\delta$ for every $\alpha<\kappa$, then the (κ,σ)-DL is the usual (κ,δ)-distributive law. B satisfies the $(<\delta,\lambda)$-DL if it satisfies the (κ,λ)-DL for every $\kappa<\delta$. B satisfies the (κ,∞)-DL if it satisfies the (κ,λ)-DL for every λ. We note that B satisfies the (κ,σ)-DL if and only if $[\![(\prod_{\alpha<\kappa} \sigma(\alpha))^{\vee} = \prod^{\vee}_{\alpha<\kappa} \check{\sigma}(\alpha)]\!]^{(B)}=1$. Let $\kappa<\mathrm{Sat}(A)$ and $\lambda\leqq|A|$. A is (κ,λ)-decomposable if there is a $P\varepsilon\mathrm{Part}(A)$ such that $|P|\geqq\kappa$ and $|A{\restriction}a|\geqq\lambda$ for every nonzero $a\varepsilon P$. A is $(<\delta,\lambda)$-decomposable if it is (κ,λ)-decomposable for every $\kappa<\delta$. A is well decomposable if it is $(<\mathrm{Sat}(A),|A|)$-decomposable. Every complete A is $(\omega,|A|)$-decomposable. In particular, every c.c.c. complete A is well decomposable. If $|A|=|A{\restriction}a|$ for every $a\varepsilon A^{+}$, then A is well decomposable. So every complete A is isomorphic to a product of well decomposable complete Boolean algebras (see[12]). Let B be complete. The Boolean power of A by B is the Boolean algebra A[B] such that

$$A[B]=\{f\varepsilon B^{A}\,|\,f(A)\varepsilon \mathrm{Part}(B)\},$$

$$f+g(a)=\bigvee\{f(b)\cdot g(c)\,|\,b+c=a\},$$

$$f\cdot g(a)=\bigvee\{f(b)\cdot g(c)\,|\,b\cdot c=a\},$$

$$\sim f(a)=f(\sim a) \text{ for every } f,g\varepsilon A[B] \text{ and } a\varepsilon A,$$

$$0_{A[B]}(0)=1 \text{ and}$$

$$1_{A[B]}(1)=1.$$

§2. Proof of the theorem and its corollaries.

In this section, we assume both A and B are complete. We state lemmas that we will use.

Lemma 1 ([18.11, 6]). Let $\Phi(x_1,\ldots,x_n)$ be a bounded set-theoretical formula(i.e., it uses only bounded quantifiers).

$\Phi(a_1,\ldots,a_n)$ if and only if $[\![\Phi(\check{a}_1,\ldots,\check{a}_n)]\!] =1$.

Lemma 2 ([20.5 ,14]).

A is complete if and only if A is Sat(A)-complete.

Lemma 3. Let $[\![p:\check{\kappa}\longrightarrow\check{A}]\!]=1$ and $a\varepsilon A$. Suppose that $[\![$ c is the supremum of $p(\check{\kappa})]\!] =1$ (i.e., $[\![c=\bigvee_{\alpha<\kappa} p(\alpha)]\!] =1$). If $[\![c=\check{a}]\!]> 0$. then $a=\bigvee_{\alpha<\kappa}\bigvee X(\alpha)$ where $X(\alpha)=\{b\mid [\![c=\check{a}]\!]\cdot[\![p(\alpha)=\check{b}]\!]> 0$ for every $\alpha<\kappa$.

Proof. $0<[\![c=\check{a}]\!]\cdot[\![p(\check{\alpha})=\check{b}]\!] \leqq [\![\check{b} \leq \check{a}]\!]$ for every $b\varepsilon X(\alpha)$. Hence, by Lemma 1, $b \leq a$ for every $b\varepsilon X(\alpha)$. So we have $a \geqq \bigvee_{\alpha<\kappa}\bigvee X(\alpha)$. On the other hand, $[\![c=\check{a}]\!] \leqq [\![p(\check{\alpha}) \varepsilon \check{X(\alpha)}]\!] \leqq [\![p(\alpha) \leqq (\bigvee X(\alpha))^{\vee}]\!]$ for every $\alpha<\kappa$. Note that $[\![(\bigvee X)^{\vee}=\bigvee \check{X}]\!]=1$ for every $X\subset A$, since ' $x = \bigvee X$ ' is a bounded formula. Therefore $[\![\check{a} \leqq (\bigvee_{\alpha<\kappa} \bigvee X(\alpha))^{\vee}]\!]\geqq [\![c=\check{a}]\!] > 0$. So $a \leqq \bigvee_{\alpha<\kappa}\bigvee X(\alpha)$.

Theorem. Let $\{a(\alpha)\varepsilon A^+ \mid \alpha<\kappa\}\varepsilon \mathrm{Part}(A)$. Suppose that $[\![\check{\kappa}<\delta$ and δ is a cardinal$]\!]=1$. The following conditions are equivalent.

(i) $[\![\check{A}$ is δ-complete $]\!]^{(B)}=1$

(ii) $[\![(A\restriction a(\alpha))^{\vee}$ is δ-complete $]\!]^{(B)}=1$ for every $\alpha<\kappa$ and B satisfies the (κ,σ)-DL where $\sigma(\alpha)=|A\restriction a(\alpha)|$ for every $\alpha<\kappa$.

Proof. (ii)$\Rightarrow$(i): Since $\{a(\alpha)\,|\,\alpha<\kappa\}\varepsilon\mathrm{Part}(A)$, $A \cong \prod_{\alpha<\kappa} A\lceil a(\alpha)$.
By the (κ,σ)-DL, $[\![(\prod_{\alpha<\kappa} A\lceil a(\alpha))^{\vee} = \prod_{\alpha<\check\kappa}\check A\lceil \check a(\alpha)]\!] = 1$. Since $[\![(A\lceil a(\alpha))^{\vee}$ is δ-complete$]\!]=1$ for every $\alpha<\kappa$, $[\![\forall\alpha<\check\kappa[(A\lceil a(\alpha))^{\vee}$ is δ-complete$]]\!]=1$.
Hence $[\![\check A$ is δ-complete$]\!] = [\![\prod_{\alpha<\check\kappa}\check A\lceil \check a(\alpha)$ is δ-complete$]\!]=1$.
(i) (ii): It is clear that $[\![(A\lceil a(\alpha))^{\vee}\delta$-complete$]\!]=1$ for every $\alpha<\kappa$. We show that B satisfies (κ,σ)-DL. It is enough to show that $[\![\prod_{\alpha<\check\kappa}\check A\lceil \check a(\alpha) \subseteq (\prod_{\alpha<\kappa} A\lceil a(\alpha))^{\vee}]\!]=1$. Suppose that $[\![p\varepsilon\prod_{\alpha<\check\kappa}\check A\lceil \check a(\alpha)]\!]=1$. Then $[\![p:\check\kappa\longrightarrow\check A]\!]=1$ and $[\![\forall\alpha<\check\kappa[p(\alpha)\leqq\check a(\alpha)]]\!]=1$.
Since $[\![\check A$ is δ-complete and $\check\kappa<\delta]\!]=1$, $[\![\bigvee p(\check\kappa)$ exists$]\!]=1$. Let $a\varepsilon A, \alpha<\kappa$ and $c=\bigvee p(\check\kappa)$. We first show that

(*) $[\![c = \check a]\!] \leqq [\![p(\check\alpha) = \check b]\!]$ for some $b\varepsilon A\lceil a(\alpha)$.

Suppose not. Then $[\![c = \check a]\!] > 0$. So, by Lemma 3, we have $a = \bigvee_{\alpha<\kappa}\bigvee X(\alpha)$. Since $|X(\alpha)| \geqq 2$, $b < \bigvee X(\alpha)$ for some $b\varepsilon X(\alpha)$.
Hence $[\![c = \check a]\!]\cdot[\![p(\check\alpha) = \check b]\!] \leqq [\![\check b < (\bigvee X(\alpha))^{\vee}]\!]$ for some $b\varepsilon X(\alpha)$.
Therefore $[\![c = \check a]\!] \leqq [\![p(\check\alpha) < (\bigvee X(\alpha))^{\vee}]\!] = [\![p(\check\alpha) < \bigvee \check X(\check\alpha)]\!]$.
Since $[\![p(\check\alpha) \leqq \check a(\check\alpha)]\!]$ and $[\![\forall\beta,\beta'[\beta\neq\beta'\longrightarrow \check a(\beta)\cdot\check a(\beta)=\check 0]]\!]=1$,
$[\![c = \check a]\!] \leqq [\![c < \bigvee_{\alpha<\check\kappa}\bigvee X(\alpha)]\!]$. Hence
$0 < [\![c = \check a]\!] \leqq [\![\check a < \bigvee_{\alpha<\check\kappa}\bigvee X(\alpha)]\!] = [\![\check a < (\bigvee_{\alpha<\kappa}\bigvee X(\alpha))^{\vee}]\!]$. But this contradicts that $a = \bigvee_{\alpha<\kappa}\bigvee X(\alpha)$. So (*) is established. Fix $b_a(\alpha)\varepsilon A\lceil a(\alpha)$ which satisfies (*). Then $b_a\varepsilon\prod_{\alpha<\kappa} A\lceil a(\alpha)$ and $[\![c = \check a]\!] \leqq [\![p = \check b_a]\!]$. Hence

$$\begin{aligned} 1 = [\![c\ \varepsilon\ \check A]\!] &= \bigvee_{a\varepsilon A}[\![c = \check a]\!] \\ &\leqq \bigvee_{a\varepsilon A}[\![p = \check b_a]\!] \\ &\leqq \bigvee_{q\ \varepsilon \prod_{\alpha<\kappa} A\lceil a(\alpha)}[\![p = \check q]\!] \\ &\leqq [\![p\ \varepsilon (\prod_{\alpha<\kappa} A\lceil a(\alpha))^{\vee}]\!]. \end{aligned}$$

Therefore $[\![\prod_{\alpha<\check\kappa}\check A\lceil \check a(\alpha) \subseteq (\prod_{\alpha<\kappa} A\lceil a(\alpha))^{\vee}]\!]=1$.

Corollary 1. Let $\{a(\alpha)\varepsilon A^{+}|\alpha<\kappa\}\varepsilon \mathrm{Part}(A)$. The following conditions are equivalent.

(i) $[\![\check{A} \text{ is complete}]\!]^{(B)}=1$.

(ii) $[\![(A\restriction a(\alpha))^{\vee} \text{ is complete}]\!]^{(B)}=1$ for every $\alpha<\kappa$ and

B satisfies the (κ,σ)-DL where $\sigma(\alpha)=|A\restriction a(\alpha)|$ for every $\alpha<\kappa$.

Proof: Since $\kappa<\mathrm{Sat}(A)$, $[\![\check{\kappa}<(\mathrm{Sat}(A))^{\vee}\leq \mathrm{Sat}(\check{A})]\!]=1$. By Lemma 2, $[\![\check{A} \text{ is complete}]\!]=[\![\check{A} \text{ is } \mathrm{Sat}(\check{A})\text{-complete}]\!]$ and $[\![(A\restriction a(\alpha))^{\vee} \text{ is complete}]\!]=[\![(A\restriction a(\alpha))^{\vee} \text{ is } \mathrm{Sat}(\check{A})\text{-complete}]\!]$.

Corollary 2. $[\![(P(\kappa))^{\vee} \text{ is complete}]\!]^{(B)}=1$ if and only if B satisfies the $(\kappa,2)$-DL, where $P(\kappa)$ is the Boolean algebra of all subsets of κ.

Proof: Let $\{a(\alpha)|\alpha<\kappa\}$ be the set of all atoms of $P(\kappa)$. $|P(\alpha)\restriction a(\alpha)|=2$, so that $[\![(P(\alpha)\restriction a(\alpha))^{\vee} \text{ is complete}]\!]=1$ for every $\alpha<\kappa$. Therefore $[\![(P(\kappa))^{\vee} \text{ is complete}]\!]=1$ if and only if B satisfies the $(\kappa,2)$-DL.

Corollary 2 gives the negative answer to the Dwinger's question in case of atomic Boolean algebras. By virtue of Corollary 1, it is enough to deal with well decomposable complete Boolean algebras.

Corollary 3. Let A be well decomposable. Suppose that $[\![(\mathrm{Sat}(A))^{\vee}=\mathrm{Sat}(\check{A})]\!]^{(B)}=1$. Then $[\![\check{A} \text{ is complete}]\!]^{(B)}=1$ if and only if B satisfies the $(<\mathrm{Sat}(A),|A|)$-DL.

Proof: ($\Rightarrow$) For every $\kappa<\mathrm{Sat}(A)$, there is a family $\{a(\alpha)\varepsilon A^{+}|\alpha<\kappa\}\varepsilon\mathrm{Part}(A)$ such that $|A\restriction a(\alpha)|=|A|$ for every $\alpha<\kappa$. Hence, by Corollary 1, B satisfies the $(\kappa,|A|)$-DL for every $\kappa<\mathrm{Sat}(A)$.

($\Leftarrow$) Since A is complete, $[\![\forall f\ \varepsilon\ \check{A}^{\kappa}[\bigvee f(\check{\kappa})\text{ exists}]]\!]=1$ for every $\kappa<\mathrm{Sat}(A)$. B satisfies the $(<\mathrm{Sat}(A),|A|)$-DL, so that $[\![\forall f\ \varepsilon\ \check{A}^{\check{\kappa}}[\bigvee f(\check{\kappa})\text{ exists}]]\!]=1$ for every $\kappa<\mathrm{Sat}(A)$. Hence

$$
\begin{aligned}
[\![\check{A}\text{ is complete}]\!] &\geqq [\![\check{A}\text{ is }\mathrm{Sat}(\check{A})\text{-complete}]\!]\\
&= [\![\check{A}\text{ is }\widecheck{\mathrm{Sat}(A)}\text{-complete}]\!]\\
&= [\![\forall\kappa<\widecheck{\mathrm{Sat}(A)}\,\forall f\ \varepsilon\ \check{A}^{\kappa}[\bigvee f(\kappa)\text{ exists}]]\!]\\
&= \bigwedge_{\kappa<\mathrm{Sat}(A)}[\![\forall f\ \varepsilon\ \check{A}^{\check{\kappa}}[\bigvee f(\check{\kappa})\text{ exists}]]\!]\\
&=1.
\end{aligned}
$$

Corollary 4. Suppose that A is c.c.c..

(1) $[\![\check{A}\text{ is countably complete}]\!]^{(B)}=1$ if and only if B satisfies the $(\omega,|A|)$-DL.

(2) In particular, $[\![\check{A}\text{ is complete}]\!]^{(B)}=1$ implies that B satisfies the $(\omega,|A|)$-DL.

(3) If $[\![\check{A}\text{ is c.c.c.}]\!]^{(B)}=1$, then $[\![\check{A}\text{ is complete}]\!]^{(B)}=1$ if and only if B satisfies the $(\omega,|A|)$-DL.

Proof: (1):($\Rightarrow$) Since every c.c.c. complete Boolean algebra is well decomposable, if $[\![\check{A}\text{ is }\omega_1\text{-complete}]\!]^{(B)}=1$, then B satisfies the $(\omega,|A|)$-DL by the theorem.

($\Leftarrow$) If B satisfies the $(\omega,|A|)$-DL, then $[\![\check{\omega}_1=\omega_1]\!]^{(B)}=1$. Hence $[\![\check{A}\text{ is }\omega_1\text{-complete}]\!]^{(B)}=[\![\check{A}\text{ is }\check{\omega}_1\text{-complete}]\!]^{(B)}=1$.

(2) and (3) are immediate from (1).

Corollary 4.(3) shows the negative answer to the Dwinger's question in case of atomless Boolean algebra.

Next we show that the property $[\![\check{A} \text{ is complete}]\!]^{(B)}=1$ is hereditary with respect to A (i.e., if $[\![\check{A} \text{ is complete}]\!]^{(B)}=1$ and C is a complete subalgebra of A, then $[\![\check{C} \text{ is complete}]\!]^{(B)}=1$). More generally, we have the following

Proposition 1. Let C be complete. Suppose that there is a function i from C to A such that

(**) $\bigvee i(W)=\bigvee i(W')$ implies $\bigvee W=\bigvee W'$ for every $W,W'\subset C$.

If $[\![\check{A} \text{ is complete}]\!]^{(B)}=1$, then $[\![\check{C} \text{ is complete}]\!]^{(B)}=1$.

Proof: First we note that i is one to one. It is enough to show that $[\![\forall p \in \check{C}^{\check{\kappa}} [\bigvee p(\check{\kappa}) \text{ exists}]]\!]=1$ for every κ. Suppose that $[\![p:\kappa \longrightarrow C]\!]=1$. Since $[\![\check{A} \text{ is complete}]\!]=1$, $[\![\bigvee \check{i}(p(\check{\kappa})) \text{ exists}]\!]=1$. Put $c=\bigvee \check{i}(p(\check{\kappa}))$. Suppose that $[\![c=\check{a}]\!] \geqq d > 0$.

Put $\quad X(d,\alpha)=\{b\in A \mid d\cdot[\![\check{i}(p(\check{\alpha}))=\check{b}]\!] > 0\}$ and

$\quad\quad Y(d,\alpha)=\{b\in C \mid d\cdot[\![p(\check{\alpha})=\check{b}]\!] > 0\}$.

By virtue of Lemma 3, it is easy to show that

$$a=\bigvee_{\alpha<\kappa} \bigvee X(d,\alpha).$$

Since i is one to one, $i(Y(d,\alpha))=X(d,\alpha)$. Hence we have $\bigvee_{\alpha<\kappa}\bigvee Y([\![c=\check{a}]\!],\alpha)=\bigvee_{\alpha<\kappa}\bigvee Y(d,\alpha)$. Put $a^*=\bigvee_{\alpha<\kappa}\bigvee Y([\![c=a]\!],\alpha)$.

There is an element $f\in V^{(B)}$ which satisfies $[\![c=a]\!] \leqq [\![f=a^*]\!]$ for every $a\in A$. We show that $[\![f=\bigvee p(\check{\kappa})]\!]=1$. Since

$[\![c=\check{a}]\!]\cdot[\![p(\check{\alpha})=\check{b}]\!] \leqq [\![\check{b} \leqq (\bigvee Y([\![c=a]\!],\alpha))^{\vee}]\!]$,

$[\![c=\check{a}]\!] \leqq [\![p(\check{\alpha}) \leqq (\bigvee Y([\![c=a]\!],\alpha))^{\vee}]\!]$.

Hence $[\![c=\check{a}]\!] \leq [\![\forall\alpha<\check{\kappa}[p(\alpha) \leq \check{a}^*]]\!]$.

$$[\![c=\check{a}]\!] \leq [\![f=\check{a}^*]\!]\cdot[\![\forall\alpha<\check{\kappa}[p(\alpha) \leq \check{a}^*]]\!]$$
$$\leq [\![\forall\alpha<\check{\kappa}[p(\alpha) \leq f]]\!] .$$

Therefore $[\![\forall\alpha<\check{\kappa}[p(\alpha) \leq f]]\!]=1$. Suppose that $[\![\forall\alpha<\check{\kappa}[p(\alpha) \leq g]]\!]=1$. We show that $[\![f \leq g]\!]=1$. Suppose not. Put $e'=[\![f \not\leq g]\!] > 0$. Then $e' \leq [\![f\cdot g < f]\!]$ and $[\![\forall\alpha<\check{\kappa}[p(\alpha) \leq f\cdot g]]\!]=1$. There is an $a\varepsilon A$ such that $e'\cdot[\![c=\check{a}]\!]> 0$. Put $e=e'\cdot[\![c=\check{a}]\!]$,

$U=\{b\varepsilon C \mid e\cdot[\![f\cdot g=\check{b}]\!] > 0\}$ and $V=\{b\varepsilon C \mid e\cdot [\![f=\check{b}]\!] > 0\}$.

$\bigvee_{\alpha<\kappa} \bigvee Y(e,\alpha) \leq \bigvee U < \bigvee V \leq \bigvee\{b \mid [\![c=\check{a}]\!]\cdot[\![f=\check{b}]\!] > 0\}$.

$0 < [\![c=\check{a}]\!]\cdot[\![f=\check{b}]\!] \leq [\![\check{b}=\check{a}^*]\!]$ implies $b=a^*$.

Hence $\bigvee_{\alpha<\kappa} \bigvee Y(e,\alpha) < a^*$. But this contradicts that $a^*=\bigvee_{\alpha<\kappa} \bigvee Y(d,\alpha)$ for every nonzero $d \leq [\![c=\check{a}]\!]$.

Therefore $[\![f = \bigvee p(\check{\kappa})]\!]=1$, so that $[\![\check{C}$ is complete $]\!]=1$.

Example 1. Let C be a complete subalgebra of A. It is well known that , in general, $\bigvee W$ is not coincide with $\bigvee i(W)$ where $W\subset C$ and i is the canonical embedding. But it is clear that i satisfies (**). So $[\![\check{A}$ is complete $]\!]=1$ implies that $[\![\check{C}$ is complete $]\!]=1$.

Example 2. Let h be a complete homomorphism from A onto C and i be a function from C to A such that $h(i(c))=c$ for every $c\in C$. $h(\bigvee i(W))=\bigvee h(i(W))=\bigvee W$, so that $[\![\check{A}$ is complete $]\!]=1$ implies that $[\![\check{C}$ is complete $]\!]=1$.

Example 3. Let X be a topological space and Reg(X) be the Boolean algebra of all regular open sets of X. Let i be the canonical injection from Reg(X) to P(X). Since $\bigvee W=cl(int(\bigcup W))=cl(int(\bigvee i(W)))$, i satisfies (**). Hence, if B satisfies the $(|X|,2)$-DL, then $[\![\check{Reg}(X)$ is complete $]\!]^{(B)}=1$.

Banaschewski and Nelson([p.33,1]) remarked that there are topological spaces X and Y such that $Reg(Y)[Reg(X)] \not\cong Reg(X \times Y)$ and Reg(X) is $(|Y|, \infty)$-distributive. It can be shown, however, that $[\![\check{Reg}(Y) \text{ is complete}]\!]^{(Reg(X))} = 1$ implies that $Reg(Y)[Reg(X)] \cong Reg(X \times Y)$ (see [5.17,15]). It seems to me that their example is not a counterexample.

§3. Sufficient conditions for $[\![\check{A} \text{ is c.c.c.}]\!]^{(B)}=1$.

In this section, we investigate sufficient conditions for $[\![\check{A} \text{ is c.c.c.}]\!]^{(B)}=1$ where A is c.c.c. and B is complete and $(\omega,|A|)$-distributive. We first define a condition which is stronger than the c.c.c..

A satisfies the (ω,ω)-chain condition if A^+ can be written in the form $A^+ = \bigcup_{n<\omega} A_n$ where, for every $n<\omega$, no countable elements of A_n are pairwise disjoint.

Proposition 2. If A satisfies the (ω,ω)-chain condition, then $[\![\check{A} \text{ is c.c.c.}]\!]^{(B)}=1$ for every $(\omega,|A|)$-distributive complete B.

Proof: Suppose that A satisfies the (ω,ω)-chain condition. Then A^+ is written in the form

$$A^+ = \bigcup_{n<\omega} A_n$$

where, for every $n<\omega$, no countable elements of A_n are pairwise disjoint.

'$\forall n<\omega \forall f \varepsilon A_n{}^{\omega} \exists i,j<\omega[i\neq j \text{ and } f(i)\cdot f(j) > 0]$' is a bounded formula, so that $[\![\forall n<\omega \forall f \varepsilon \check{A_n{}^{\omega}}\ i,j<\omega[i\neq j \text{ and } f(i)\cdot f(j) > \check{0}]]\!]=1$.

Since B satisfies $(\omega,|A|)$-DL, $[\![\check{A_n{}^{\omega}} = (\check{A}_n)^{\check{\omega}} = (\check{A}_n)^{\omega}]\!]=1$.

Therfore $[\![\forall n<\omega \forall f \varepsilon (\check{A}_n)^{\omega} \exists i,j<\omega[i\neq j \text{ and } f(i)\cdot f(j) > \check{0}]\!]=1$.

Hence $[\![\check{A}$ satisfies the (ω,ω)-chain condition$]\!]=1$. Every Boolean algebra which satisfies the (ω,ω)-chain condition is c.c.c.. So $[\![\check{A} \text{ is c.c.c.}]\!]^{(B)}=1$ for every $(\omega,|A|)$-distributive complete B.

There are many Boolean algebra which satisfies the (ω,ω)-chain condition. For example, a Boolean algebra having a strictly positive measure satisfies the (ω,ω)-chain condition. For more details, see [3]. Galvin and Hajnal ([5]) showed that there is a Boolean algebra A which has calibre ω_1 but does not satisfies the (ω,ω)-chain condition. It can be shown that their example is a.c.c.c.. Thus, there is a Boolean algebra A which is a.c.c.c. but does not satisfies the (ω,ω)-chain condition. Therefore the converse of Proposition 2 is false.

Next we define a condition which is stronger than the (ω,∞)-DL.

B satisfies D_{ω_1} if for every $b\varepsilon B^+$ and every $\{P(\alpha)\,|\,\alpha<\omega_1\}\subseteq \mathrm{Part}(B\upharpoonright b)$, there is an $h\varepsilon\prod_{\alpha<\omega_1} P(\alpha)$ such that $\bigwedge_{\beta<\alpha} h(\beta)\neq 0$ for every $\alpha<\omega_1$.

Proposition 3. If B satisfies D_{ω_1}, then $[\![\check{A}$ is c.c.c.$]\!]^{(B)}=1$ for every c.c.c. A.

Proof: Suppose that $[\![\check{A}$ is c.c.c.$]\!]<1$. There is an $f\varepsilon V^{(B)}$ such that $[\![f:\omega_1\longrightarrow(A)^+$ and $\forall\alpha,\beta<\omega_1[\alpha\neq\beta\longrightarrow f(\alpha)\cdot f(\beta)=\check{0}]\,]\!]>0$.

Note that $[\![(\check{A})^+=\check{A^+}]\!]=1$.

Put $b=[\![f:\omega_1\longrightarrow(\check{A})^+$ and $\forall\alpha,\beta<\omega_1[\alpha\neq\beta\longrightarrow f(\alpha)\cdot f(\beta)=\check{0}]\,]\!]>0$.

Let $P(\alpha)=\{[\![f(\check{\alpha})=\check{a}]\!]\cdot b\,|\,a\varepsilon A^+\}$ for every $\alpha<\omega_1$.

Since
$$b\leqq[\![\forall\alpha<\omega_1\exists a\varepsilon(A)^+[f(\alpha)=a]]\!]$$
$$\leqq\bigwedge_{\alpha<\omega_1}\bigvee_{a\varepsilon A^+}[\![f(\check{\alpha})=\check{a}]\!]$$
and $a\neq b$ implies that $[\![f(\check{\alpha})=\check{a}]\!]\cdot[\![f(\check{\alpha})=\check{b}]\!]\leqq[\![\check{a}=\check{b}]\!]=0$, $P(\alpha)\varepsilon\mathrm{Part}(B\upharpoonright b)$ for every $\alpha<\omega_1$.

Hence there is an $h \in \prod_{\alpha<\omega_1} P(\alpha)$ such that $\bigwedge_{\beta<\alpha} h(\beta)\neq 0$ for every $\alpha<\omega_1$. Suppose that $h(\alpha)=b\cdot[\![f(\check{\alpha})=\check{a(\alpha)}]\!]$ where $a(\alpha)\in A^+$. Then

$$0 < h(\alpha)\cdot h(\beta) = b\cdot [\![f(\check{\alpha})=\check{a(\alpha)}]\!]\cdot[\![f(\check{\beta})=\check{a(\beta)}]\!]$$
$$\leqq [\![f(\check{\alpha})\cdot f(\check{\beta})=\check{0}]\!]\cdot[\![f(\check{\alpha})=\check{a(\alpha)}]\!]\cdot[\![f(\check{\beta})=\check{a(\beta)}]\!]$$
$$\leqq [\![\check{a(\alpha)}\cdot\check{a(\beta)}=\check{0}]\!].$$

Hence $a(\alpha)\cdot a(\beta)=0$. Thus $\{a(\alpha)\in A^+|\alpha<\omega_1\}$ is a pairwise disjoint family of A. But this contradicts that A is c.c.c..

In [7], Jech considered the following infinite game G_B played on a Boolean algebra between two players I and II. I and II alternatively play $b_0,b_1,b_2,b_3,\cdots$ from B^+ such that $b_0\geqq b_1\geqq b_2\geqq b_3\geqq\cdots$. II wins if and only if $\prod_{n<\omega} b_n\neq 0$. A positional winning strategy for II is a function σ from B to B such that he wins every play according to σ (i.e., if $b_{2n+1}=\sigma(b_{2n})$ for every $n<\omega$, then $\prod_{n<\omega} b_n\neq 0$). Jech showed that if II has a positional winning strategy, then B satisfies D_{ω_1}.

Finally, we consider the following chain condition which is first defined by Mansfield([10]). It is closedly related to $[\![\check{A}$ is c.c.c.$]\!]^{(B)}=1$.

A satisfies the $<\omega_1$-c.c. with respect to B (($<\omega_1$,B)-c.c.) if for every function Q from A to B such that

(†) $Q(a)\cdot Q(a') > 0$ implies $a=a'$ or $a\cdot a'=0$ for every $a,a'\in A$,

there is a family $\{b_i|i\in I\}$ such that $\bigvee\{b_i|i\in I\}=1$ and $|\{a|Q(a)\cdot b_i > 0\}|<\omega_1$ for every $i\in I$.

if A is complete

Proposition 4. If A satisfies the $(<\omega_1, B)$-c.c. and B satisfies the $(\omega,2)$-DL, then $[\![\check{A} \text{ is c.c.c.}]\!]^{(B)}=1$. Moreover, it follows $[\![\check{A} \text{ is complete}]\!]^{(B)}=1$, so that B satisfies $(\omega, |A|)$-DL.

Proof: Suppose that $[\![\check{A} \text{ is c.c.c.}]\!]^{(B)} < 1$. Let f,b and $P(\alpha)$ be as in the proof of Proposition 3. We define $Q: A \longrightarrow B$ by $Q(a)=b \cdot \bigvee_{\alpha<\omega_1} [\![f(\check{\alpha})=\check{a}]\!]$. It is easy to show that Q satisfies (†). Hence there is a family $\{b_i | i \varepsilon I\}$ such that $\bigvee\{b_i | i\varepsilon I\}=1$ and $|\{a | Q(a)\cdot b_i > 0\}| < \omega_1$ for every $i\varepsilon I$. Since $\bigvee\{b_i | i\varepsilon I\}=1$, $b\cdot b_i > 0$ for some $i\varepsilon I$. Let i be such a element and $a\varepsilon X=\{c\varepsilon A^+ | Q(c)\cdot b_i > 0\}$. Put

$q(a)=\{ [\![f(\check{\alpha})=\check{a}]\!]\cdot b\cdot b_i | \alpha<\omega_1 \} \quad \{\sim\left(\bigvee_{\alpha<\omega_1} [\![f(\check{\alpha})=\check{a}]\!]\right)\cdot b\cdot b_i\}$.

Note that the $(\omega,2)$-DL implies the (ω,ω_1)-DL. Since $|X| \leqq \omega$, there is a common refinement $\{d_j | j\varepsilon J\}$ of $\{q(a) | a\varepsilon X\}$ by the (ω,ω_1)-DL. Pick $d_j > 0$. Then, for every $\alpha<\omega_1$, there is an $a(\alpha)\varepsilon A^+$ such that $d_j \leqq [\![f(\check{\alpha})=\check{a(\alpha)}]\!]$.

$\{a(\alpha) | Q(a(\alpha))\cdot b_i > 0\} \subseteq X$. Hence $|\{a(\alpha) | Q(a(\alpha))\cdot b_i > 0\}| < \omega_1$. So, there are $\alpha,\beta<\omega_1$ such that $\alpha\neq\beta$ and $a(\alpha)=a(\beta)$. Therefore $\alpha\neq\beta$ and $[\![f(\check{\alpha})=f(\check{\beta})]\!] \geqq d_j > 0$. This contradicts $d_j \leqq b \leqq [\![\forall\alpha,\beta<\omega_1 [\alpha\neq\beta \longrightarrow f(\alpha)\cdot f(\beta)=\check{0}]]\!]$ and $b \leqq [\![f(\check{\alpha})\varepsilon(A)^+]\!]$. Hence $[\![\check{A} \text{ is c.c.c.}]\!]=1$.

Now we show that $[\![\check{A} \text{ is complete}]\!]=[\![\check{A} \text{ is } \omega_1\text{-complete}]\!]=1$. Since B satisfies the $(\omega,2)$-DL, $[\![\check{\omega}_1=\omega_1]\!]=1$. Let $[\![p:\omega\longrightarrow\check{A}]\!]=1$. It is enough to show that $[\![\bigvee p(\omega) \text{ exists}]\!]=1$. Without loss of generality, we can assume that p satisfies $[\![\forall n,m<\omega[n\neq m \longrightarrow p(n)\cdot p(m)=\check{0}]]\!]=1$ (see [20.1,14]). We define $Q': A\longrightarrow B$ by $Q'(a)=\bigvee_{n<\omega} [\![p(\check{n})=\check{a}]\!]$.

Q' satisfies (†). So there is a family of $\{b_i | i\varepsilon I\}$ such that $\bigvee\{b_i | i\varepsilon I\}=1$ and $|\{a | Q'(a)\cdot b_i > 0\}| < \omega_1$ for every $i\varepsilon I$. Put $X(n)=\{ [\![p(\check{n})=\check{a}]\!]\cdot b_i | a\varepsilon\{c|Q'(c)\cdot b_i > 0\}\}$. $|X(n)| \leqq \omega$. B satisfies the (ω,ω)-DL, so that $\{X(n)|n<\omega\}$ has a common refinement. Now it is easy to show that $[\![\bigvee p(\omega) \text{ exists}]\!]=1$.

There is a complete Boolean algebra B which is $(\omega,2)$-distributive but is not $(\omega,(2^{\omega})^{+})$-distributive (see[11]). Let F be the free Boolean algebra with $(2^{\omega})^{+}$ generators. $[\![\check{F} \text{ is c.c.c.}]\!]^{(B)}=1$ and F does not satisfy the $(<\omega_1,B)$-c.c. by the Proposition 4. Hence it is false that ' $[\![\check{A} \text{ is c.c.c.}]\!]^{(B)}=1$ implies that A satisfies the $(<\omega_1,B)$-c.c.'. The following theorem, however, has been obtained by the author.

Theorem ([16]).

(1) If B satisfies the (ω,∞)-DL, then $[\![\check{A} \text{ is c.c.c.}]\!]^{(B)}=1$ if and only if A satisfies the $(<\omega_1,B)$-c.c..

(2) Suppose that A is complete and c.c.c.. The following conditions are equivalent.

(i) B satisfies the (ω,∞)-DL and $[\![\check{A} \text{ is c.c.c.}]\!]^{(B)}=1$.

(ii) $[\![\check{A} \text{ is complete}]\!]^{(B)}=1$ and $[\![\check{\kappa}^{(\check{A})}=\check{\kappa}^{(A)}]\!]^{(B)}=1$ for every κ.

(iii) $[\![\check{A} \text{ is complete}]\!]^{(B)}=1$ and $(\kappa^{(A)})^{(B)}=\kappa^{(A[B])}$ for every κ.

($\kappa^{(A)}$ is the Boolean power of κ by A and so on.)

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