数理解析研究所講究録
第 540 巻 1984 年 189-204

Finite images and elementary equivalence of algebraic structures

F. OGER

In section I, we give some classical results about the properties of finitely generated groups which have the same finite images without being isomorphic. These results were published between 1956 and 1982, mainly after 1970.

In section II, we study the problem of the elementary equivalence of such groups. In section III, we generalize to diagrams of groups some results of section II.

I/ Finite images of finitely generated groups.

1°) Definitions.

The finite images of a group G are the finite groups H such that there is a surjective homomorphism from G to H.

If x and y are elements of G, the commutator of x and y is the element $[x,y]=x^{-1}y^{-1}xy$.

If H is a subset of G, we note $[G,H]$ the subgroup of G which is generated by the elements $[x,y]$ for $x\in G$ and $y\in H$.

A subgroup H of G is said to be <u>normal</u> in G if we have $[G,H] \subset H$.

G is <u>abelian</u> if and only if $[G,G]=\{1\}$. We say that G is <u>finite-by-abelian</u> if $[G,G]$ is finite.

The subgroup $\Gamma_n(G)$ is defined by induction on $n \geqslant 1$ with $\Gamma_1(G)=G$ and $\Gamma_{n+1}(G)=[G,\Gamma_n(G)]$ for each integer $n \geqslant 1$.

G is said to be <u>nilpotent</u> (resp. <u>finite-by-nilpotent</u>) if there exists an integer $n \geqslant 1$ such that $\Gamma_n(G)=\{1\}$ (resp. $\Gamma_n(G)$ is finite).

G is <u>polycyclic-by-finite</u> if there is a sequence of subgroups of G: $\{1\}=G_0 \subset G_1 \subset \ldots \subset G_n \subset G_{n+1}=G$ with G_i normal in G_{i+1} for $0 \leqslant i \leqslant n$, G_{i+1}/G_i cyclic for $0 \leqslant i \leqslant n-1$ and G_{n+1}/G_n finite.

Finitely generated finite-by-nilpotent groups are polycyclic-by-finite.

A group G is said to be <u>torsion-free</u> if, for any elements $x, y \in G$ and for each integer $n \geqslant 1$, $x^n=y^n$ implies $x=y$.

<u>2°) Finite images and profinite completion.</u>

At the beginning of the sixties, Hirsch and other group theorists proposed the following problem: If G and H are polycyclic-by-finite groups with same finite images, are they necessarily isomorphic?

The answer is positive if G and H are finitely generated abelian groups. On the other hand, it is negative for polycyclic-by-finite groups and even for finitely generated nilpotent groups. Examples of finitely generated nilpotent groups which have the same finite images without being isomorphic were given by Remeslennikov (1967), Baumslag, Mislin,....

Anyhow, Grünewald, Pickel and Segal proved the following result in [GPS]:

Theorem 1.1. Any class of polycyclic-by-finite groups with same finite images is the union of a finite number of isomorphism classes.

If G is a group and $n \geq 1$ an integer, we denote by G^n the subgroup of G which is generated by $\{x^n \mid x \in G\}$. If G is polycyclic-by-finite, we have $\bigcap_{n \in \mathbb{N}^*} G^n = \{1\}$ and G/G^n is finite for each integer $n \geq 1$.

The profinite completion $\hat{G}$ of a polycyclic-by-finite group G is the completion of G for the separated topology which is obtained by taking $\{G^n \mid n \geq 1\}$ as a fundamental system of neighbourhoods of 1; we have $G \subset \hat{G}$.

The following result was also proved in [GPS]:

Theorem 1.2. If G and H are polycyclic-by-finite groups, the following properties are equivalent:
1) G and H have the same finite images,
2) $G/G^n \cong H/H^n$ for each integer $n \geqslant 1$,
3) $\hat{G} \cong \hat{H}$.

3°) Isomorphic direct products.

If G is a finite group (or a finitely generated abelian group), if $G \cong A_1 \times \ldots \times A_m \cong B_1 \times \ldots \times B_n$ and if none of the groups A_i, B_j is trivial or isomorphic to a direct product of non trivial groups, then, we have $m = n$ and there exists a permutation σ of $\{1, \ldots, n\}$ such that $A_i \cong B_{\sigma(i)}$ for each $i \in \{1, \ldots, n\}$. This is the Remak-Krull-Schmidt theorem.

The same result is not true for finitely generated nilpotent groups or finitely generated finite-by-abelian group: For instance, there are examples of non isomorphic finitely generated finite-by-abelian nilpotent groups A, B such that $A \times \mathbb{Z} \cong B \times \mathbb{Z}$ or $\underbrace{A \times \ldots \times A}_{n \text{ times}} \cong \underbrace{B \times \ldots \times B}_{n \text{ times}}$. The first example was given by Walker in 1956 for $A \times \mathbb{Z} \cong B \times \mathbb{Z}$.

Then, Warfield proved the following theorem in [W]:

<u>Theorem 1.3.</u> Let G and H be finitely generated finite-by-abelian groups. The following properties are equivalent:
1) G and H have the same finite images,
2) $G\times\mathbb{Z}\cong H\times\mathbb{Z}$,
3) there exists an integer $n\geqslant 1$ such that $\underbrace{G\times\ldots\times G}_{n\ \text{times}}\cong\underbrace{H\times\ldots\times H}_{n\ \text{times}}$.

Hirshon proved afterwards that 2)$\Leftrightarrow$3) and 2)$\Rightarrow$1) remain true for larger classes of groups which include polycyclic-by-finite groups (see [H1], [H2] and [H3] for proofs). On the other hand, there are examples of finitely generated torsion-free nilpotent groups which have the same finite images while 2) and 3) are not true for them.

II/Finite images and elementary equivalence of groups.

We say that an elementary extension N of a structure M is <u>M-saturated</u> if it realizes each 1-type over M (i.e. each consistent set $\Sigma(v)$ of formulas with 1 free variable v and parameters in M).

We first observe that, if G is a polycyclic-by-finite group and $n\geqslant 1$ an integer, then, there is an integer $k(n)\geqslant 1$ such that G^n is defined in G by the formula

$(\exists x_1 \ldots \exists x_{k(n)})(x = x_1^n \ldots x_{k(n)}^n)$. If a group S is elementarily equivalent to G, then, S^n is defined in S by the same formula and S/S^n is isomorphic to the finite group G/G^n.

We use this remark to prove the following theorem:

Theorem 2.1. ([O1]). Let G be a polycyclic-by-finite group, S a G-saturated elementary extension of G and $E_S = \bigcap_{n \in \mathbb{N}^*} S^n$. Then, S/E_S is isomorphic to the profinite completion $\hat{G}$ of G.

Remark. It easily follows from this theorem that, if two polycyclic-by-finite groups G, H are elementarily equivalent, then, $\hat{G}$ and $\hat{H}$ are isomorphic and G and H have the same finite images.

If G is a finitely generated finite-by-nilpotent group and S a G-saturated elementary extension of G, then, for each integer $n \geqslant 1$, the map $E_S \to E_S$ given by $x \to x^n$ is bijective. In particular, if G is a finitely generated finite-by-abelian group, the abelian group E_S is divisible and therefore isomorphic to the additive structure of a vector space over $\mathbb{Q}$.

We also prove that, if G is a finitely generated finite-by-abelian group and S a G-saturated elementary extension of G, then, E_S is in the center of S and there is a subgroup T of S such that $E_S \cap T = \{1\}$ and S is generated by E_S and T. So, we have $S \cong T \times E_S \cong (S/E_S) \times E_S \cong \hat{G} \times E_S$.

Then, we can prove the following result:

Theorem 2.2. ([O1]). For two finitely generated finite-by-abelian groups G and H, the following properties are equivalent:

1) G and H are elementarily equivalent,

2) G and H have the same finite images,

3) $G \times \mathbb{Z} \cong H \times \mathbb{Z}$.

Remark. The finitely presented groups $G = \langle a, b ; a^{25} = 1, [a, b] = a^5 \rangle$ and $H = \langle c, d ; c^{25} = 1, [c, d] = c^{10} \rangle$ are finite-by-abelian (and nilpotent). Baumslag proved in [B] that they have the same finite images without being isomorphic. It follows from theorem 2.2. that they are elementarily equivalent.

The same remark can be applied to examples of non isomorphic finitely generated finite-by-abelian groups with same finite images which were given by Brigham, Dyer, Mislin,...

<u>Proof of theorem 2.2.</u> (summary). The equivalence of 2) and 3) follows from theorem 1.3. We saw that 1)$\Rightarrow$2) is true for polycyclic-by-finite groups.

We prove 2)$\Rightarrow$1) by showing that, if G and H have the same finite images, then, for every non trivial ultrafilter U over $\mathbb{N}$, $S=G^U$ and $T=H^U$ are isomorphic.

S is a G-saturated elementary extension of G and T is a H-saturated elementary extension of H, so, we have $S \cong \hat{G} \times E_S$ and $T \cong \hat{H} \times E_T$. As $\hat{G}$ and $\hat{H}$ are isomorphic, we only have to prove that E_S and E_T are isomorphic. But E_S and E_T are isomorphic to the additive structures of vector spaces over $\mathbb{Q}$ and we prove that the dimension of these vector spaces is 2^ω by showing that their cardinal is 2^ω.

In [04], we use the results above to describe the models of the theories of finitely generated finite-by-abelian groups and the elementary embeddings between these models. In particular, we prove that, if G is such a group, then, $\hat{G}$ is an elementary extension of G.

Anyhow, implications 2)$\Rightarrow$1) and 2)$\Rightarrow$3) of theorem 2.2. cannot be generalized to polycyclic-by-finite groups, or even to finitely generated finite_by_nilpotent groups.

In order to show that, we consider examples, given by Grünewald_Scharlau and Remeslennikov, of finitely generated torsion_free nilpotent groups G, H which have the same finite images without being isomorphic. We prove in [O2] and [O5] that, in each of these examples, G and H are not elementarily equivalent. It follows that $G\times\mathbb{Z}$ and $H\times\mathbb{Z}$ are not isomorphic, according to the following result, which is a partial generalization of theorem 2.2:

<u>Theorem 2.3.</u> ([O3]). If G and H are groups such that $G\times\mathbb{Z}\cong H\times\mathbb{Z}$, then, G and H are elementarily equivalent.

<u>Remark.</u> In [H2], Hirshon gave an example of non isomorphic finitely generated torsion_free nilpotent groups G, H such that $G\times\mathbb{Z}\cong H\times\mathbb{Z}$. These groups are elementarily equivalent according to theorem 2.3.

<u>III/ Finite images and elementary equivalence of diagrams of groups.</u>

We consider a <u>type of diagram</u> $D=(\dot{I},\dot{J},c,d)$, where

$\dot{I}$, $\dot{J}$ are sets and c, d are maps from $\dot{J}$ to $\dot{I}$. A D_diagram of groups $G=((G_i)_{i\in\dot{I}},(g_j)_{j\in\dot{J}})$ is a system made of:

- pairwise disjoined groups G_i,
- homomorphisms of groups: $g_j : G_{c(j)} \to G_{d(j)}$.

We define a language for G by interpreting the multiplications of the groups G_i and the homomorphisms g_j as relations on $\bigcup_{i\in\dot{I}} G_i$.

An homomorphism f from a D_diagram $G=((G_i)_{i\in\dot{I}},(g_j)_{j\in\dot{J}})$ to a D_diagram $H=((H_i)_{i\in\dot{I}},(h_j)_{j\in\dot{J}})$ is the union of a family of maps $(f_i)_{i\in\dot{I}}$ such that:

- for each $i\in\dot{I}$, f_i is an homomorphism from G_i to H_i,
- for each $j\in\dot{J}$, the following diagram is commutative:

$$\begin{tikzcd} G_{c(j)} \arrow[r, "f_{c(j)}"] \arrow[d, "g_j"'] & H_{c(j)} \arrow[d, "h_j"] \\ G_{d(j)} \arrow[r, "f_{d(j)}"] & H_{d(j)} \end{tikzcd}$$

The finite images of a D_diagram G are the finite D_diagrams which are images of G by surjective homomorphisms (if $\dot{I}$ is infinite, G has no finite image).

For each integer $n\geqslant 1$, we denote by G/G^n the D_diagram of groups $((G_i/G_i^n)_{i\in\dot{I}},(g_j(n))_{j\in\dot{J}})$, where the homomorphisms $g_j(n)$ are obtained from the homomor_ phisms g_j by the commutative diagrams:

$$\begin{tikzcd} G_{c(j)} \arrow[r, "g_j"] \arrow[d] & G_{d(j)} \arrow[d] \\ G_{c(j)}/G^n_{c(j)} \arrow[r, "g_j(n)"] & G_{d(j)}/G^n_{d(j)} \end{tikzcd}$$

The canonical surjections $G_i \to G_i/G_i^n$ define a surjective homomorphism from G to G/G^n.

If I is finite and if the groups G_i are polycyclic-by-finite, then, $G/G^n = \bigcup_{i\in I} G_i/G_i^n$ is finite for each integer $n \geq 1$.

The following result is a generalization of theorem 2.2. to diagrams of groups:

<u>Theorem 3.1.</u> If $D=(I,J,c,d)$ is a type of diagram associated to a finite set I and if G, H are D-diagrams of finitely generated finite-by-abelian groups, then, the following properties are equivalent:

a) G and H are elementarily equivalent,

b) G and H have the same finite images,

c) for each integer $n \geq 1$, G/G^n and H/H^n are isomorphic.

<u>Method of proof.</u> We generalize to D-diagrams of groups

the notion of profinite completion and the arguments which were used in the proof of theorem 2.2.

In each of the following examples, we prove that the diagrams of groups which we consider are elementarily equivalent by showing that c) is true for them. This is quite easy to do since, under the hypotheses of theorem 3.1, G/G^n and H/H^n are finite for each integer $n \geqslant 1$.

<u>Application 1</u>. In [Z], Zil'ber has given an example of two finitely generated commutative semigroups S, T which are elementarily equivalent without being isomorphic (a <u>semigroup</u> is a set with an associative multiplication).

The elementary equivalence of S and T can be easily deduced from theorem 3.1 since they are respectively obtained from the diagrams of groups

$$
\begin{tikzcd}
 & \mathbb{Z}/5\mathbb{Z} & \\
\mathbb{Z}_1 \arrow[ur, "g_1"] & & \mathbb{Z}_2 \arrow[ul, "g_2"']
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
 & \mathbb{Z}/5\mathbb{Z} & \\
\mathbb{Z}_1 \arrow[ur, "h_1"] & & \mathbb{Z}_2 \arrow[ul, "h_2"']
\end{tikzcd}
$$

where $\mathbb{Z}_1, \mathbb{Z}_2$ are copies of $\mathbb{Z}$ with $\mathbb{Z}_1, \mathbb{Z}_2, \mathbb{Z}/5\mathbb{Z}$ pairwise disjoined and g_1, g_2, h_1, h_2 are homomorphisms

with $g_1(1)=1, g_2(1)=2, h_1(1)=1, h_2(1)=4$. The addition of the commutative semigroup S (resp. T) is defined as follows:
$x+y=$ sum of x and y in $\mathbb{Z}_1, \mathbb{Z}_2$ or $\mathbb{Z}/5\mathbb{Z}$ if x and y both belong to $\mathbb{Z}_1, \mathbb{Z}_2$ or $\mathbb{Z}/5\mathbb{Z}$,
$x+y=g_1(x)+g_2(y)$ (resp. $x+y=h_1(x)+h_2(y)$) if $x\in\mathbb{Z}_1$ and $y\in\mathbb{Z}_2$,
$x+y=g_1(x)+y$ (resp. $x+y=h_1(x)+y$) if $x\in\mathbb{Z}_1$ and $y\in\mathbb{Z}/5\mathbb{Z}$,
$x+y=x+g_2(y)$ (resp. $x+y=x+h_2(y)$) if $x\in\mathbb{Z}/5\mathbb{Z}$ and $y\in\mathbb{Z}_2$.

More generally, any completely regular inverse semigroup can be defined from a diagram of groups and theorem 3.1 can therefore be applied to such semigroups (a <u>completely regular inverse semigroup</u> is a semigroup $(S,.)$ which is the union of a family of pairwise disjoined subgroups $(S_i)_{i\in I}$ such that, for any elements $i,j\in I$, there is an element $k\in I$ for which $G_i.G_j\subset G_k$).

<u>Application 2</u>. Remeslennikov has proved that the matrices

$$A=\begin{pmatrix} 2 & 2\\ -3 & 3\end{pmatrix} \qquad \text{and} \qquad B=\begin{pmatrix} 2 & 1\\ -6 & 3\end{pmatrix}$$

are not conjugate as matrices with coefficients in $\mathbb{Z}$, while their images modulo n are conjugate as matrices with coefficients in $\mathbb{Z}/n\mathbb{Z}$, for each integer $n\geqslant 1$. If g

and h are the endomorphisms of $\mathbb{Z}\times\mathbb{Z}$ which respectively admit A and B as matrices in the canonical basis of $\mathbb{Z}\times\mathbb{Z}$, then, $G=(\mathbb{Z}\times\mathbb{Z},+,g)$ and $H=(\mathbb{Z}\times\mathbb{Z},+,h)$ are such that $G/nG\cong H/nH$ for each integer $n\geq 1$; thus, G and H are elementarily equivalent without being isomorphic.

Application 3. If R is a (possibly non commutative) ring, each left (resp. right) R-module is completely determined by its additive structure and the endomorphisms $x\to rx$ (resp. $x\to xr$) for $r\in R$. So, R-modules can be interpreted as diagrams of groups and we can apply theorem 3.1:

If M and N are two finitely generated abelian groups with structures of left (resp. right) R-modules, then, the following properties are equivalent:

a) M and N are elementarily equivalent as R-modules,

b) M and N have the same finite images (i.e. each finite left (resp. right) R-module which is an image of one of them is also an image of the other),

c) for each integer $n\geq 1$, the finite R-modules M/M^n and N/N^n are isomorphic.

The following facts are consequences of this result:

1°) If g and h are the endomorphisms of $\mathbb{Z}\times\mathbb{Z}$ that we considered in application 2, then, the $\mathbb{Z}[X]$-modules

M, N respectively defined on $\mathbb{Z}\times\mathbb{Z}$ by $X.u = g(u)$ and $X.u = h(u)$ for each $u \in \mathbb{Z}\times\mathbb{Z}$ are elementarily equivalent without being isomorphic.

2°) If R is the ring of integers in a finite-degree extension of $\mathbb{Q}$, then, all non null ideals of R are elementarily equivalent as R-modules.

References

[B] G. Baumslag, Residually finite groups with the same finite images, Compositio Math. 29, 1974, 249-252.

[GPS] F.J. Grünewald, P.F. Pickel and D. Segal, Polycyclic groups with isomorphic finite quotients, Annals of Math. 111, 1980, 155-195.

[H1] R. Hirshon, The cancellation of an infinite group in direct products, Arch. Math. 26, 1975, 134-138.

[H2] R. Hirshon, Some cancellation theorems with applications to nilpotent groups, J. Austral. Math. Soc. 23, series A, 1977, 147-165.

[H3] R. Hirshon, The equivalence of $\times^t C \cong \times^t D$ and $J\times C \cong J\times D$, Trans. Amer. Math. Soc. 249, 1979, 331-340.

[O1] F. Oger, Equivalence élémentaire entre groupes finispar-abéliens de type fini, Comment. Math. Helv. 57, 1982, 469-480.

[O2] F. Oger, Des groupes nilpotents de classe 2 sans torsion de type fini ayant les mêmes images finies peuvent ne pas être élémentairement équivalents, C. R. Acad. Sci. Paris 294, série I, 1982, 1-4.

[O3] F. Oger, Cancellation and elementary equivalence of groups, J. Pure and Applied Algebra 30, 1983, 293-299.

[O4] F. Oger, The model theory of finitely generated finite-by-abelian groups, J. Symbolic Logic, to appear in 1984.

[O5] F. Oger, Elementary equivalence and isomorphism of finitely generated nilpotent groups, Communications in Algebra, to appear in 1984.

[W] R.B. Warfield, Genus and cancellation for groups with finite commutator subgroup, J. Pure and Applied Algebra 6, 1975, 125-132.

[Z] B.I. Zil'ber, On the isomorphism and elementary equivalence of commutative semigroups, Algebra i Logika 9, 1970, 667-671.

Francis OGER

Département de Mathématiques, Université Paris VII,

75251 Paris Cedex 05, FRANCE.