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THE CAUCHY PROBLEM FOR THE COUPLED MAXWELL-SCHRÖDINGER EQUATIONS

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Abstract: The Cauchy problem for the coupled Maxwell-Schrödinger equations in $\mathbb{R}^d$ in the Lorentz gauge is considered. The viscosity method is used to establish local existence. In one and two space dimensions, global solutions are obtained.

1. Introduction

Due in part to the developments of lasers, there has been a revived interest in the theory of the interaction of the radiation and non-relativistic charged particles in recent years [3]. In this paper we shall study the Cauchy problem for the closely related minimally coupled Maxwell-Schrödinger equations, by specializing to the Lorentz gauge. These equations are the classical approximation to the quantum field equations for an electrodynamical non-relativistic many body system [7], and may be written as

(1.1) $$\partial^{\mu}F_{\mu\nu} = J_{\nu}, \quad F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu},$$

(1.2) $$(iD_0 + D_jD_j)\psi = V\psi, \quad D_{\mu} = \partial_{\mu} - iA_{\mu},$$

with the components $A_{\mu}(t,x)$'s of the electromagnetic real vector potential and the complex scalar field $\psi(t,x)$ of non-relativistic charged partilces. Here $x \in \mathbb{R}^d$, μ,ν rnage over $0,1,\ldots,d$, j ranges over $1,\ldots,d$ (repeated indices always imply summation), $\partial^0 = \partial_0 = \partial/\partial t$, $(-\partial^1,\ldots,-\partial^d) = (\partial_1,\ldots,\partial_d) = \nabla$, $V = V(x)$ is a given real external potential, and the J_ν are the charge-current densities given by

$$J_0 = -\overline{\psi}\psi, \quad J_j = -i(\overline{\psi}D_j\psi - \psi\overline{D_j\psi}), \quad j = 1,\ldots,d.$$

The Lorentz gauge condition is expressed as

$$\partial^\mu A_\mu = 0. \tag{1.3}$$

In Sect.2, we shall show that the Cauchy problem for Eqs.(1.1)-(1.3) with $d \geqq 1$ has a local solution, provided the initial data and the external potential V are sufficiently regular. Our local existence argument uses the viscosity method (see e.g. [4][8][9]) to deal with the difficulty arising from the presence of highly singular derivative coupling terms in the Schrodinger equation, which is an unwelcome feature of the equations.

In Sect.3, we prove the global existence of solutions in the cases of one and two space dimensions. The needed a priori estimates for the solutions will be obtained by using the energy method in the form developed in [5], together with the covariant Sobolev inequalities.

Throughout this paper we shall use Greek indices $\mu,\nu,\cdots$ to run from 0 to d and Latin indices $j,k,\cdots$ to run from 1 to d, and the summation convention for both types of indices. We use the standard notation H^s for the Sobolev space of order s and exponent 2. If X is a normed space, we write $\|\cdot\|_X$ for its norm, and if X is also an inner product space, $(\cdot, \cdot)_X$ for its inner product. The L^p-norm will be denoted simply by $\|\cdot\|_p$.

2. Local Existence

Introducing the momenta $P_\mu = \partial_0 A_\mu$, and the vector notation $\vec{f} = (f_1, \ldots, f_d)$, we write Eq.(1.1)-(1.3), with initial data A_0^0, P_0^0, $\vec{A}^0$, $\vec{P}^0$, ψ^0, in the form

$$\text{(2.1)} \qquad \frac{du}{dt} + Zu = K(u), \quad u = (A_0, P_0, \vec{A}, \vec{P}, \psi)$$

$$\text{(2.2)} \qquad P_0 - \partial_j A_j = 0$$

$$u(0) = u_0 = (A_0^0, P_0^0, \vec{A}^0, \vec{P}^0, \psi^0),$$

where the components A_μ, P_μ, and ψ of the unknown u take values in L^2, and the operator Z and the function K(u) are respectively defined by

$$Zu = (-P_0, -\Delta A_0, -\vec{P}, -\Delta\vec{A}, -i\Delta\psi),$$

$$K(u) = (0, J_0, 0, \vec{J}, -iV\psi + iA_0\psi + P_0\psi + 2A_j\partial_j\psi - iA_jA_j\psi),$$

the indicated differential operators being defined by Fourier transformation. Use has been made of the side condition Eq.(1.3) to convert Eqs.(1.1)-(1.2) into Eq.(2.1). Eqs.(2.1)-(2.2) impose the following initial value constraints on the components of the data u_0:

$$(2.3) \qquad P^0_0 - \partial_j A_j = 0, \quad \partial_j P^0_j - \Delta A^0_0 + \overline{\psi^0}\psi^0 = 0.$$

Next we form for $s \geqq 1$ the direct sum Hilbert spaces

$$X^s = H^{s+1}(\mathbb{R}^d;\mathbb{R}) \oplus H^s(\mathbb{R}^d;\mathbb{R}) \oplus H^s(\mathbb{R}^d;\mathbb{R}^d) \oplus H^{s-1}(\mathbb{R}^d;\mathbb{R}^d) \oplus H^s(\mathbb{R}^d;\mathbb{C}),$$

$$Y^s = H^{s+1}(\mathbb{R}^d;\mathbb{R}) \oplus H^s(\mathbb{R}^d;\mathbb{R}) \oplus H^s(\mathbb{R}^d;\mathbb{R}^d) \oplus H^{s-1}(\mathbb{R}^d;\mathbb{R}^d) \oplus H^{s-1}(\mathbb{R}^d;\mathbb{C}).$$

We now state the main result of this section. We will denote by [p] the integer part of $p \in \mathbb{R}$. Let $d \geqq 1$.

Theorem 2.1. Let m be an integer satisfying $m \geqq [\frac{d}{2} + 2]$, and let $V \in H^m(\mathbb{R}^d;\mathbb{R})$. Let u_0 be any initial data lying in X^m, not necessarily satisfying (2.3). Then Eq.(2.1) has a unique solution u on [0,T) for some T, $0 < T \leqq \infty$, such that $u \in C([0,T); X^m) \cap C^1([0,T); Y^{m-1})$ and $u(0) = u_0$, where we may assume that either $T = \infty$ or $\lim_{t\to T} \|u(t)\|_{X^{[\frac{d}{2}+2]}} = \infty$; the solution u depends continuously on the initial data u_0, in the sense that if $\|u\|_{X^{[\frac{d}{2}+2]}} \leqq C$ on [0,T'] for some fixed C, T' > 0 when u_0 converges to some $u_0' \in X^m$ in X^m, then the solution u' of Eq.(2.1) corresponding to the data u_0' also exists on [0,T'], and u converges to u' weakly in X^m uniformly on [0,T']. Furthermore, if in addition the data u_0 satisfies the initial value constraints (2.3), the solution u satisfies Eq.(2.2).

The proof of Theorem 2.1 depends on

<u>Lemma 2.2.</u> Let m and V be as in Theorem 2.1, and let $0 \leqq \alpha \leqq 1$. Then

(2.4) $\|K(u) - K(u')\|_{Y^{m-\alpha}} \leqq \omega(\|u\|_{X^{m-\alpha}}, \|u'\|_{X^{m-\alpha}})\|u - u'\|_{X^{m-\alpha}}, \quad u,u' \in X^{m-\alpha},$

(2.5) $\mathrm{Re}(K(u), u)_{X^m} \leqq c\{1 + \|u\|_{X^{[\frac{d}{2}+2]}} + \|u\|^2_{X^{[\frac{d}{2}+2]}}\}\|u\|^2_{X^m}, \quad u \in X^{m+1},$

(2.6) $\mathrm{Re}(K(u) - K(u'), u - u')_{X^{m-1}} \leqq \omega(\|u\|_{X^m}, \|u'\|_{X^m})\|u - u'\|^2_{X^{m-1}}, \ u,u' \in X^m,$

where $\omega(a, b) = c\{1 + a + b + a^2 + b^2\}$.

Proof. The inequality (2.4) immediately follows from the multiplication lemma:

(2.7) $\|fg\|_{H^s} \leqq c\,\|f\|_{H^{s_1}}\|g\|_{H^{s_2}}$ if $s_1, s_2 \geqq s \geqq 0$, and $s_1 + s_2 - \frac{d}{2} > s$,

which implies that $\|fg\|_{H^{s-1}} \leqq c\,\|f\|_{H^s}\|g\|_{H^{s-1}}$ for $s > \frac{d}{2}$.

In proving (2.5) and (2.6), we may assume that u and u' are C_0^∞-vectors in view of (2.4). Let m be an integer satisfying $m \geqq [\frac{d}{2} + 2]$. To show (2.5), we make use of the following inequality:

(2.8) $\|\partial^\alpha(fg) - f\partial^\alpha g\|_2 \leqq c\{\|f\|_{H^m}\|g\|_{H^{[\frac{d}{2}+1]}} + \|f\|_{H^{[\frac{d}{2}+2]}}\|g\|_{H^{m-1}}\}, \quad f,g \in C_0,$

where $|\alpha| \leqq m$. For $|\alpha| \leqq \frac{d}{2} + 1$, (2.8) (the right side becomes simplar in this case) results from (2.7) after an application of Leibnitz rule. For the case $\frac{d}{2}+1<|\alpha|\leqq m$, see e.g. [6]. Now let k be any integer satisfying $0\leqq k\leqq m$. Then (2.8) in particular implies that

$$(2.9)\quad \|fg\|_{H^k}\leqq c\{\|f\|_{H^m}\|g\|_{H^{[\frac{d}{2}+1]}}+\|f\|_{H^{[\frac{d}{2}+2]}}\|g\|_{H^{m-1}}+\|f\|_{H^{[\frac{d}{2}+1]}}\|g\|_{H^k}\},$$

since for $|\alpha|=k$, $\|f\partial^\alpha g\|_2\leqq\|f\|_\infty\|g\|_{H^k}\leqq c\|f\|_{H^{[\frac{d}{2}+1]}}\|g\|_{H^k}$. A repeated application of (2.9) yields

$$(2.10)\quad \|f_1f_2f_3\|_{H^k}\leqq c\sum\|f_{j_1}\|_{H^{[\frac{d}{2}+2]}}\|f_{j_2}\|_{H^{[\frac{d}{2}+2]}}\|f_{j_3}\|_{H^m},\qquad f_1,f_2,f_3\quad C_0^\infty,$$

where the sum is taken over all cyclic permutations (j_1,j_2,j_3)'s of (1,2,3). Except for the term $\mathrm{Re}(A_j\partial_j\psi,\psi)_{H^m}$, the inequality (2.5) can be proved by using (2.9) and (2.10) in the left hand side, after an application of Schwartz inequalities. To estimate the term $\mathrm{Re}(A_j\partial_j\psi,\psi)_{H^m}$, we shall use (2.8) directly. We write

$$(2.11)\quad \mathrm{Re}(A_j\partial_j\psi,\psi)_{H^m}=\mathrm{Re}\sum_{|\alpha|\leqq m}(\partial^\alpha(A_j\partial_j\psi)-A_j\partial^\alpha\partial_j\psi,\partial^\alpha\psi)_{L^2}+\mathrm{Re}\sum_{|\alpha|\leqq m}(A_j\partial^\alpha\partial_j\psi,\partial^\alpha\psi)_{L^2}.$$

Applying (2.8) to the first term on the right side in the obvious mannar, and noting in the second term that, by an integration by parts,

$$\mathrm{Re}(A_j\partial^\alpha\partial_j\psi,\partial^\alpha\psi)_{L^2}=-\frac{1}{2}(\partial_jA_j\partial^\alpha\psi,\partial^\alpha\psi)_{L^2},$$

and that $\|\partial_jA_j\|_\infty\leqq c\|\partial_jA_j\|_{H^{[\frac{d}{2}+1]}}$, we find that the right side of (2.11) is bounded by a constant times $\|u\|_{X^{[\frac{d}{2}+2]}}\|u\|^2_{X^m}$. Thus (2.5) follows.

The proof of the inequality (2.6) is, except for the term

$\mathrm{Re}(A_j\partial_j\psi - A'_j\partial_j\psi', \psi - \psi')_{H^{m-1}}$, straightforward, and in fact reduces to that of (2.4). We now write

$$\mathrm{Re}(A_j\partial_j\psi - A'_j\partial_j\psi', \psi - \psi')_{H^{m-1}} = \mathrm{Re}((A_j - A'_j)\partial_j\psi', \psi - \psi')_{H^{m-1}}$$

$$+ \mathrm{Re}\sum_{\substack{|\alpha|\leq m-1\\ 0<|\beta|\leq|\alpha|}} c_{\alpha\beta}(\partial^\beta A_j\partial^{\alpha-\beta}\partial_j(\psi - \psi'), \partial^\alpha(\psi - \psi'))_{L^2}$$

$$+ \mathrm{Re}\sum_{|\alpha|\leq m-1}(A_j\partial^\alpha\partial_j(\psi - \psi'), \partial^\alpha(\psi - \psi'))_{L^2}.$$

Using (2.7) in the first and the second term on the right side after the application of Schwartz inequalities, and estimating the last term in the same way as in the last part of the proof of (2.5), one finds that the right side of the above equality is bounded by a constant times $\{\|u\|_{X^m} + \|u'\|_{X^m}\}\|u - u'\|^2_{X^{m-1}}$, and obtains (2.6).

Proof of Theorem 2.1. Let m and V be as in the statements of Theorem 2.1, and assume first that the initial data u_0 is an arbitrary element of X^m. In order to construct the solution of Eq.(2.1) corresponding to the data u_0, we shall introduce the following approximate Cauchy problem:

$$\text{(2.12)}\qquad \frac{du}{dt} + B_\varepsilon u = F(u),$$

$$u(0) = u_0,$$

where $\varepsilon > 0$, and $B_\varepsilon = \varepsilon T + S$, $F(u) = -Mu + K(u)$, with $T = I - \Delta$ and

$$Mu = (-P_0, -\Delta A_0, -\vec{P}, -\Delta\vec{A}, 0),\quad Su = (0, 0, 0, 0, -i\Delta\psi),$$

(so that $M + S = Z$). Note that the norms $\|\cdot\|_{X^{1+2\alpha}}$ and $\|B_\varepsilon^\alpha(\cdot)\|_{X^1}$ are equivalent for each $\alpha \geqq 0$, B_ε^α's being the fractional powers of B_ε. The operator B_ε^α generates a holomorphic semigroup on $X^{[\frac{d}{2}+1]}$. On the other hand, from (2.4) we have

$$\|F(u) - F(u')\|_{X^s} \leqq \omega(\|u\|_{X^{s+1}}, \|u'\|_{X^{s+1}}) \|u - u'\|_{X^{s+1}}$$

for any $s \geqq [\frac{d}{2} + 1]$, where $\omega(a, b) = c\{1 + a + b + a^2 + b^2\}$. Thus, by the well established theory of semilinear parabolic equations (see e.g. [1]), Eq.(2.12) has a unique solution u_ε on $[0,T_\varepsilon)$, for some $0 < T_\varepsilon \leqq \infty$, such that $u_\varepsilon \in C([0,T_\varepsilon); X^m) \cap C^1([0,T_\varepsilon); X^{m-1})$ and $u_\varepsilon(0) = u_0$, and here we may assume that either $T_\varepsilon = \infty$ or $\lim_{t\to T_\varepsilon} \|u_\varepsilon(t)\|_{X^{[\frac{d}{2}+2]}} = \infty$.

We shall now consider the convergence of u_ε, $\varepsilon > 0$, in the limit $\varepsilon \longrightarrow 0$. Taking the X^m-inner product of Eq.(2.12) for u_ε with u_ε and adding the complex conjugate of the result, we have

$$\text{(2.13)} \qquad \frac{1}{2}\frac{d}{dt}\|u_\varepsilon\|^2_{X^m} + \varepsilon(Tu_\varepsilon, u_\varepsilon)_{X^m} = \mathrm{Re}(J(u_\varepsilon), u_\varepsilon)_{X^m}.$$

Using (2.5) and noting that the second term on the left side is non-negative, we obtain

$$\frac{d}{dt}\|u_\varepsilon\|_{X^m} \leqq P(\|u_\varepsilon\|_{X^{[\frac{d}{2}+2]}})\|u_\varepsilon\|_{X^m},$$

where $P(a) = c\{1 + a + a^2\}$. It follows that

$$\text{(2.14)} \quad \|u_\varepsilon(t)\|_{X^m} \leqq \|u_\varepsilon(0)\|_{X^m}\exp[\int_0^t P(\|u_\varepsilon(s)\|_{X^{[\frac{d}{2}+2]}})ds] \quad \text{on } [0,T_\varepsilon).$$

With the solution b of the scalar Cauchy problem

$$\frac{db}{dt} = P(b), \quad b(0) = L \geqq \|u_0\|_{X^{[\frac{d}{2}+2]}},$$

which exists and is bounded on a time interval $[0,T_0]$, $T_0 = T_0(L) > 0$, it also follows that $\|u_\varepsilon(t)\|_{X^{[\frac{d}{2}+2]}} \leqq b(t)$ on $[0,T_\varepsilon) \cap [0,T_0]$, from which we may assume that $T_\varepsilon > T_0$. Then by (2.14),

$$\|u_\varepsilon(t)\|_{X^m} \leqq \|u_0\|_{X^m} e^{Ct} \quad \text{on } [0,T_0], \tag{2.15}$$

where C is a positive constant independent of ε.

Next let $0 < \varepsilon_1 < \varepsilon_2$, and put $w = u_{\varepsilon_1} - u_{\varepsilon_2}$. From (2.12) we have

$$\frac{dw}{dt} + B_{\varepsilon_1} w = (\varepsilon_2 - \varepsilon_1) T_{\varepsilon_2} - Mw + J(u_{\varepsilon_1}) - J(u_{\varepsilon_2}). \tag{2.16}$$

Taking the X^{m-1}-inner product of this equation with w and adding the complex conjugate of the result, we get

$$\frac{1}{2}\frac{d}{dt}\|w\|^2_{X^{m-1}} + \varepsilon_1 (Tw, w)_{X^{m-1}}$$

$$= (\varepsilon_1 - \varepsilon_2)\mathrm{Re}(Tu_{\varepsilon_2}, w)_{X^{m-1}} + \mathrm{Re}(J(u_{\varepsilon_1}) - J(u_{\varepsilon_2}), w)_{X^{m-1}}.$$

Noting that the second term on the left side is non-negative and using (2.6) and (2.15), we obtain

$$\frac{1}{2}\frac{d}{dt}\|w\|^2_{X^m} \leqq (\varepsilon_2 - \varepsilon_1)\|u_{\varepsilon_2}\|_{X^m}\|w\|_{X^m} + \omega(\|u_{\varepsilon_1}\|_{X^m}, \|u_{\varepsilon_2}\|_{X^m})\|w\|^2_{X^{m-1}} \tag{2.17}$$

$$\leq c\varepsilon_2 + c\|w\|^2_{X^{m-1}}.$$

Application of Gronwall's inequality then gives $\|w(t)\|^2_{X^{m-1}} \leq c\varepsilon_2$ on $[0,T_0]$ since $w(0) = 0$. Thus by letting $\varepsilon_2 \longrightarrow 0$, we find a function $u \in C([0,T_0];\ X^{m-1})$ such that $u_\varepsilon \longrightarrow u$ in X^{m-1} uniformly on $[0,T_0]$ as $\varepsilon \longrightarrow 0$. By (2.15), it also follows that $u \in L^\infty([0,T_0];\ X^m)$ with $\|u(t)\|_{X^m} \leq \|u_0\|_{X^m} e^{Ct}$, that $u_\varepsilon \longrightarrow u$ weakly in X^m uniformly on $[0,T_0]$ as $\varepsilon \longrightarrow 0$, and that u is weakly continuous from $[0,T_0]$ to X^m.

Now let ϕ be a smooth element of Y^{m-1}. Then we have

$$(-B_\varepsilon u_\varepsilon + F(u_\varepsilon),\ \phi)_{Y^{m-1}} = -\ \varepsilon(u_\varepsilon,\ T\phi)_{Y^{m-1}} + (u_\varepsilon,\ Z\phi)_{Y^{m-1}} + (J(u_\varepsilon),\ \phi)_{Y^{m-1}}.$$

This together with (2.4) implis that $(-\ B_\varepsilon u_\varepsilon + F(u_\varepsilon),\ \phi)_{Y^{m-1}}$ converges to $(-\ Zu + J(u),\ \phi)_{Y^{m-1}}$ uniformly on $[0,T_0]$. Thus, integrating the equality $(-\ \frac{du_\varepsilon}{dt} - B_\varepsilon u_\varepsilon + F(u_\varepsilon),\ \phi)_{Y^{m-1}} = 0$ on a time interval in $[0,T_0]$, and changing the order of the inner product and the time integral after taking the limit $\varepsilon \longrightarrow 0$, we find that u is a solution of Eq.(2.1) lying in the class $C([0,T_0];\ X^{m-1}) \cap L^\infty([0,T_0];\ X^m)$. By taking the X^{m-1}-inner product of Eq.(2.1) with u and using (2.6) and Gronwall's inequality, we also find that the solution u is unique in this class. Note that u is strongly continuous in X^m at $t = 0$ since u is weakly continuous in X^m at $t = 0$ and $\limsup_{t\to 0} \|u(t)\|_{X^m} \leq \|u_0\|_{X^m}$. The fact that Eq.(2.1) is time translational then implies that the solution corresponding to the initial data $u(t_0)$, given at $t = t_0 > 0$, is also right continuous in X^m at $t = t_0$. By the above uniqueness result, it follows that u is right continuous in X^m at any t in $[0,T_0]$. Since Eq.(2.1) is also time reverssible (in a suitable sense), we deduce that $u \in C([0,T_0]; X^m)$. Note that the choice of T_0 was uniform for all initial data u_0 satisfying $\|u_0\|_{X^{[\frac{d}{2}+2]}} \leqq L$, for each fixed $L > 0$. Thus the above solution u extends to some larger interval $[0,T)$ in such a way that $u \in C([0,T); X^m)$ with either $T = \infty$ or $\lim_{t\to T} \|u(t)\|_{X^{[\frac{d}{2}+2]}} = \infty$. From the equation, it also follows that $u \in C^1([0,T); Y^{m-1})$.

To prove the continuous dependence of the solution u of Eq.(2.1) on the initial data u_0, let $\{u_{n0}\}_{n=1}^{\infty}$ be a sequence in X^m that converges to u_0 in X^m, and let u_n be the solution of Eq.(2.1) corresponding to the initial data u_{n0}, for each n. Suppose that $\{u_n\}$ satisfies $\|u_n\|_{X^{[\frac{d}{2}+2]}} \leqq C$ on $[0,T']$ for some constants $C, T' > 0$ independently of n. Now an argument similar to that which led from (2.13) to (2.15) (but now setting $\varepsilon = 0$ and replacing u_0 by u_{n0}) shows that $\|u_n\|_{X^m} \leqq \|u_{n0}\|_{X^m} e^{Ct}$ on $[0,T']$ for some constant $C > 0$ independent of n. An analysis similar to that which led from (2.17) to (2.18) then shows that $\|u_n(t) - u_{n'}(t)\|_{X^{m-1}} \leqq C \|u_{n0} - u_{n'0}\|_{X^{m-1}}$ on $[0,T']$ again for some constant $C > 0$ independent of n and n'. Then by the same argument as above, we deduce that $u_n \longrightarrow u$ in X^m uniformly on $[0,T']$.

It remains to show the last part of the theorem. To see this, assume further that u_0 satisfies (2.3), and let $u = (A_0, P_0, \vec{A}, \vec{P}, \psi)$ be the corresponding solution of Eq.(2.1). Put

$$f = P_0 - \partial_j A_j, \quad g = \partial_j P_j - \Delta A_0 + \overline{\psi}\psi.$$

From Eq.(2.1) we have

$$\frac{df}{dt} = -g, \quad \frac{dg}{dt} = -\Delta f + 2\bar{\psi}\psi f.$$

Using these equation, we obtain

$$\frac{1}{2}\frac{d}{dt}\{\|f\|_2^2 + \|\nabla f\|_2^2 + \|g\|_2^2 = \int_{\mathbb{R}^d} (2\bar{\psi}\psi - 1) f g \, dx$$

$$\leqq \{\tfrac{1}{2} + \|\psi\|_\infty^2\}\{\|f\|_2^2 + \|g\|_2^2\},$$

on the interval of existence $[0,T)$ of u. Since $\|\psi\|_\infty \leqq \|\psi\|_{H^{[\frac{d}{2}+1]}} \leqq c(T')$ on each $[0,T']$, $T' < T$, and $f(0) = g(0) = 0$ by assumption, if follows that $\|f\|_2^2 + \|\nabla f\|_2^2 + \|g\|_2^2 = 0$ on $[0,T)$. Thus $f \equiv 0$, which is the desired result.

3. Global Existence in One and Two Space Dimensions

In this section we shall prove

Theorem 3.1. Let $d \in \{1,2\}$, m an integer satisfying $m \geqq [\frac{d}{2} + 4]$, and $V \in H^m(\mathbb{R}^d; \mathbb{R})$. Let u_0 be any initial data lying in X^m and satisfying the constraints (2.3). Then Eqs.(2.1)-(2.2) has a unique solution u on $[0,\infty)$ such that $u \in C([0,\infty); X^m) \cap C^1([0,\infty); Y^{m-1})$ and $u(0) = u_0$.

The proof of Theorem 3.1 depends on the energy and the charge conservation laws, which we shall state here as a lemma. Note that we have the identities

$$\partial_\mu(f\bar{g}) = D_\mu f\bar{g} + f\overline{D_\mu g}, \quad D_\mu(fg) = \partial_\mu fg + fD_\mu g,$$

$$D_\mu D_\nu f = D_\nu D_\mu f + iF_{\nu\mu} f,$$

$$\partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} + \partial_\lambda F_{\mu\nu} = 0,$$

whenever f, g and the A_μ are smooth in (t,x) and the A_μ are real valued.

Lemma 3.2. Let (A_0, A_1,..., A_d, ψ) be a smooth solution of Eqs.(1.1)--(1.2) with V smooth, and assume that A_0, A_1,..., A_d, ψ, V and their derivatives (of suitable order) are square integrable on $\mathbb{R}^d$. Then the energy E_1 and the charge Q of the solution, that is,

$$E_1 = \int_{\mathbb{R}^d} \{D_j\psi\overline{D_j\psi} + V\psi\bar{\psi} + \frac{1}{2}F_{j0}F_{j0} + \frac{1}{4}F_{jk}F_{jk}\} dx ,$$

$$Q = \int_{\mathbb{R}^d} \psi\bar{\psi}\, dx ,$$

are finite constant functions of time.

Proof. In fact, we have from the equations that $\frac{d}{dt}E_1 = \frac{d}{dt}Q = 0$. The proof is facilitated by using the above identities for the D_μ and the $F_{\mu\nu}$.

Proof of Theorem 3.1. We shall prove the theorem under slight weaker assumptions. We replace the condition $m \geqq [\frac{d}{2} + 4]$ by $m \geqq [\frac{d}{2} + 2]$, and assume the existence of a sequence $\{u_{n0}\}$ of initial data in X^k, with $k \geqq \max\{m, [\frac{d}{2} + 4]\}$, such that $u_{n0} \longrightarrow u_0$ in X^m, and that each u_{n0}

satisfies the constraints (2.3). Let $\{V_n\}$ be a sequence of external potentials in $H^k(\mathbb{R}^d; \mathbb{R}^d)$, k being as above, that converges to V in H^m, and for each n, $u_n \in C([0,T_n); X^k)$ the solution of Eqs.(2.1)-(2.2) corresponding to the initial data u_{n0} and the external potential V_n, given by Thorem 2.1. We shall show that for such $\{u_n\}$, there is a locally bounded function $C(\cdot)$ on $[0,\infty)$ which can be chosen independently of n, such that $\|u_n(t)\|_{X^{[\frac{d}{2}+2]}} \leqq C(t)$ on $[0,T_n)$. One can then show, by an obvious change of the proof of the previous result on continuous dependence of solutions on initial data (to include the dependence on the external potential) that, for every $T > 0$, the solution u of Eqs. (2.1)-(2.2) corresponding to the initial data u_0 and the external potential V exists in $C([0,T]; X^m)$ as the uniform limit of $\{u_n\}$ on $[0,T]$ in the weak topology of X^m, and thus conclude the desired global existence result.

To derive the above estimate, we will use the covariant Sobolev inequality (see e.g. [2] Appendix):

$$\|f\|_p \leqq K\{\sum_{1\leqq j\leqq d}\|D_j f\|_q\}^a\|f\|_r^{1-a}, \quad D_j = \partial_j - iA_j,$$

where $\frac{1}{p} = a(\frac{1}{q} - \frac{1}{d}) - (1 - a)\frac{1}{r}$, with $d \geqq 1$, $1 \leqq p \leqq \infty$, $1 \leqq q \leqq \infty$, $1 \leqq r < \infty$ and $0 \leqq a \leqq 1$ (if $p = \infty$, only $a < 1$ is allowed), $K = K(d,p,q,r)$, and the A_j are real and f is a complex valued, with $f \in L^r$, $\partial_j f \in L^q$ and $A_j f \in L^q$. We will need the following particular estimates:

(3.1) $$\|f\|_4 \leqq K\|Df\|_2^{d/4}\|f\|_2^{1-d/4}, \quad d = 1,2,$$

(3.2) $$\|f\|_\infty \leqq K\,\|Df\|_2^{1/2}\|f\|_2^{1/2}, \quad d = 1,$$

(3.3) $$\|f\|_\infty \leqq K\,\|D^2f\|_2^{\varepsilon/2}\|Df\|_2^{1-\varepsilon}\|f\|_2^{\varepsilon/2}, \quad 0 < \varepsilon < 1, \quad d = 2,$$

and also use the usual estimates obtained by setting $A_j = 0$ for all j in (3.1)-(3.3). Here as in the following, we write $\|D^s f\|_2$ for $\{\sum_{j_1,\dots,j_s} \|D_{j_1}\cdots D_{j_s} f\|_2^2\}^{1/2}$. We will also use the notation $\|\partial^s f\|_2$ to designate $\{\sum_{j_1,\dots,j_s} \|\partial_{j_1}\cdots\partial_{j_s} f\|_2^2\}^{1/2}$.

We shall denote u_n simply as u, and any positive locally bounded function of $t \in [0,\infty)$ (including any positive constant) which can be chosen independenty of n by the same letter C. Note that $u = (A_0, P_0, \vec{A}, \vec{P}, \psi) \in C([0,T_n); X^k)$ implies that $A_0 \in C^\ell([0,T_n)\ H^{k+1-\ell})$, $A_j \in C^\ell([0,T_n); H^{k-\ell})$ $(j = 1,\dots,d)$ and $\psi \in C^\ell([0,T_n); H^{k-2\ell})$, for $\ell = 0,1,\dots,[\frac{k}{2}]$. The idintities for the D_μ and the $F_{\mu\nu}$ will be freely used in the following arguments.

Lemma 3.2 and the fact that the sequence of initial data with which we are concerned is bounded in X^m first give

(3.4) $$E_1 < C, \quad \|\psi\|_2 < C.$$

Consider now the second order pseudo-energy E_2 defined by

$$E_2 = \int_{\mathbb{R}^d} \{D_jD_j\psi\overline{D_kD_k\psi} + \tfrac{1}{2}\partial_kF_{j0}\partial_kF_{j0} + \tfrac{1}{2}\partial_kF_{j\ell}\partial_kF_{j\ell}\}\,dx\,.$$

We note that (3.1) and (3.4) imply

$$\|D^2\psi\|_2^2 = \int_{\mathbb{R}^d} D_j D_k \psi \overline{D_j D_k \psi}\, dx$$

$$= \int_{\mathbb{R}^d} \{D_j D_j \psi \overline{D_k D_k \psi} + 2iF_{jk} D_j \psi \overline{D_k \psi} + i\partial_j F_{jk} \psi \overline{D_k \psi}\}\, dx$$

$$\leqq \|D_j D_j \psi\|_2^2 + \|F_{jk}\|_2 \|D_j \psi\|_4 \|D_k \psi\|_4 + \|\partial_j F_{jk}\|_2 \|\psi\|_4 \|D_k \psi\|_4$$

$$\leqq E_2 + C\|D^2\psi\|_2^{d/2} + CE_2^{1/2}\|D^2\psi\|_2^{d/4},$$

so that

$$\|D^2\psi\|_2^2 \leqq CE_2 + C. \tag{3.5}$$

We compute the time derivative of E_2 using Eqs.(1.1)-(1.2), and then estimate the result by means of (3.1)-(3.5). It results that

$$\frac{1}{2}\frac{d}{dt}E_2 = \mathrm{Re}\int_{\mathbb{R}^d} \{2iF_{j0} D_k \psi \overline{D_k D_j \psi} - i\partial_j \partial_j V \psi \overline{D_k D_k \psi}$$

$$- 2i\partial_j V D_j \psi \overline{D_k D_k \psi} - \partial_k F_{j0} F_{jk} \psi \overline{\psi}\}\, dx$$

$$\leqq 2\|F_{j0}\|_4 \|D_k \psi\|_4 \|D_k D_j \psi\|_2 + \|\partial_j \partial_j V\|_2 \|\psi\|_\infty \|D_k D_k \psi\|_2$$

$$+ 2\|\partial_j V\|_4 \|D_j \psi\|_4 \|D_k D_k \psi\|_2 + \|\partial_k F_{j0}\|_2 \|F_{jk}\|_4 \|\psi\|_4 \|\psi\|_\infty$$

$$\leqq CE_2 + C.$$

($\|\partial_j V\|_4$ should be estimated by the standard Sobolev inequality). Recalling that the sequence of initial data is bounded in X^m, we obtain

(3.6) $$E_2 < C.$$

In a completely analogous way, we further estimate for d = 2 the third order pseudo-energy

$$E_3 = \int_{\mathbb{R}^2} \{D_\ell D_j D_j \psi \overline{D_\ell D_k D_k \psi} + \tfrac{1}{2}\partial_\ell \partial_k F_{j0} \partial_\ell \partial_k F_{j0} + \tfrac{1}{4}\partial_\ell \partial_k F_{jn} \partial_\ell \partial_k F_{jn}\} \, dx ,$$

by using the covariant Sobolev inequalities and the above estimates. One first finds that

$$\begin{aligned} \|D^3\psi\|_2^2 &= \int_{\mathbb{R}^2} D_j D_k D_\ell \psi \overline{D_j D_k D_\ell \psi} \, dx \\ &= \int_{\mathbb{R}^2} \{D_\ell D_j D_j \psi \overline{D_\ell D_k D_k \psi} + iF_{kj} D_\ell \psi \overline{D_j D_k D_\ell \psi} - iF_{kj} D_j D_\ell \psi \overline{D_k D_\ell \psi} \\ &\quad + i\partial_j F_{\ell j} \psi \overline{D_k D_k D_\ell \psi} + 2iF_{\ell j} D_j \psi \overline{D_k D_k D_\ell \psi} - i\partial_k F_{\ell k} D_\ell D_j D_j \psi \overline{\psi} \\ &\quad - 2iF_{\ell k} D_\ell D_j D_j \psi \overline{D_k \psi}\} \, dx \\ &\leqq CE_3 + C\|D^3\psi\|_2 + C \end{aligned}$$

and thus obtains

(3.7) $$\|D^3\psi\|_2^2 \leqq CE_3 + C.$$

Using Eqs.(1.1)-(1.2) and noting this estimate, one has

$$\frac{1}{2}\frac{d}{dt}E_3 = \int_{\mathbb{R}^d} \{(2iF_{j0}D_\ell D_j\psi + iF_{\ell 0}D_jD_j\psi + 2i\partial_\ell F_{j0}D_j\psi + i\partial_j F_{j0}D_\ell\psi$$

$$+ i\partial_\ell\partial_j F_{j0}\psi - i\partial_\ell\partial_j\partial_j V\psi - i\partial_j\partial_j VD_\ell\psi - 2i\partial_\ell\partial_j VD_j\psi$$

$$- 2i\partial_j VD_\ell D_j\psi - i\partial_\ell VD_jD_j\psi)\overline{D_\ell D_k D_k\psi} + (iD_j\psi\overline{D_\ell D_k\psi}$$

$$+ iD_\ell D_j\psi\overline{D_k\psi} + iD_kD_j\psi\overline{D_\ell\psi} + iD_\ell D_kD_j\psi\overline{\psi})\partial_\ell\partial_k F_{j0}\}\,dx$$

$$\leqq CE_3 + C,$$

which gives

(3.8) $$E_3 < C.$$

The results (3.4)-(3.8) show that

(3.9) $$\|\psi\|_2 < C, \quad \|D^s\psi\|_2 < C, \quad s = 1,\ldots,[\tfrac{d}{2}+2],$$

(3.10) $$\|F_{\mu\nu}\|_{H^{[\frac{d}{2}+1]}} < C, \quad \mu,\nu = 0,1,\ldots,d,$$

for d = 1,2. To derive the needed bound on $(A_0,\ \partial_0 A_0,\ \vec{A},\ \partial_0\vec{A},\ \psi)$, we shall consider the following quantities:

$$E_{A_0} = \|\vec{A}\|_2^2 + \|A_0\|^2_{H^{[\frac{d}{2}+3]}} + \|\partial_0 A_0\|^2_{H^{[\frac{d}{2}+2]}}$$

$$= \int_{\mathbb{R}^d} A_\mu A_\mu \, dx + \sum_{|\alpha| \leq [\frac{d}{2}+2]} \int_{\mathbb{R}^d} \partial^\alpha \partial_\mu A_0 \partial^\alpha \partial_\mu A_0 \, dx$$

$$E_A = \|\partial \vec{A}\|^2_{H^{[\frac{d}{2}+1]}} + \|\partial_0 A\|^2_{H^{[\frac{d}{2}+1]}}$$

$$= \sum_{|\alpha| \leq [\frac{d}{2}+1]} \int_{\mathbb{R}^d} \partial^\alpha \partial_\mu A_j \partial^\alpha \partial_\mu A_j \, dx .$$

Here we use the notation ∂^α in the usual sense; thus $\partial^\alpha = \partial_{j_1} \cdots \partial_{j_s}$ with $|\alpha| = s$. Eq.(1.1), Eq.(1.3) and (3.10) give

$$\frac{1}{2} \frac{d}{dt} E_{A_0} = - \int_{\mathbb{R}^d} F_{j0} A_j \, dx - \sum_{|\alpha| \leq [\frac{d}{2}+2]} \int_{\mathbb{R}^d} \partial^\alpha \partial_0 A_0 \partial^\alpha \partial_j F_{j0} \, dx$$

$$\leq \|F_{j0}\|_2 \|A_j\|_2 + \|\partial_0 A_0\|_{H^{[\frac{d}{2}+2]}} \| |\psi|^2 \|_{H^{[\frac{d}{2}+2]}}$$

$$\leq \{C + \| |\psi|^2 \|_{H^{[\frac{d}{2}+2]}}\} E_{A_0}^{1/2} .$$

But from (3.1)-(3.3) and (3.9) one obtains

$$\| |\psi|^2 \|_2 \leq \|\psi\|_4^2 < C,$$

$$\|\partial |\psi|^2 \|_2 \leq 2\|\psi\|_4 \|D\psi\|_4 < C,$$

$$\|\partial^2 |\psi|^2 \|_2 \leq 2\|\psi\|_\infty \|D^2\psi\|_2 + 2\|D\psi\|_4^2 < C,$$

$$\|\partial^3 |\psi|^2 \|_2 \leq 2\|\psi\|_\infty \|D^3\psi\|_2 + 6\|D\psi\|_4 \|D^2\psi\|_4 < C,$$

the last estimate being needed only for d = 2. Thus it follows that

$E_{A_0} < C$. Eq.(1.3), (3,10) and this result yield

$$E_{\vec{A}} = \sum_{|\alpha| \leqq [\frac{d}{2}+1]} \int_{\mathbb{R}^d} \{\partial^\alpha F_{\mu j}\partial^\alpha \partial_\mu A_j + \partial^\alpha \partial_j A_0 \partial^\alpha \partial_0 A_j + \partial^\alpha \partial_0 A_0 \partial^\alpha \partial_0 A_0\}\, dx$$

$$\leqq \|F_{\mu j}\|_{H^{[\frac{d}{2}+1]}} \|\partial_\mu A_j\|_{H^{[\frac{d}{2}+1]}} + \|\partial_j A_0\|_{H^{[\frac{d}{2}+1]}} \|\partial_0 A_0\|_{H^{[\frac{d}{2}+1]}}$$

$$+ \|\partial_0 A_0\|^2_{H^{[\frac{d}{2}+1]}}$$

$$\leqq C E_{\vec{A}}^{1/2} + C,$$

giving $E_{\vec{A}} < C$. Therefore,

$$\|A_0\|_{H^{[\frac{d}{2}+3]}} < C, \quad \|\partial_0 A_0\|_{H^{[\frac{d}{2}+2]}} < C, \quad \|\vec{A}\|_{H^{[\frac{d}{2}+2]}} < C, \quad \|\partial_0 \vec{A}\|_{H^{[\frac{d}{2}+1]}} < C$$

Finally, from the definition of the D_μ, one has

$$\partial_j \psi = D_j \psi + i A_j \psi,$$

$$\partial_j \partial_k \psi = D_j D_k \psi + i \partial_j A_k \psi + i A_k \partial_j \psi + i A_j D_k \psi,$$

$$\partial_j \partial_k \partial_\ell \psi = D_j D_k D_\ell \psi + i \partial_j \partial_k A_\ell \psi + i \partial_k A_\ell \partial_j \psi + i \partial_j A_\ell \partial_k \psi + i A_\ell \partial_j \partial_k \psi$$

$$+ i \partial_j A_k D_\ell \psi + i A_k \partial_j \partial_\ell \psi + A_k \partial_j A_\ell \psi - A_k A_\ell \partial_j \psi - i A_j D_k D_\ell \psi.$$

Using these expressions, one finds, with the help of (3.1)-(3.3) and the usual Sobolev inequalities, that (3.9) and the above estimate on $\vec{A}$ imply that $\|\psi\|_{H^{[\frac{d}{2}+2]}} < C$, which completes the proof of the desired global result.

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