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On the global classical solutions to some classes of semilinear wave equations

福岡大 理 蛯原 幸義 (Yukiyoshi Ebihara)

1. Introduction

In this paper we are concerned with the initial-boundary value problem for the following semilinear wave equation :

$$\begin{cases} u_{tt} + (\gamma - \Delta)u + f(x, t, u, u_t) = 0, & x \in \Omega,\ t > 0, \\ u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), & x \in \Omega, \\ u(\xi, t) = 0, \quad \xi \in \partial\Omega, & t \geqq 0, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, γ is a nonnegative constant and $f(x, t, u, v)$ is a polynomial in u, v more precisely,

$$f(x, t, u, v) \equiv c_0 v + f_0(x, t, u, v),$$

$$f_0(x, t, u, v) \equiv g(x, t)\{c_1 u^{p_1} + c_2 v^{p_2} + c_3 u^{q_1} v^{q_2}\},$$

here c_i are constants, p_i, q_i are positive integers and $g(x, t)$ satisfies following condition : Let m be an integer with $m \geqq [\frac{n}{2}] + 1$. ① If $p_1 \geqq m + 1$, $p_2 \geqq m + 1$, $q_1 + q_2 \geqq m + 1$, then $g(x, t) \in C^2(0, \infty; C^{m+3}(\Omega))$. ② If, $p_1 < m + 1$ or $p_2 < m + 1$ or $q_1 + q_2 < m + 1$, then $g(x, t) \in C^2(0, \infty; C_0^{m+3}(\Omega))$. Our purpose is to find a sufficient condition to initial data and $f(x, t, u, v)$ under which the problem (*) admits a classical solution existing for all $t > 0$. We know some examples of equations which have such global solutions under so-called monotonicity conditions, small initial data conditions and so on, though we do not report them here. For the purpose, we construct (m)-solution, modified (m)-solution defined in this paper. Then we show that (*) has modified (m)-solution and it satisfies a certain variational inequality. And for the special cases :

$$u_{tt} + (\gamma - \Delta)u + c_0 u' + g(x)(u')^{2p+1} = 0$$

$$u_{tt} + (\gamma - \Delta)u + c_0 u' + g(x)u^{2p}u' = 0 ,$$

we show that there exists a set $W \subset V^{m+3} \times V^{m+2}$ such that if the initial values belong W, then the above equations admits (m)-solutions, where V^k are so-called escalated energy spaces (See the definitions in section 2). The interesting point of the set W is that it is not bounded in the space $V^{m+3} \times V^{m+2}$. Roughly speaking, the above equations admit global classical solution giving special initial data in $V^{m+3} \times V^{m+2}$ which are not bounded in this class.

Apart from this section, in Section 2, we prepare some notations of function spaces and make definitions of solutions of (*) and state main theorems, in Section 3, we make approximating equations, approximating solutions and make some lemmas, in final section 4 we give the sketch of proofs of Theorems.

2. Preliminaries

Let us put

$$(u, v) \equiv \int_\Omega uv\, dx , |u|^2 \equiv (u, u),$$

$$(u, v)_k \equiv ((-\Delta)^{\frac{k}{2}}u, (-\Delta)^{\frac{k}{2}}v), \quad |u|_k^2 \equiv (u, u)_k , k = 1, 2, \cdots.$$

Let $\{\phi_j\}$ be a system of eigen functions of $(-\Delta)$ considered on $\overset{\circ}{H}{}^1(\Omega) \cap H^2(\Omega)$. Then, we put V as a set of all finite linear combinations of $\{\phi_j\}$ and put V_k as a completion of V by the norm $|\cdot|_k$. Then we know that $V_k \subset \overset{\circ}{H}{}^1(\Omega) \cap H^k(\Omega)$ and the norm $|\cdot|_k$ is equivalent in the space V_k to the standard norm of $H^k(\Omega)$. We now make definitions of solutions of (*).

[Def. 1] A function $u(x, t)$ is said to be an (m)-solution of (*)

$\Longleftrightarrow$ (1) $u(x, t) \in C^1(0, \infty; V_{m+1}) \cap C^2(0, \infty; V_m)$

(2) $u(x, t)$ satisfies for any $\phi \in V$

$$\begin{cases} \frac{d^2}{dt^2}(u(t), \phi) + (\nabla u(t), \nabla\phi) + \gamma(u(t), \phi) \\ \qquad + (f(\cdot, t, u(t), u'(t)), \phi) = 0 \\ u(0) = u_0, \quad u_t(0) = u_1 \end{cases}$$

Note. If there exists such (m)-solution $u(x, t)$, then it is a global classical solution of (*) since $m \geqq [\frac{n}{2}] + 1$.

[Def. 2] Let $K(t) \in C^0[0, \infty)$, $K(t) > 0$ $(t \geqq 0)$. A function $u(x, t)$ is said to be a modified (m)-solution controlled by $K(t)$

$\Longleftrightarrow$ (1) $u(x, t) \in C^1(0, \infty; V_m)$,

(2) $u(0) = u_0$, $u_t(0) = u_1$,

(3) $|u_1|_m^2 < K(0)$, $|u_t(t)|_m^2 \leqq K(t)$ $(t > 0)$,

(4) if there exists $[\tau_1, \tau_2] \subset [0, \infty)$ such that $|u_t(t)|_m^2 < K(t)$, $t \in [\tau_1, \tau_2]$, then $u(t)$ satisfies the equation classically in the interval $[\tau_1, \tau_2]$.

We next define :

[Def. 3] A set $W \subset V_{m+3} \times V_{m+2}$ is said to be an (m)-admissible set for (*) if it holds that for any $(u_0, u_1) \in W$ there exists an (m)-solution whenever (u_0, u_1) is chosen as an initial value of (*).

[Def. 4] A set $W \subset V_{m+3} \times V_{m+2}$ is said to be an unbounded (m)-admissible set for (*) if W is an (m)-admissible set for (*) and further it is not bounded in $V_{m+3} \times V_{m+2}$.

We now state our main assertions.

[Th. 1] For any $(u_0, u_1) \in V_{m+3} \times V_{m+2}$, we have a modified (m)-solution controlled by $K(t) \equiv K$, where K is a constant with $|u_1|_m^2 < K$. Further, the function $u(x, t)$ satisfies,

(i) $u(x, t) \in C^0(0, \infty; V_{m+2}) \cap C^1(0, \infty; V_{m+2})$

$u(x, t) \in L^\infty_{\ell oc}(0, \infty; V_{m+3})$, $u_t(x, t) \in L^\infty_{\ell oc}(0, \infty; V_{m+2})$

$u_{tt}(x, t) \in L^\infty_{\ell oc}(0, \infty; V_{m+1})$,

(ii) $|u_t(\tau_1) - u_t(\tau_2)|_m \leqq C(t_1, t_2)|\tau_1 - \tau_2|$,

$$0 < t_1 < \tau_1 < \tau_2 < t_2 < \infty.$$

[Th. 2] We can construct the function $u(x, t)$ in Theorem 1 which satisfies the following modified variational inequality,

$$\int_0^T (u_{tt} + (\gamma - \Delta)u - g(\cdot, t, u, u_t), v(t) - u_t)_m dt \geqq 0,$$

$$\{u_{tt}(t) + (\gamma - \Delta)u(t) - g(\cdot, t, u(t), u_t(t))\}u_t(t) \leqq 0,$$

$$\text{a. e. } t \geqq 0.$$

for any $T > 0$ and $v(t) \in D \cap L^\infty(0, T; V_m)$, where D is an arbitrary bounded set in V_m. Further, if $D \equiv \{v \in V_m ; |v|_m^2 < K\}$ and $|u_1|_m^2 < K$, where K is large enough, then a function which satisfies above inequality is uniquely determined in the class if we put the condition $u_t(t) \in D$, $t \geqq 0$.

[Th. 3] If $m \geqq [\frac{n}{2}] + 2$, and $f(x, t, u, v)$ is the form

$$f(x, t, u, v) \equiv c_0 v + g(x)v^{2p+1}$$

or

$$f(x, t, u, v) \equiv c_0 v + g(x) u^{2p} v ,$$

then we can construct unbounded (m)-admissible set for (*).

3. Approximate equations and the solutions

Let $F(y)$ be a function which satisfies

(i) $F(y) \in C^2(0, \infty)$, (ii) $F(y) \geqq A/y^\beta$, $y \in (0, \alpha]$, $A > 0$ $\alpha > 0$, $\beta \geqq 1$ (iii) $F'(y) \leqq 0$, $y \in (0, \infty)$, (iv) $F(y) \equiv 1$ $y \in [1, \infty)$.

Now we consider the following problems :

$$(*)_\varepsilon \begin{cases} \mathcal{L}_\varepsilon(u) \equiv u'' + (\gamma - \Delta)u + f(x, t, u, u') + \varepsilon F(\frac{K-E(t)}{\varepsilon})u' = 0 \\ \qquad\qquad t > 0, \quad x \in \Omega \\ u(x, 0) = u_0(x), \quad u'(x, 0) = u_1(x) \\ u'(\cdot, t) \in V_m, \quad t \geqq 0, \end{cases}$$

where $\varepsilon > 0$, $E(t) \equiv |u'(t)|_m^2$, $u' \equiv D_t u$, and $K > 0$ is a constant with $|u_1|_m^2 < K$. Let $u_j^\varepsilon(t) \equiv \sum_{i=1}^j \lambda_{j,i}^\varepsilon(t)\phi_i(\cdot)$ be a unique solution of the problem :

$$(*)_{\varepsilon j} \begin{cases} (\mathcal{L}_\varepsilon(u_j^\varepsilon(t)), \phi_i) = 0, \quad t > 0, \quad i = 1, 2, \cdots, j \\ u_j^\varepsilon(0) = u_{0j} \equiv \sum_{i=1}^j a_i\phi_i, \quad (u_j^\varepsilon(0))' = u_{1j} \equiv \sum_{i=1}^j b_i\phi_i, \end{cases}$$

where, $(u_{0j}, u_{1j}) \longrightarrow (u_0, u_1)$ strongly in $V_{m+3} \times V_{m+2}$. Then we have the following lemmas.

[Lemma 1] Suppose the above assumptions. Then there exists j_0 such that

$$E_j^\varepsilon(t) \equiv |(u_j^\varepsilon(t)'|_m^2 < K , \quad t \in [0, \infty)$$

for each $j \geqq j_0$, $\varepsilon > 0$. (See lemma 2.1 in [1].)

[Lemma 2] Under the same assumptions, we have

(i) $$\sup_{\substack{t\in[0,T]\\ \varepsilon j}} \{|u_j^\varepsilon(t)|_{m+3} + |(u_j^\varepsilon(t))'|_{m+2} + |(u_j^\varepsilon(t))''|_{m+1}\} < C,$$

(ii) $$\varepsilon F\left(\frac{K-E_j^\varepsilon(t)}{\varepsilon}\right) \leqq C ,$$

for each fixed $T > 0$, where C are not depend on ε, j. (See lemma 2.3~2.4 in [1].)

[Lemma 3] Under the same assumptions, if we have

$$\sup_{t\in[\tau_1,\tau_2]} \limsup_{\substack{\varepsilon\to 0\\ j\to\infty}} E_j^\varepsilon(t) < K$$

for an interval $[\tau_1, \tau_2]$, then we obtain a function $u(t)$ belonging to $C^1(\tau_1, \tau_2 ; V_{m+1}) \cap C^2(\tau_1, \tau_2 ; V_m)$ and satisfying our equation in $[\tau_1, \tau_2]$. (See lemma 2.5 in [1])

[Cor. 1] If $\limsup_{\substack{\varepsilon\to 0\\ j\to\infty}} E_j^\varepsilon(t) < K$ for any $t > 0$, then we can construct (m)-solution of (*).

[Cor. 2] If $K > 0$ is large enough for u_0, u_1 and $g(x, t)$, then by these approximate solutions $\{u_j^{\varepsilon}(t)\}$, we can construct a local classical solution of (*).

4. Proofs of Theorems

4.1. Proof of Theorem 1

Apply to the results of Lemma 1～3, then with the use of standard compactness argument we have the assertion.

Q.E.D.

4.2. Proof of Theorem 2

Since D is bounded in V_m, we can find a positive number K large enough such that $K > \max\{|u_1|_m^2, \sup_{u \in D} |u|_m^2\}$ and which guarantees the condition of Corollary 2. Using this number K, we can construct modified (m)-solution $u(x, t)$ by Theorem 1. Then, from (ii) of Lemma 2, we know that

$$\varepsilon F\left(\frac{K-E_j^{\varepsilon}(t)}{\varepsilon}\right) \longrightarrow \chi(t) \equiv \frac{-1}{E(t)}\{(u'', u')_m + \gamma(u, u')_m + (u, u')_{m+1} - (g(\cdot, t, u, u'), u')_m\}$$

$$(\varepsilon \to 0,\ j \to \infty),$$

this implies that $u(t)$ should satisfy

$$u'' + (\gamma - \Delta)u + g(x, t, u, u') + \chi(t)u' = 0$$

for a.e. $t \geqq 0$ and $x \in \Omega$. Here we know that $\chi(t) \geqq 0$ a.e. $t \geqq 0$ and further $\chi(t_0) = 0$ for every t_0 with $E(t_0) < K$. Because, if $E(t_0) < K$, then we can prove that for some $\delta > 0$

$$\sup_{|t-t_0|\leqq\delta} \limsup_{\substack{\varepsilon\to 0 \\ j\to\infty}} E_j^{\varepsilon}(t) < K$$

holds. Then from Lemma 3, $\chi(t) = 0$ should follow for $t_0 - \delta \leqq t \leqq t_0 + \delta$. Thus, for every $T > 0$, $u(t)$ satisfies

$$0 = \int_0^T (u'' + (\gamma - \Delta)u - g(\cdot, t, u, u') + \chi(t)u', v - u')_m dt$$

for $v(t) \in D \cap L^\infty(0, T; V_m)$.

This shows

$$\int_0^T (u'' + (\gamma - \Delta)u - g(\cdot, t, u, u'), v - u')_m dt$$

$$\geqq \int_0^T \chi(t)\{|u'|_m^2 - |u'|_m|v|_m\}dt \geqq 0.$$

And

$$\{u'' + (\gamma - \Delta)u - g(\cdot, t, u, u')\}u' \leqq 0 \quad \text{a.e. } t \geqq 0$$

is obvious.

The final assertion of this theorem follows by setting $w(x, t)$ as another function in this class and

$$v_1(s) \equiv \begin{cases} w'(s) & (0 \leqq s \leqq t) \\ u'(s) & (0 < s \leqq T) \end{cases} \qquad v_2(s) \equiv \begin{cases} u'(s) & (0 \leqq s \leqq t) \\ w'(s) & (0 < s \leqq T). \end{cases}$$

In fact, since $v_1(s)$, $v_2(s) \in D \cap L^{\infty}(0, T, V_m)$, we get

$$\int_0^T (u'' + (\gamma - \Delta)u - g(\cdot, t, u, u'), v_1 - u')_m dt \geqq 0$$

$$\int_0^T (w'' + (\acute{\gamma} - \Delta)w - g(\cdot, t, w, w'), v_2 - w')_m dt \geqq 0 .$$

Therefore, $U(s) \equiv u(s) - w(s)$ should satisfy

$$\int_0^T (U'' + (\gamma - \Delta)U - g(\cdot, s, u, u') + g(\cdot, s, w, w'), U')_m ds \leqq 0.$$

From this, we can lead to the Gronwall type inequality.

Q.E.D.

4.3. Proof of Thoerem 3

Suppose that

$$f(x, t, u, v) \equiv c_0 v + g(x) v^{2p+1}$$

where $c_0 > 0$ and $g(x)$ is a function with our fundamental conditions in section 1 and further assume $g(x) \geqq 0$ in Ω. Then, the modified (m)-solution $u(x, t)$ satisfies

$$\frac{1}{2}\{|u'|_0^2 + \gamma|u|_0^2 + |u|_1^2\}' + c_0|u'|_0^2 + \int_\Omega g(x)(u')^{2p+2} dx \leqq 0,$$

$$\frac{1}{2}\{|u'|_m^2 + \gamma|u|_m^2 + |u|_{m+1}^2\}' + c_0|u'|_m^2 + (g(\cdot)(u')^{2p+2}, u')_m \leqq 0$$

for a.e. $t \geqq 0$.

From the first inequality, we get

$$|u'|_0 \longrightarrow 0 \quad (t \to \infty).$$

And since we have for some $\rho > 0$

$$|(g(\cdot)(u')^{2p+2}, u')_m| \leqq c|u'|_m^2 |u'|_{m-1}^{2p}$$

$$\leqq cK^{p-\rho}|u'|_0^{2\rho}|u'|_m^2 ,$$

Applying this to the second inequality, we have

$$\frac{1}{2}\{|u'|_m^2 + \gamma|u|_m^2 + |u|_{m+1}^2\}' \leqq |u'|_m^2\{-c_0 + cK^{p-\rho}|u'|_0^{2\rho}\}.$$

This implies $E(t) \equiv |u'|_m^2 \to 0$ $(t \to 0)$. Thus, for any $(u_0, u_1) \in V_{m+3} \times V_{m+2}$, we have the time $T \geqq 0$ such that

$$|u_1|_m^2 = |u'(T)|_m^2 , \quad |u'(t)|_m^2 < |u_1|_m^2 \quad (t > T).$$

Then, the modified (m)-solution $u(t)$ becomes genuine in $[T, \infty)$ and therefore if we write the correspondence as

$$X_T : \quad (u_0, u_1) \longrightarrow (u(T), u'(T))$$

and set

$$W \equiv \{X_T(u_0, u_1) ; (u_0, u_1) \in V_{m+3} \times V_{m+2}\} ,$$

then we know W is an unbounded (m) admissible set for (*).

In the next, differentiating by t the equation

$$u'' + (\gamma - \Delta)u + c_0 u' + g(x)(u')^{2p+1} = 0$$

we obtain

$$u^{(3)} + (\gamma - \Delta)u' + c_0 u'' + (2p + 1)g(x)(u')^{2p}u'' = 0 .$$

Then setting $U \equiv u'$, we have

$$U'' + (\gamma - \Delta)U + c_0 U' + (2p + 1)g(x)U^{2p}U' = 0 .$$

Therefore, applying previous result, we should have an unbounded (m)-admissible set in this case.

Q.E.D.

Note. This speech is based on the speaker's papers : [1] On solutions of semilinear wave equations, Nonlinear Analysis, T.M.A., 6, 467 - 486 (1982), [2] Modified variational inequalities to semilinear wave equations, ibd., 7, 821 - 826 (1983).