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# NORMALITY OF BLOWING-UP

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## §1. Introduction.

Let $A$ be a Noetherian local ring with maximal ideal $m$ and $d = \dim A > 0$. Let $q = (a_1, \ldots, a_d)$ be a parameter ideal in $A$ and put $R = \bigoplus_{n \geq 0} q^n$, the Rees ring of $q$. In this lecture we shall explore when the scheme Proj $R$ is normal and our result is stated as follows:

Theorem(1.1). Suppose that depth $A > 0$. Then the following conditions are equivalent.

(1) Proj $R$ is normal.

(2) $A$ is a regular local ring and $l_A(q + m^2/m^2) \geq d - 1$.

When this is the case, the ring $R$ is a normal ring with divisor class group $\mathbb{Z}$. (Here $l_A(q + m^2/m^2)$ stands for the length of the $A$-module $q + m^2/m^2$.)

In [6] K. Yamagishi also tackled with this theme and mentioned the equivalence of (1) and (2) in (1.1) with a rather strong assumption that $A$ is Cohen-Macaulay (cf. [6, Chap. 4, (1.3)]); our theorem guarantees his assumption can be replaced by the weaker one that depth $A > 0$ .

We will prove Theorem(1.1) in the next section. As is noted in (1.1), the ring $R$ is normal if (and only if) Proj $R$ is normal and depth $A > 0$ . The normality of $R$ itself is characterized in divers manners; especially, appealing to a recent result of J. Watanabe [7] on $m$-full ideals, we can prove that $R$ is normal if and only if $q$ is $m$-full. As the fact may have its own significance, in §3 we will discuss this subject a little more closely.

Throughout this lecture let $A$ denote a Noetherian local ring with maximal ideal $m$ . We assume that $\dim A = d > 0$ and fix a parameter ideal $q = (a_1, \ldots, a_d)$ in $A$ . Let $R = \bigoplus_{n \geq 0} q^n$ be the Rees ring of $q$ .

## §2. Proof of Theorem(1.1).

Let $B = A[x/a_1 \mid x \in q]$ and $P = mB$ . To begin with we note

Proposition(2.1). (1) $\dim B = d$ .

(2) $P$ is a height one prime ideal of $B$ and $P = \sqrt{a_1 B}$ .

(3) The elements $a_i/a_1$ mod $P$ ( $2 \leq i \leq d$ ) of $B/P$ are algebraically independent over $A/m$ .

Let $p \in \operatorname{Spec} A$ with $p \not\ni a_1$ . We put $I(p) = pA[1/a_1] \cap B$ . Then $I(p) \in \operatorname{Spec} B$ , $I(p) \cap A = p$ , and $B_{I(p)} = A_p$ . Let $x^* = x \bmod p$ for each $x \in A$ .

Lemma(2.2). $B/I(p) = A/p[x^*/a_1^* \mid x \in q]$ as $A$-algebras. In particular $\dim B/I(p) \leq \dim A/p$ .

Proof. By definition we get an embedding $B/I(p) \hookrightarrow A/p[1/a_1^*]$ of $A$-algebras, whose image coincides with $A/p[x^*/a_1^* \mid x \in q]$ .

As $\dim A/p[x^*/a_1^* \mid x \in q] \le \dim A/p$ by dimension formula, the second assertion follows from the first.

Corollary(2.3). (1) $\mathrm{Ass}\, B = \{ I(p) \mid p \in \mathrm{Ass}\, A$ and $p \not\ni a_1 \}$.
(2) $\{ I \in \mathrm{Spec}\, B \mid \dim B/I = d \} = \{ I(p) \mid p \in \mathrm{Spec}\, A$ and $\dim A/p = d \} = \{ I \in \mathrm{Spec}\, B \mid I \subsetneq P \}$.

Proof. Let $I \in \mathrm{Spec}\, B$ and put $p = I \cap A$. Then if $p \not\ni a_1$, we have $I = I(p)$ as $A[1/a_1] = B[1/a_1]$. Hence we get (1), because $a_1$ is $B$-regular and $A_p = B_{I(p)}$. Consider (2). First of all take $I \in \mathrm{Spec}\, B$ with $\dim B/I = d$. Then as $\dim B_I = 0$, we may write $I = I(p)$ with $p = I \cap A$. Notice $\dim B/I = d \le \dim A/p$ by (2.2) and we get $\dim A/p = d$. Conversely let $p \in \mathrm{Spec}\, A$ and assume $\dim A/p = d$. Then $p \not\ni a_1$ clearly. We put $I = I(p)$. Recall that $B/I = A/p[x^*/a_1^* \mid x \in q]$ as $A$-algebras and we see by (2.1) that the ring $B/P+I = (B/I)/m(B/I)$ is a polynomial ring with $d-1$ variables over the field $k = A/m$. Hence the canonical epimorphism $B/P \to B/P+I$ of $k$-algebras must be an isomorphism, because $B/P$ and $B/P+I$ are $k$-isomorphic; thus $P \supsetneq I$. Finally let $I \in \mathrm{Spec}\, B$ with $I \subsetneq P$. Then $\dim B/I = d$, as $\dim B/P = d-1$ —— this completes the proof of (2).

Let $e(A)$ (resp. $e(B_P)$ ) denote the multiplicity of $A$ (resp. $B_P$ ).

Lemma(2.4). $e(B_P) \ge e(A)$.

Proof. Let $h : A[T_2,\dots,T_d] \to B$ be the $A$-algebra map defined by $h(T_i) = a_i/a_1$ $(2 \le i \le d)$, where $T_2,\dots,T_d$ are indeterminates over $A$. Let $K = \mathrm{Ker}\, h$ and put $f_i = a_1T_i - a_i$ $(2 \le i \le d)$. Then $K \supset (f_2,\dots,f_d)$. Notice $a_1^nK \subset (f_2,\dots,f_d)$ for some integer $n \ge 1$, because $A[1/a_1] = B[1/a_1]$. Now let $C = A[T_2,\dots,T_d]_M$

where $M = mA[T_2,\ldots,T_d]$ and consider the exact sequence $0 \to L \to C/(f_2,\ldots,f_d)C \to B_P \to 0$ of $C$-modules. Then as $a_1^nL = 0$ and as $a_1, f_2, \ldots, f_d$ form a system of parameters for the local ring $C$, we have $l_C(L) < \infty$ and therefore $e(B_P) = e(C/(f_2,\ldots,f_d)C$. Recalling that $e(C/(f_2,\ldots,f_d)C) \geq e(C)$, we get the required inequality $e(B_P) \geq e(A)$, as $e(C) = e(A)$.

We say that $A$ is unmixed if $\dim \hat{A}/p = d$ for any $p \in \mathrm{Ass}\, \hat{A}$, where $\hat{A}$ denotes the completion of $A$. We shall use the following criterion of regularity.

Proposition(2.5)([4,(40.6)]). $A$ is a regular local ring if and only if $e(A) = 1$ and $A$ is unmixed.

The next result (2.6) is a key theorem in this lecture.

Theorem(2.6). Suppose that $A$ is unmixed. Then the following conditions are equivalent.

(1) $A$ is a regular local ring and $l_A(q + m^2/m^2) \geq d - 1$.

(2) $B_P$ is a DVR.

(3) Proj $R$ is normal.

Proof. (3)$\Rightarrow$(2) Since Spec $B$ appears as one of the affine charts of Proj $R$, this implication is clear.

(2)$\Rightarrow$(1) As $e(A) = 1$ by (2.4), we get $A$ is regular (cf. (2.5)). We will prove that $l_A(q + m^2/m^2) \geq d - 1$. Let us maintain the same notation as in Proof of (2.4). Notice that $K = (f_2,\ldots,f_d)$ in our case, since $a_1, \ldots, a_d$ is an $A$-regular sequence. Hence $f_2, \ldots, f_d$ is a part of a minimal system of generators for the maximal ideal $mC$ of $C$, because $B_P = C/(f_2,\ldots,f_d)C$ is a DVR by our assumption. Thus $l_A(q + m^2/m^2) = l_C(qC + m^2C/m^2C) \geq$

$d-1$ , as $qC = (a_1, f_2, \ldots, f_d)C$ .

(1)$\Rightarrow$(3) As the ring $R$ is Cohen-Macaulay (cf. [1]), the scheme $\operatorname{Proj} R$ satisfies the condition $(S_2)$. So it is enough to check that all the rings $A[x/a_i \mid x \in q]$ $(1 \leq i \leq d)$ satisfy the condition $(R_1)$. We may assume without loss of generality that $i = 1$ . Let $I \in \operatorname{Spec} B$ with $\dim B_I = 1$ . Then if $I \not\ni a_1$ , $B_I = A_p$ where $p = I \cap A$ and $B_I$ is a DVR in this case. Suppose that $I \ni a_1$ . Then we get $I = P$ by (2.1)(2). We must show that $B_P$ is a DVR. First of all notice that $m = (a_1, \ldots, \hat{a}_i, \ldots, a_d, b)$ for some $1 \leq i \leq d$ and $b \in m$ , because $l_A(q + m^2/m^2) \geq d-1$ . We put $J = bB_P$ . Assume $i \geq 2$ and write $a_i = \sum_{j \neq i} a_j x_j + by$ with $x_j$ , $y \in A$ . Then as $a_k = a_1 a_k/a_1$ for all $k$ , we get $a_1(a_i/a_1 - \sum_{j \neq 1,i} (a_j/a_1)x_j - x_1) \in J$ . Hence $a_1 \in J$ , because $a_i/a_1 - \sum_{j \neq 1,i} (a_j/a_1)x_j - x_1 \notin P$ (cf. (2.1)(3)). We can similarly prove that $a_1 \in J$ for the case $i = 1$ too. Thus $mB_P = bB_P$ , which gurantees that $B_P$ is a DVR. This completes the proof of (2.6).

Remark(2.7). Unless $A$ is unmixed, the implication (2)$\Rightarrow$(1) in (2.6) is not true in general even though $A$ is an integral domain and $B$ is a regular ring. In fact according to M. Nagata [4], there exist a Noetherian local integral domain $A$ of $\dim A = 2$ and a system $a$ , $b$ of parameters for $A$ which satisfy the following conditions: (1) $A$ is not a regular ring;

(2) $B = A[b/a]$ is a regular ring.

Proof. Take a Noetherian local integral domain $A$ of $\dim A = 2$ so that (1) the normalization $\bar{A}$ of $A$ is a regular ring and only has two maximal ideals, say $M$ and $N$ ; (2) $m = M \cap N$ ; (3) $\bar{A}$ contains elements $x$ and $z$ such that $M = (x-1, z)\bar{A}$ , $N = x\bar{A}$ , $z \in N$ , and $\bar{A} = A + Ax$ . (Such a ring $A$ must exist, see

[4, p.204].) Then $A$ is not regular as $A \neq \bar{A}$. We put $a = xz$ and $b = x(x-1)$. Then $a, b$ form a system of parameters in $A$. Let us check that $B = A[b/a]$ is a regular ring. Recall that $z \in m = M \cap N$. Then we see $B \supset \bar{A}$, as $B$ contains $b/a = (x-1)/z$ and as $\bar{A} = A + A(x-1)$ by (3); hence $B = \bar{A}[(x-1)/z]$. Let $Q$ be a prime ideal of $B$ and put $p = Q \cap \bar{A}$. If $Q \ni z$, then $Q$ contains $x-1 = z(x-1)/z$ and so we have $p = M$ by (3). Therefore we get $B_p = \bar{A}_M[(x-1)/z]$, which is a regular ring because $x-1, z$ is a regular system of parameters for $\bar{A}_M$ (cf. e.g. [2,(4.6)]). Hence the local ring $B_Q = (B_p)_{QB_p}$ is regular. If $Q \not\ni z$, then $B_Q = \bar{A}_p$ as $B[1/z] = \bar{A}[1/z]$ and we have $B_Q$ is a regular ring also in this case.

Corollary(2.8). The following conditions are equivalent.

(1) $B_P$ is a DVR.

(2) The completion $\hat{A}$ of $A$ contains a unique prime ideal $p$ such that $\dim \hat{A}/p = d$. Furthermore $\hat{A}/p$ is a regular local ring, $l_{\hat{A}}(q\hat{A} + m^2\hat{A} + p/m^2\hat{A} + p) \geq d - 1$, and $\hat{A}_p$ is a field.

Proof. We put $C = \hat{A}[x/a_1 \mid x \in q]$ and $Q = mC$. Then $C$ is a flat extension of $B$ as $C = \hat{A} \otimes_A B$. Notice $Q = PC$ and $P = Q \cap B$. Then we get $C_Q$ is a DVR if and only if $B_P$ is; thus we may assume that $A$ is complete.

(1)$\Rightarrow$(2) By (2.4) we get $e(A) = 1$. Consequently by the formula $e(A) = \sum_{p \in \operatorname{Spec} A,\, \dim A/p = d} l(A_p) \cdot e(A/p)$ (cf. [4,(23.5)]), we find that $A$ contains a unique prime ideal $p$ with $\dim A/p = d$. Furthermore $A/p$ is a regular local ring by (2.5), as $e(A/p) = 1$. Clearly $A_p$ is a field. Now we will prove that $l_A(q + m^2 + p/m^2 + p) \geq d - 1$. Let $I = I(p)$. Then by (2.3)(2) we see $I \subsetneq P$, whence $IB_P = 0$ as $B_P$ is a DVR. On the other hand we have by (2.2) an isomorphism $B/I = A/p[x^*/a_1^* \mid x \in q]$ of $A$-algebras. Thus the local ring $(A/p[x^*/a_1^* \mid x \in q])_P$ is a DVR and we conclude by (2.6) that $l_A(q + m^2 + p/m^2 + p) \geq d - 1$ .

(2)⇒(1) Let $I = I(p)$ . Notice that $I$ is, by (2.3)(2), a unique prime ideal of $B$ such that $I \subsetneq P$ . Then we get that $IB_P = 0$ , as $B_P$ is a Cohen-Macaulay ring and as $B_I = A_p$ is a field. Recalling the isomorphism $B/I = A/p[x^*/a_1^* \mid x \in q]$ , we have that $B_P = B_P/IB_P$ is a DVR because by (2.6) so is the local ring $(A/p[x^*/a_1^* \mid x \in q])_P$ .

Corollary(2.9). Assume that $A$ is a homomorphic image of a Cohen-Macaulay ring and let $a = a_1$ be a non-zerodivisor of $A$ . Then the following conditions are equivalent.

(1) $B$ is a normal ring.

(2) (a) $A[1/a]$ is a normal ring.

(b) $A$ contains a unique prime ideal $p$ such that $\dim A/p = d$ . Furthermore $A/p$ is regular and $l_A(q + m^2 + p/m^2 + p) \geq d - 1$ .

(c) For each $Q \in \operatorname{Ass} A$ with $Q \neq p$ , there is an integer $N \geq 1$ such that $a^N \in q^{N+1} + Q$ .

Proof. (1)⇒(2) As $A[1/a] = B[1/a]$ , we see $A[1/a]$ is a normal ring; hence $A$ is reduced as $A \subset A[1/a]$ . Notice that $B$ is integrally closed in the total quotient ring of $A$ , as it is normal. Then we get by (2.2) and (2.3)(1) an isomorphism $B = \prod_{Q \in \operatorname{Ass} A} A/Q[x^*/a^* \mid x \in q]$ (#) of $A$-algebras. Recall that $e(A) = 1$ by (2.4) and we find by the formula $e(A) = \sum_{p \in \operatorname{Spec} A,\ \dim A/p = d} l(A_p) \cdot e(A/p)$ that $A$ contains a unique prime ideal $p$ of $\dim A/p = d$ . Moreover $A/p$ is, by (2.5), a regular local ring because $A/p$ is unmixed by our standard assumption. Let $Q \in \mathrm{Ass}\, A$ such that $Q \neq p$. Then we get by the isomorphism (#) that $A/Q[x^*/a^* \mid x \in q] = m\cdot(A/Q[x^*/a^* \mid x \in q])$, since $P = mB$ is a prime ideal of $B$ and since $A/p[x^*/a^* \mid x \in q] \neq m\cdot(A/p[x^*/a^* \mid x \in q])$ by (2.1)(2). Hence we find that $B_P = (A/p[x^*/a^* \mid x \in q])_P$ is a DVR and that the element $a^* = a \bmod Q$ is invertible in the ring $A/Q[x^*/a^* \mid x \in q]$. Thus $l_A(q + m^2 + p/m^2 + p) \geq d - 1$ by (2.6) and $a^N \in q^{N+1} + Q$ for some $N \geq 1$.

(2)$\Rightarrow$(1) Let $J \in \mathrm{Spec}\, B$ and we will show that the local ring $B_J$ is normal. If $J \not\ni a$, this follows from (a) because $B[1/a] = A[1/a]$. Assume $J \ni a$, or equivalently $J \supset P$.

Claim. $J \not\supset I(Q)$ for any $Q \in \mathrm{Ass}\, A$ such that $Q \neq p$.

For, suppose that $J \supset I(Q)$ for some $Q \in \mathrm{Ass}\, A$ with $Q \neq p$. Then as $a^* = a \bmod Q$ is invertible in the ring $A/Q[x^*/a^* \mid x \in q]$ (cf. (c)), we find by (2.2) that $I(Q) + P = B$ whence $J = B$ —— this is a contradiction.

By this claim and the embedding $B \subset \prod_{Q \in \mathrm{Ass}\, A} B/I(Q)$ (recall that $\bigcap_{Q \in \mathrm{Ass}\, A} I(Q) = 0$ in $B$, see (2.3)(1)), we get that the ring $B_J$ appears as a local ring of $C = A/p[x^*/a^* \mid x \in q]$. Hence $B_J$ is normal by (2.6).

Example(2.10). Let $S = k[[X,Y,Z,W]]$ be a formal power series ring over a field $k$ and let $I = (X) \cap (Y,Z) \cap (X-Y,Z,W)$ in $S$. We put $A = S/I$, $a = Z^2 - X^2 \bmod I$, $b = Y - X \bmod I$, and $c = W - X \bmod I$. Then $q = (a,b,c)$ is a parameter ideal in $A$ and $B = A[b/a, c/a]$ is a normal ring.

Proof. To check that $q$ is a parameter ideal in $A$ is routine. To see that $B$ is normal, let $x = X \bmod I$, $y = Y \bmod I$ and $z = Z \bmod I$. Then $m = (x,b,c,z)$; hence $b$, $c$ form a part of a regular system of parameters for the ring $A/xA$ . As $a \in (q^2+(y,z)) \cap (q^2+(x-y,z,w))$ and as $A[1/a]$ is normal, our assertion follows from (2.9).

Lemma(2.11). Let $I = (b_1,\dots,b_s)$ be an $m$-primary ideal of $A$ . Then $A[x/b_i \mid x \in I] \neq m \cdot A[x/b_i \mid x \in I]$ for some $1 \leq i \leq s$ .

Proof. Assume the contrary and take an integer $N \geq 1$ so that $b_i^N \in mI^N$ for all $i$ . Let $G = \bigoplus_{n\geq 0} I^n/I^{n+1}$ and put $f_i = b_i \bmod I^2$ . Then as $f_i^N \in mG$ , we find that all the $f_i$'s are nilpotent in $G$ , whence $d = \dim G = 0$ —— this is a contradiction.

In the situation of (2.11) we don't have always $A[x/b_i \mid x \in I] \neq m \cdot A[x/b_i \mid x \in I]$ . (For instance, consider $A = k[[t^2,t^3]]$ and $I = (t^2,t^3)$ .) This is, of course, the case when $b_1,\dots,b_s$ is a system of parameters in $A$ , cf. (2.1).

We now prove Theorem(1.1).

Proof of Theorem(1.1). (2)$\Rightarrow$(1) See (2.6).

(1)$\Rightarrow$(2) According to (2.6) we have only to show that $A$ is unmixed. We put, as in Proof of (2.8), $C = \hat{A}[x/a_1 \mid x \in q]$ and $Q = mC$ . Let $N$ be a maximal ideal of $C$ such that $N \supset Q$ .

Claim 1. $\dim C_N/QC_N = d-1$ .

Proof. The ideal $N/Q$ is maximal in the ring $C/Q$ and so we have that $\dim C_N/QC_N = d-1$ , since $C/Q$ is a polynomial ring with $d-1$ variables over the field $A/m$ , cf. (2.1)(3).

Claim 2. $\dim C_N/I = d$ for any $I \in \mathrm{Ass}\, C_N$ .

Proof. Notice that $\mathrm{Ass}_B B/a_1B = \{P\}$ as $B$ is normal. Then we have $\mathrm{Ass}_C C/a_1C = \{Q\}$ as $B/a_1B \cong C/a_1C$ . Let $I \in \mathrm{Ass}\, C_N$ and take $J \in \mathrm{Ass}_{C_N} C_N/a_1C_N$ so that $J \supset I$ (this choice is possible as $a_1$ is $C_N$-regular, cf. e.g. [3,(15.D)]). Then we must have , as

$\mathrm{Ass}_{C_N} C_N/a_1C_N = \{ QC_N \}$ , that $J = QC_N$ whence $\dim C_N/J = \dim C_N/QC_N = d-1$ by Claim 1. Thus $\dim C_N/I = d$ since $J \supsetneq I$ .

Let us check that $A$ is unmixed. Assume the contrary and pick $p \in \mathrm{Ass}\, \hat{A}$ so that $\dim \hat{A}/p < d$ . Then we get by (2.11) $\hat{A}/p[x^*/a_i^* \mid x \in q] \neq m\cdot(\hat{A}/p[x^*/a_i^* \mid x \in q])$ (#) for some $1 \leq i \leq d$ , where $x^* = x \bmod p$ for each $x \in \hat{A}$ . We may assume $i = 1$ . Recall that the ideal $q$ is generated by non-zerodivisors of $A$ , because depth $A > 0$ by our standard assumption. Hence we may further assume that $a = a_1$ is a non-zerodivisor of $A$ . Then as $p \not\ni a$ , we get by (2.2) an isomorphism $C/I = \hat{A}/p[x^*/a^* \mid x \in q]$ of $\hat{A}$-algebras, where $I = I(p)$ . According to (#) this isomorphism guarantees that $Q + I \neq C$ , whence we may choose a maximal ideal $N$ of $C$ so that $N \supset Q + I$ . Then as $I \in \mathrm{Ass}\, C$ by (2.3)(1), we get $\dim C_N/IC_N = d$ by Claim 2 —— this is quite impossible since by (2.2) $\dim C_N/IC_N \leq \dim C/I \leq \dim \hat{A}/p < d$ . Thus $A$ is unmixed. The proof of the last assertion of (1.1) shall be given in the next section, see (3.1).

Remark(2.12). Proj $R$ is not necessarily regular even though Proj $R$ is normal and depth $A > 0$ . In fact, provided $d \geq 2$ and depth $A > 0$ , Proj $R$ is regular if and only if $A$ is a regular local ring and $q = m$ (cf. [2,(4.6)]).

## §3. Normality of the ring R .

In this section we discuss the normality of the ring $R = \bigoplus_{n\geq 0} q^n$ and our goal is

Theorem(3.1). The following conditions are equivalent.

(1) $A$ is regular and $l_A(q + m^2/m^2) \geq d-1$ .

(2) A is an integral domain and q is integrally closed.

(3) q is m-full.

(4) R is normal.

When this is the case, the divisor class group C(R) of R is an infinite cyclic group.

To begin with we recall the definition of m-full ideals.

Let I be an ideal of A. Then we say that I is m-full if $mI : x = I$ for some $x \in m$.

The concept of m-full ideals was introduced by D. Rees [5] and basic properties of such ideals are discussed in [7], a few of which we need to prove (3.1).

Let $v_A(M)$ denote the number of elements in a minimal system of generators for a finitely generated A-module M.

Proposition(3.2)([7, Theorem 2 and 3]). Let I be an m-primary ideal of A and assume that I is m-full. Let $x \in m$ such that $mI : x = I$. Then $v_A(J) \leq v_A(I) = l_A(A/I+xA) + v_A(I+xA/xA)$ for any ideal J of A containing I.

Let I be an ideal of A. Then an element x of A is called integral over I if x satisfies an equation $x^N + c_1 x^{N-1} + \cdots + c_N = 0$ with $c_i \in I^i$ ($1 \leq i \leq N$). Recall that I is said to be integrally closed if every element of A which is integral over I belongs to I.

The next result is due to D. Rees and a proof may be found in [7] (cf. Theorem 5).

Proposition(3.3). Suppose that A is an integral domain with infinite residue class field. Then every integrally closed ideal of $A$ is $m$-full.

Proof of Theorem(3.1). (4)$\Rightarrow$(2) Let $N = mR + R_+$ and $P \in \operatorname{Ass} R$. Then $P \subset N$, as $P$ is graded and as $N$ is a unique graded maximal ideal of $R$. As $R_N$ is normal, it is an integral domain and so $PR_N = 0$, whence $P = 0$. Thus $R$ is an integral domain and so $A$ is. Let us identify $R$ with the $A$-subalgebra $A[cT \mid c \in q]$ of $A[T]$ where $T$ is an indeterminate over $A$. Let $c \in A$ which is integral over $q$. Then as $cT$ is integral over $R$, we get $cT \in R$; hence $cT \in qT$, that is $c \in q$. Thus $q$ is integrally closed.

(3)$\Rightarrow$(1) Take $x \in m$ so that $mq : x = q$. Then by (3.2) we find that $v_A(m) \leq v_A(q) = l_A(A/q+xA) + v_A(q+xA/xA)$. So $A$ is a regular local ring, since $v_A(m) \leq v_A(q) = d$. Furhtermore we get $l_A(A/q+xA) = 1$, because $l_A(A/q+xA) \geq 1$ and $v_A(q+xA/xA) \geq d-1$. Thus $q + xA = m$, that is $l_A(q+m^2/m^2) \geq d-1$.

(2)$\Rightarrow$(1) Passing to the ring $A[U]_{mA[U]}$ where $U$ is an indeterminate over $A$, we may assume that the field $A/m$ is infinite. Then as $q$ is $m$-full by (3.3), our implication follows from (3)$\Rightarrow$(1).

(1)$\Rightarrow$(3) Let $x \in m$ with $m = (a_1,\ldots,\hat{a}_i,\ldots,a_d,x)$ for some $1 \leq i \leq d$. Then we get $l_A(A/q+xA) + v_A(q+xA/xA) = l_A(A/m) + v_A(m/xA) = d$ (a). Recalling the exact sequence $0 \to mq : x/mq \to A/mq \xrightarrow{x} A/mq \to A/mq+xA \to 0$ of $A$-modules, we have $l_A(mq : x/mq) = l_A(A/mq+xA)$ (b). Notice that $l_A(A/q+xA) + v_A(q+xA/xA) = l_A(A/q+xA) + l_A(q+xA/mq+xA) = l_A(A/mq+xA)$. Then we get by (a) and (b) that $l_A(mq : x/mq) = l_A(q/mq)$ (c), as $l_A(q/mq) = v_A(q) = d$.

Since $mq : x \supset q$ , it follows from (c) that $mq : x = q$ . Thus $q$ is $m$-full.

(1)$\Rightarrow$(4) Let $N = mR + R_+$ . Then as Proj $R$ is normal (cf. (2.6)), we get that the local ring $R_P$ is normal for any prime ideal $P$ ( $P \neq N$ ) of $R$ . On the other hand as $R$ is a Cohen-Macaulay ring (cf. [1]), we see depth $R_N = \dim R_N = d + 1 \geq 2$ ; so the local ring $R_N$ must be normal too. This completes the proof of the equivalence of the conditions in (3.1).

Let us compute the divisor class group $C(R)$ of $R$ . We may assume $d \geq 2$ . Let $e = e_q(A)$ and choose a minimal system $b_1, \ldots, b_d$ of generators for $m$ so that $q = (b_1^e, b_2, \ldots, b_d)$ . We put $p = b_1 A$ , $P = pA[T] \cap R$ , and $Q = mR$ . Then we have

Claim. (1) $P$ and $Q$ are height one prime ideals in $R$ .
(2) $P \cap A = p$ and $Q \cap A = m$ .
(3) $b_1 R = P \cap Q$ .

Proof. (1) As $R/P \cong R_{A/p}(q+p/p)$ , we get $\dim R/P = d$ ; hence $\dim R_P = 1$ . Recall that $R \cong A[T_1, \ldots, T_d]/I$ as $A$-algebras, where $C = A[T_1, \ldots, T_d]$ is a polynomial ring over $A$ and $I$ denotes the ideal of $C$ generated by all the $2 \times 2$ minors of the matrix $\begin{pmatrix} T_1 & T_2 & \cdots & T_d \\ b_1^e & b_2 & \cdots & b_d \end{pmatrix}$ . Hence $R/Q \cong A/m[T_1, \ldots, T_d]$ and we get $Q = mR$ is a height one prime ideal of $R$ .

(2) This is clear.

(3) It suffices to show that $\mathrm{Ass}_R R/b_1 R = \{ P , Q \}$ , $b_1 R_P = PR_P$ , and $b_1 R_Q = QR_Q$ . As $P \cap Q \ni b_1$ , $\mathrm{Ass}_R R/b_1 R \supset \{ P , Q \}$ clearly. Let $P' \in \mathrm{Ass}_R R/b_1 R$ and put $p' = P' \cap A$ . Notice $\dim R_{P'} = 1$ since $R$ is normal. If $p' = m$ , then $P' \supset Q$ ; hence $P' = Q$ . Assume $p' \neq m$ . Then $R_{p'} = A_{p'}[T]$ and $P'R_{p'} \supset p'R_{p'} \neq 0$ .

Consequently $P'R_{p'} = p'R_{p'}$ (as $\dim R_{p'} = 1$ ) and so we find $\dim A_{p'} = 1$ . Thus $p' = p$ (recall $p' \ni b_1$ ) and therefore $P'R_p = pR_p$ . Because $P \in \mathrm{Ass}_R R/b_1R$ and $P \cap A = p$ , we see $P'R_p = PR_p$ whence $P' = P$ . Thus $\mathrm{Ass}_R R/b_1R = \{P, Q\}$ . Furthermore as $PR_p = pR_p$ ( $= b_1R_p$ ), we find $b_1R_P = PR_P$ . Let $\tilde{R} = R[1/b_1^eT]$ and we get $Q\tilde{R} = b_1R$ (recall $b_i = b_1^e b_i T/b_1^e T$ for each $2 \le i \le d$ ). Therefore $b_1R_Q = QR_Q$ as $Q \not\ni b_1^eT$ , which completes the proof of Claim.

By this claim and the fact that $R[1/b_1] = A[1/b_1][T]$ is a UFD, we see that $C(R)$ is generated by $\mathrm{cl}(Q)$ ( $= -\mathrm{cl}(P)$ ). We must show that the order of $\mathrm{cl}(Q)$ is not finite. Assume the contrary and choose an integer $n > 0$ so that $n \cdot \mathrm{cl}(Q) = 0$ . Then $Q^{(n)} = bR$ for some $b \in R$ and, as $b_1^n \in Q^{(n)}$ , we may write $b_1^n = bc$ with $c \in R$ . Notice that $b, c \in A = R_0$ , because $b_1^n \neq 0$ and $R$ is a graded integral domain. Moreover we find $c \notin Q$ since $b_1^nR_Q = Q^nR_Q = bR_Q$ . Therefore $c$ is a unit of $A$ and we get $Q^{(n)} = bR = b_1^nR$ ; consequently $P \supset Q$ (as $P \ni b_1$ ). This is of course impossible and we conclude that $C(R) = \mathbb{Z}$ as required.

Remark(3.4). Let $B_i = A[x/a_i \mid x \in q]$ for $1 \le i \le d$ . We put $e = e_q(A)$ and assume $R$ is normal. Then we get for each $1 \le i \le d$ , similarly as in Proof of (3.1), that the divisor class group $C(B_i)$ of $B_i$ is a finite cyclic group and $|C(B_i)| \mid e$ . The proof is not complicated which we leave to readers.

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Added in proof. After giving this lecture, the author was told that a similar result as the equivalence of the conditions (1) and (2) in Theorem(3.1) had been obtained also by D. Katz (A criterion for complete-intersections to be self-radical, Arch. Math., 42 (1984), 423 - 425). However according to our Theorem(1.1), his main result Theorem 1 is not correct and therefore the proof of Corollary 6 and 7 in the paper is not complete. The author guarantees that they are immediate consequences of our Theorems (1.1) and (3.1).