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Microlocal Analysis for Nonlinear Equations Describing Incompressible Fluids

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§0. Introduction.

The purpose of this paper is to study the microlocal properties of the equations of incompressible fluids, that is, the microlocal hypoellipticity of the Navier-Stokes equations

$$\text{(NS)} \quad \begin{cases} \dfrac{\partial u}{\partial t} - \Delta u + \displaystyle\sum_{k=1}^{n} \frac{\partial}{\partial x_k}(u_k \cdot u) + \nabla p = f & \text{in} \quad (0,T) \times \Omega, \\ \nabla \cdot u = 0 & \text{in} \quad (0,T) \times \Omega \end{cases}$$

and the propagation of local and microlocal regularity of the Euler equations

$$\text{(E)} \quad \begin{cases} \dfrac{\partial u}{\partial t} + \displaystyle\sum_{k=1}^{n} \frac{\partial}{\partial x_k}(u_k \cdot u) + \nabla p = f & \text{in} \quad (0,T) \times \Omega, \\ \nabla \cdot u = 0 & \text{in} \quad (0,T) \times \Omega. \end{cases}$$

Here $0 < T < \infty$, Ω is an open set in $\mathbb{R}^n$ ($n \geqq 2$), the external force $f = (f_1, \cdots, f_n)$ is a given real-valued function of $t \in I = [0,T]$ and $x \in \Omega$, and the velocity $u = (u_1, \cdots, u_n)$ and the pressure p are unknown real-valued functions of t and x.

Microlocal analysis for nonlinear equations has been recently studied by many authors. See Lascar [13], Beals [2], [3], [4], Rauch [15], Bony [6], Meyer [14], Beals-Reed [5], Rauch-Reed [16],

[17], [18], [19], [20]. They supposed the existence of a solution with some regularity, and analyzed the solution microlocally. We work on a similar assumption; that is, we suppose the existence of a solution (u,p) which is "strong" in some sense. Indeed, it seems too difficult to discuss on such properties of the "weak" solutions at irregular points.

First we introduce a notation. For a function space $E \subset \mathcal{D}'(\Omega)$ with a stronger topology than that of $\mathcal{D}'(\Omega)$, let $B(I,E)$ denote the set of distributions $v(t,\cdot)$ with a parameter $t \in I$ such that

$$v(t,\cdot) \in E \quad \text{for all} \quad t \in I,$$

and that

$$\{\, v(t,\cdot) : t \in I \,\} \quad \text{is bounded in} \quad E.$$

Next we define some notions which will be used to describe our results.

<u>Definition.</u>

For a subset $C \subset I\times\Omega$ and a distribution $v(t,x) \in B(I,\mathcal{D}'(\Omega))$, we say that $v(t,x)$ is <u>locally in</u> E <u>on</u> C if, for every compact subset K of C, there exists a function $\phi(t,x) \in C_0^\infty(I\times\Omega)$ such that $\phi(t,x) \equiv 1$ holds on some neighborhood of K and that $\phi(t,x)v(t,x) \in B(I,E)$ holds.

For a subset $\Gamma \subset I\times(T^*\Omega\setminus 0) = I\times\Omega\times(\mathbb{R}^n\setminus\{0\})$ and $v(t,x)$ as above, we say that $v(t,x)$ is <u>microlocally in</u> E <u>on</u> Γ if, for every compact subset K of Γ, there exist functions $\phi(t,x) \in C_0^\infty(\mathbb{R}^n)$ and $\psi(t,x,\xi) \in C^\infty(I,S^0)$ such that the following three conditions hold:

$$\begin{cases} \phi(t,x) \equiv 1 & \text{on some neighborhood of } \pi(K). \\ \psi(t,x,\xi) \equiv 1 & \text{on some conic neighborhood of } K. \\ \psi(t,x,D)(\phi u)(t,x) \in B(I,E). \end{cases}$$

Here S^0 denotes the class of the symbols of the zeroth order pseudodifferential operators on $\mathbb{R}^n_x$, π denotes the natural projection of $I\times(T^*\Omega\setminus 0)$ onto $I\times\Omega$, and we say that U is a conic neighborhood of K if there exists an open subset V of $I\times(T^*\Omega\setminus 0)$ such that $K \subset V \subset U$ and that $(t,x,\lambda\xi) \in V$ holds for $(t,x,\xi) \in V$ and $\lambda \geqq 1$.

Now we can state the main theorem for (NS). We suppose that (u,p) is a solution of (NS), and that all u_j and p belong to the space $B(I,\mathcal{D}'(\Omega))$.

<u>Theorem 1.</u> (Microlocal hypoellipticity of (NS))

<u>Suppose</u> $0 < \mathring{t} \leq T$, $\mathring{x} \in \Omega$, $\mathring{\xi} \neq 0$ <u>and</u> $s > \max\{0, n/r-1\}$. <u>If each</u> u_j <u>is locally in</u> W^s_r <u>on the set</u> $\{(\mathring{t},\mathring{x})\}$ <u>and each</u> f_j <u>is microlocally in</u> $W^{2s-1-n/r}_r$ <u>on</u> $\{(\mathring{t},\mathring{x},\mathring{\xi})\}$, <u>then each</u> u_j <u>is microlocally in</u> $W^{2s+1-n/r-\delta}_r$ <u>on</u> $\{(\mathring{t},\mathring{x},\mathring{\xi})\}$ <u>for every positive number</u> δ.

Statements for p and $\frac{\partial u}{\partial t}$ will be given in Theorems 4 and 5 later.

To describe our results for (E), we must put further assumptions. Let σ be a number greater than 1. A function $v(t,x)$ on $I\times\Omega$ is said to belong to the class $C^{0,\sigma}(I\times\Omega)$ if the following two conditions (1) and (2) are satisfied:

(1) $\partial_x^\alpha u(t,x)$ exists and is bounded, continuous on $I\times\Omega$ for any $\alpha \in \mathbb{N}^n$ such that $|\alpha| < \sigma$.

(2) $|\partial_x^\alpha u(t,x)-\partial_x^\alpha u(t,y)|/|x-y|^{\sigma-k}$ is bounded on $I\times\Omega$ for any $\alpha \in \mathbb{N}^n$, where k is the greatest integer less than σ.

In the next definition and Theorems 2 and 3, we suppose that (u,p) is a solution of (E) such that $p \in B(I,\mathcal{D}'(\Omega))$ and that $u_j \in C^{0,\sigma}(I\times\Omega)$ for every $j = 1,\cdots,n$.

Remark 1.

For $n = 2$, Kato [10] proved the existence of the time-global solution of (E) satisfying the above assumptions. His results can be summarized as follows:

Suppose that Ω is a bounded domain with smooth boundary $\partial\Omega$ and that σ is a positive number such that $\sigma \notin \mathbb{N}$ and $\sigma > 1$. Put $f \equiv 0$, and let $u_0(x)$ be a function in $C^\sigma(\Omega)$ satisfying $u_0(x)\cdot n_x = 0$, where n_x is the normal vector of $\partial\Omega$ at x. Then there exists uniquely a pair (u,p) which satisfies

$$u(0,x) = u_0(x) \quad \text{on} \quad \Omega,$$

$$(0.1) \quad u(t,x)\cdot n_x = 0 \quad \text{on} \quad I\times\partial\Omega$$

and is a solution of (E) satisfying the above conditions.

For $n \geqq 3$, the existence of the solutions satisfying the above conditions have been obtained by Ebin-Marsden [8], Swann [22], Kato [11], Bourguignon-Brezis [7], Temam [23] and Kato-Lai [12]. But, in this case, the number T depends on $u_0(x)$.

These results suggest that our assumption is not unnatural.

Next, to state our results, we introduce some notions.

Definition. We call a connected integral curve of the vector field

$$\frac{\partial}{\partial t} + \sum_{k=1}^{n} u_k \cdot \frac{\partial}{\partial x_k} \qquad \text{in} \quad I\times\Omega$$

a trajectory curve, and a connected integral curve of

$$\frac{\partial}{\partial t} + \sum_{k=1}^{n} u_k \cdot \frac{\partial}{\partial x_k} - \sum_{j,k=1}^{n} \xi_j \cdot \frac{\partial u_j}{\partial x_k} \cdot \frac{\partial}{\partial \xi_k} \qquad \text{in} \quad I\times(T^*\Omega\setminus 0)$$

a bicharacteristic. That is, a curve $\{(t,X(t))\} = C$ is a trajectory curve and $\{(t,X(t),\Xi(t))\} = \Gamma$ is a bicharacteristic if and only if $X(t)$ and $\Xi(t)$ satisfy the system

$$\begin{cases} (0.2) \quad \dfrac{\partial X_j}{\partial t} = u_j(t,X(t)), \\ (0.3) \quad \dfrac{\partial \Xi_j}{\partial t} = -\displaystyle\sum_{k=1}^{n} \Xi_k(t)\frac{\partial u_k}{\partial x_j}(t,X(t)). \end{cases}$$

Remark 2.

To solve the above system, we first solve the equations (0.2). Owing to the Lipschitz condition of $u_j(t,x)$ with respect to x, the system (0.2) can be solved uniquely, at least locally in time. Then the linear system (0.3) can be solved as long as $X(t)$, the solution of (0.2), exists, and the solution $\Xi(t)$ is homogeneous of degree 1 with respect to the initial value. Especially, if the initial value is not equal to zero, then $\Xi(t)$ never vanishes.

If $u(t,x)$ satisfies the boundary condition (0.1), then the system (0.2) can always be solved for whole $t \in I$.

Roughly speaking, our results for the equation (E) are as follows: let C be a trajectory curve and Γ be a bicharacteristic. Then the local regularity of the solution $u(t,\cdot)$, where $t \in I$ is regarded as a parameter, propagates along C, provided the external force f is sufficiently smooth along C. Similarly, if f is sufficiently smooth along Γ, then the microlocal regularity of $u(t,\cdot)$ propagates along Γ; that is, for two different times s and t, the wave front set (modulo an appropriate function space) of $u(s,\cdot)$ is mapped onto that of $v(t,\cdot)$ by the transformation of $T^*\Omega$ induced by the diffeomorphism of Ω determined by the trajectory curves.

More strictly, we have the following theorems.

Theorem 2. (Propagation of local regularity in (E))

Suppose that f_j *is locally in* W_r^s *on a trajectory curve* C *for every* j, *where* $s > 1$. *Then, if there exists a point* $(\mathring{t},\mathring{x}) \in C$ *such that every* $u_j(\mathring{t},\cdot)$ *is locally in* W_r^s *at* $\mathring{x}$, *the solution* $u_j(x)$ *is locally in* W_r^s *on* C *for every* j.

Theorem 3. (Propagation of microlocal regularity in (E))

Suppose that $\sigma > 2$, *that every* $f_j(x)$ *is microlocally in* W_r^s *on a bicharacteristic* Γ, *and every* $u_j(x)$ *is locally in* $W_r^{s+2-\sigma}$ *on the trajectory curve* $\pi(\Gamma)$.

Then, if there exists a point $(\mathring{t},\mathring{x},\mathring{\xi}) \in \Gamma$ *such that every* $u_j(\mathring{t},\cdot)$ *is microlocally in* W_r^s *at* $(\mathring{x},\mathring{\xi})$, *the solution* $u_j(t,\cdot)$ *is microlocally in* W_r^s *on* Γ.

Remark 3.

The trajectory curve and the bicharacteristic play the same roles as those of the bicharacteristic curve and the bicharacteristic strip respectively, in the theory of linear equations. Usually, for higher order differential equations or first order systems, local regularity does not propagate along bicharacteristic curves. But in this case, the equation is essentially first order, hence all bicharacteristics passing through the fiber of a base point are mapped onto the same trajectory curve by the projection π. Owing to this fact, our local propagation theorem is valid.

Finally we shall consider the regularity of $\frac{\partial u}{\partial t}(t,\cdot)$ and $p(t,\cdot)$. For this purpose, we put

$$(0.4)\qquad \begin{cases} \text{any number greater than } \max\{0,\ n/r-n/2\} \\ \qquad \text{if } s \leq \max\{-n/r,\ n/r-n\}-1, \\ (s+n/r+1)/2 \quad \text{if } s > \max\{-n/r,\ n/r-n\}-1 \end{cases}$$

and

$$(0.5)\qquad \rho = \max\{s,\ \tau\}$$

for a real number s.

Then we have the following two theorems.

Theorem 4.

Let (u,p) be a solution of (NS) or (E) such that all u_j and p belong to the space $B(I,\mathcal{D}'(\Omega))$, and suppose that C is a subset of $I\times\Omega$. If all f_j are locally in W_r^s on C and if all u_j are locally in W_r^ρ on C, where ρ is determined by (0.5), then p is locally in W_r^{s+1} on C and all $\frac{\partial u_j}{\partial t}$ is locally in W_r^{s-1} on C if (u,p) is the solution of (E), and is in W_r^{s-2} on C if (u,p) is the solution of (NS).

Theorem 5.

Let (u,p) be as in the previous theorem, and suppose that Γ is a subset of $I\times(T^*\Omega\backslash 0)$. If the conditions

$$\begin{cases} \text{Every } f_j \text{ is microlocally in } W_r^s \text{ on } \Gamma. \\ \text{Every } u_j \text{ is microlocally in } W_r^s \text{ on } \Gamma. \\ \text{Every } u_j \text{ is locally in } W_r^\tau \text{ on } \pi(\Gamma), \text{ where } \tau \text{ is determined} \\ \quad \text{by } (0.4). \end{cases}$$

are satisfied, then p is microlocally in W_r^{s+1} on Γ. Every $\frac{\partial u_j}{\partial t}$ is microlocally in W_r^{s-1} on Γ if (u,p) is a solution of (E), and is microlocally in W_r^{s-2} on Γ if (u,p) is a solution of (NS).

Remark 4.

Using the results of [25] and [26], we can replace the Sobolev space W_r^s by the Besov space B_{pq}^s and the Triebel-Lizorkin space F_{pq}^s, which are generalizations of the Hölder space and the Sobolev space respectively. For the definitions and the basic properties of these spaces, see Triebel [24]. The local propagation theorem for the equation (E) in the Hölder space was, as far as the author knows, first obtained by Giga [9], and the propagation of local analyticity was proved by Alinhac-Métivier [1].

Proofs of Theorems 2 and 3 are given in Yamazaki [28], so we shall only prove Theorems 1, 4 and 5. The proof is a combination of the method of vorticity (see Serrin [21]) and the symbol calculus of paradifferential operators.

§1. Some results on paradifferential operators.

In this section we give a brief sketch of the theory of paradifferential operators used later. In the sequel let r be a number satisfying $1 < r < \infty$, and let $L^r = L^r(\mathbb{R}^n)$ and $L^\infty = L^\infty(\mathbb{R}^n)$ with respect to the Lebesgue measure. And we assume that the domain of integration is the whole space $\mathbb{R}^n$ unless otherwise specified.

First we fix a function $\Psi(t) \in C^\infty(\mathbb{R})$ such that $0 \leqq \Psi(t) \leqq 1$, $\Psi(t) \equiv 1$ for $t \leqq 1$, $\Psi(t) \equiv 0$ for $t \geqq 4/3$. Next we introduce functions $\Psi_j(\xi)$, $\Phi_j(\xi) \in C_0^\infty(\mathbb{R}^n)$ for $j \in \mathbb{N}$ by $\Psi_j(\xi) = \Psi(2^{-j}|\xi|)$, $\Phi_0(\xi) = \Psi_0(\xi)$ and $\Phi_j(\xi) = \Psi_j(\xi) - \Psi_{j-1}(\xi)$ for $j \geqq 1$.

Then, for a symbol $P(x,\xi)$ and a tempered distribution $u(x)$, we define $\pi_1(P(X,D),u)(x)$ as follows:

$$\pi_1(P(X,D),u)(x) = \sum_{j=2}^{\infty} F^{-1}\left[\int \Psi_{j-2}(\xi-\eta)\Phi_j(\eta)\hat{P}(\xi-\eta,\eta)\hat{u}(\eta)đ\eta\right](x).$$

Here $đ\eta = (2\pi)^{-n}d\eta$ and $\hat{P}(\xi,\eta)$ denotes the Fourier transform of $P(x,\eta)$ with respect to x; that is, $\hat{P}(\xi,\eta) = \int e^{-ix\cdot\xi}P(x,\eta)dx$. We call the operator $u(x) \to \pi_1(P(X,D),u)(x)$ the paradifferential operator associated with the symbol $P(x,\xi)$.

Then we have the following lemma, which provides a linearization adapted for the microlocal analysis.

Lemma 1.

Let $m \in \mathbb{N}$, $1 < r < \infty$, $1 < r', r'' \leqq \infty$, *and* $s', s'' \in \mathbb{R}$. *Suppose* $1/r = 1/r' + 1/r''$ *and* $s = s' + s'' > 0$.

If $u \in W^{s'}_{r'}$ *and* $v \in W^{s''}_{r''}$, *then* $w = uv - \pi_1(u,v) - \pi_1(v,u)$ *belongs to* W^s_r *and it satisfies the estimate* $\|w\|_{W^s_r} \leqq \|u\|_{W^{s'}_{r'}} \|v\|_{W^{s''}_{r''}}$.

Here, if r' *or* $r'' = \infty$, *the space* $W^{s'}_{r'}$ *or* $W^{s''}_{r''}$ *is replaced by* $C^{s'}$ *or* $C^{s''}$ *respectively.*

Next we introduce some classes of symbols. For a positive number σ, we define a function space C^σ by
$C^\sigma = \{f(x) \in L^\infty;\ \|f(x)\|_{C^\sigma} < \infty\}$, where

$$\|f\|_{C^\sigma} = \max_{|\alpha| \leqq [\sigma]} \|\partial^\alpha f\|_{L^\infty} + \max_{|\alpha| = [\sigma]} \sup_{x,y \in \mathbb{R}^n} \frac{|f(x) - f(y)|}{|x-y|^{\sigma - [\sigma]}}$$

if $\sigma \notin \mathbb{N}$, and

$$\|f\|_{C^\sigma} = \max_{|\alpha| < \sigma} \|\partial^\alpha f\|_{L^\infty} + \max_{|\alpha| = \sigma - 1} \sup_{x,y \in \mathbb{R}^n} \frac{|f(x+y) - 2f(x) + f(x-y)|}{|y|}$$

if $\sigma \in \mathbb{N}$.

For a nonpositive number σ, we define

$$C^\sigma = \{\ f(x) \in S';\ \|f\|_{C^\sigma} = \|(1-\Delta)^{-N} f\|_{C^{2N+\sigma}} < \infty\ \},$$

where N is the least positive integer greater than $-\sigma/2$.

Next, let E denote L^∞ or C^σ, and we put

$$S(E)^m = \{P(x,\xi);\ \text{For every } \alpha \quad \mathbb{N}^n \text{ there exists } C_\alpha > 0 \text{ such that } \|\partial_\xi^\alpha P(x,\xi)\|_E \leqq C_\alpha \langle\xi\rangle^{m-|\alpha|}\}$$

for a real number m. Here $\langle\xi\rangle = (1+|\xi|^2)^{1/2}$ and $\partial_\xi^\alpha P(x,\xi)$ is regarded as a function of x with ξ as a parameter.

Then we have the following results on the boundedness of paradifferential operators.

Lemma 2.

If a symbol $P(x,\xi)$ *belongs to the class* $S(L^\infty)^m$ [*resp.* $S(C^\sigma)^m$, $\sigma < 0$], *then the operator* $\pi_1(P(x,D),\cdot)$ *is bounded of* W_r^s *onto* W_r^{s-m} [*resp. onto* $W_r^{s+\sigma-m}$] *for every* $1 < r < \infty$ *and real numbers* m, s.

Lemma 3.

If a symbol $P(x,\xi)$ *belongs to the class* $S(C^\sigma)^m$ *and* $s+\sigma-m > 0$, *then the operator* A *defined by* $Au(x) = P(x,D)u(x)-\pi_1(P(x,D),u)(x)$ *is bounded of* W_r^s *onto* W_r^κ , *where* $\kappa = \sigma+\min\{0,s-m\}$.

These three lemmas are special cases of Theorem 5 in Yamazaki [25], and the proof is given in Yamazaki [27].

Finally we remark that paradifferential operators have the same symbol calculus as that of pseudodifferential operators. Namely, we have the following

Lemma 4.

Suppose $\sigma \geq 0$ *and that* $P(x,\xi)$ *and* $Q(x,\xi)$ *are symbols belonging to* $S(C^{\sigma})^{m_1}$ *and* $S(C^{\sigma})^{m_2}$ *respectively. In case* $\sigma = 0$, *we also assume that either* $P(x,\xi)$ *or* $Q(x,\xi)$ *belongs to* $S(C^{\varepsilon})^{m_1}$ *for some* $\varepsilon > 0$. *If we put*

$$R(x,\xi) = \sum_{|\alpha| \leq \sigma} \frac{1}{i^{|\alpha|}\alpha!} \partial_{\xi}^{\alpha} P(x,\xi) \cdot \partial_{x}^{\alpha} Q(x,\xi),$$

then the operator A *defined by*

$$Au = \pi_1(R(x,D),u) - \pi_1(P(x,D),\pi_1(Q(x,D),u))$$

is bounded of W_r^s *into* $W_r^{s+\sigma-m_1-m_2}$.

This lemma is proved in [27].

§2. Proof of Theorem 1.

In the proof we identify all vector-valued functions $v(t,x) = (v_1(t,x),\cdots,v_n(t,x))$ with 1-forms $v = v_1(t,x)dx_1+\cdots+v_n(t,x)dx_n$, and let d denote the exterior differentiation with respect to the x-variables, and let d^* be the dual of d. Then the system (NS) can be written as

$$(2.1)\quad \begin{cases} \dfrac{\partial}{\partial t} u + d^* du + \displaystyle\sum_{k=1}^{n} \dfrac{\partial}{\partial x_k}(u_k u) + dp = f. \\ d^* u = 0. \end{cases}$$

First, it follows from the hypotheses that there exist an open neighborhood V of $(\overset{\circ}{t},\overset{\circ}{x})$ in $I\times\Omega$ and a conic open neighborhood Γ' of ξ such that we can take smooth functions $\phi''_j(t,x) \in C_0^\infty(I\times\Omega)$ and symbols $\psi''_j(t,x,\xi) \in C^\infty(I,S^0)$ for $j = 1,\cdots,n$ satisfying

$$(2.2)\quad \begin{cases} \phi''_j(t,x) \equiv 1 \quad \text{on} \quad V^-, \\ \psi''_j(t,x,\xi) \equiv 1 \quad \text{on} \quad (V\times\Gamma')^-, \\ \psi''_j(t,x,D)(\phi''_j f_j)(t,x) \in B(I,W_r^{2s-1-n/r}) \quad \text{for every} \\ \qquad j = 1,\cdots,n. \end{cases}$$

Then it is sufficient to prove the following

Proposition 1.

Suppose that $s \leqq \kappa < 2s+1-n/r$ and that there exist an open neighborhood V' of $(\overset{\circ}{t},\overset{\circ}{x})$ in $I\times\Omega$ and a conic open neighborhood Γ'' of $\overset{\circ}{\xi}$ such that we can take $\phi'(t,x) \in C_0^\infty(I\times\Omega)$ and $\psi'(t,x,\xi) \in C^\infty(I,S^0)$ satisfying

$$(2.3)\quad \begin{cases} \phi'(t,x) \equiv 1 \quad \text{on} \quad (V')^-, \\ \psi'(t,x,\xi) \equiv 1 \quad \text{on} \quad (V'\times\Gamma'')^-, \\ \psi'(t,x,D)(\phi' u_j)(t,x) \in B(I,W_r^\kappa) \quad \underline{\text{for every}} \quad j = 1,\cdots,n. \end{cases}$$

<u>Then there exist</u> $\tilde{\phi}(t,x) \in C_0^\infty(I\times\Omega)$ <u>and</u> $\tilde{\psi}(t,x,\xi) \in C^\infty(I,S^0)$ <u>such that</u>

$$\begin{cases} \tilde{\phi}(t,x) \equiv 1 \quad \underline{\text{on some neighborhood of}} \quad (\mathring{t},\mathring{x}), \\ \tilde{\psi}(t,x,\xi) \equiv 1 \quad \underline{\text{on some conic neighborhood of}} \quad (\mathring{t},\mathring{x},\mathring{\xi}), \\ \tilde{\psi}(t,x,D)(\tilde{\phi} u_j)(t,x) \in B(I,W_r^\mu) \quad \underline{\text{holds for every}} \quad j = 1,\cdots,n \\ \quad \underline{\text{and every}} \quad \mu < \min\{\ 2s-n/r+1,\ \kappa+1\ \}. \end{cases}$$

First we derive Theorem 1 from this proposition. The hypotheses of the proposition are satisfied for $\kappa = s$ if we take $\phi'(t,x)$ suitably and put $\psi'(t,x,\xi) \equiv 1$.

If $s \leqq n/r$, then $s+1 \geqq 2s-n/r+1$ and the conclusion of Theorem 1 immediately follows from the proposition.

Otherwise, we take $\mu = s+1/2$, and apply the proposition again with κ, ϕ', ψ' replaced by μ, $\tilde{\phi}$, $\tilde{\psi}$ respectively. Then we conclude that each u_j is microlocally in W_r^ν at $(\mathring{t},\mathring{x},\mathring{\xi})$ for every $\nu < \min\{\ 2s-n/r+1,\ s+3/2\ \}$, and if $s \leqq n/r+1/2$ we are finished. Since $s \leqq n/r+k/2$ holds for some $k \in \mathbb{N}$, we obtain the conclusion by repeated use of Proposition 1.

Proof of the proposition. Take $\phi(t,x) \in C_0^\infty(V \cap V')$ satisfying $\phi(t,x) \equiv 1$ on U', where U' is an open neighborhood of $(\overset{\circ}{t},\overset{\circ}{x})$ in $V \cap V'$, and put $v(t,x) = \phi(t,x)u(t,x)$ and $q(t,x) = \phi(t,x)p(t,x)$.

Next we take $\Psi(t,x,\xi)$ $C^\infty(I,S^0)$ such that $\Psi(t,x,\xi) \equiv 0$ out of $U'\times(\Gamma' \cap \Gamma'')$, and that $\Psi(t,x,\xi) \equiv 1$ on $U\times\Gamma$, where U is an open neighborhood of $(\overset{\circ}{t},\overset{\circ}{x})$ in U' and Γ is a conic neighborhood of $\overset{\circ}{\xi}$ in $\Gamma' \cap \Gamma''$.

Then we have

$$\begin{aligned}
&\Psi(t,x,D)d^*d(\phi f_j)(t,x)\\
&= \Psi(t,x,D)d^*d(\phi\cdot\phi_j f_j)(t,x)\\
&= \Psi(t,x,D)d^*d\{\phi(t,x)\cdot(1-\psi_j(t,x,D))(\phi_j f_j)(t,x)\}+\\
&\qquad +\Psi(t,x,D)d^*d\{\phi(t,x)\cdot\psi_j(t,x,D)(\phi_j f_j)(t,x)\}
\end{aligned}$$

for all $j = 1,\cdots,n$.

Since $\operatorname{supp}\Psi \cap \operatorname{supp}(1-\psi_j) = \emptyset$, the first term belongs to $B(I,W_r^\sigma)$ for all $\sigma \in \mathbb{R}$. On the other hand, it follows from (2.2) that the second term belongs to $B(I,W_r^{2s-3-n/r})$. Thus

$$(2.4) \quad \Psi(t,x,D)d^*d(\phi f_j)(t,x) \in B(I,W_r^{2s-3-n/r}) \quad \text{for all } j = 1,\cdots,n.$$

In the same manner, from (2.3) and the equation $d^*u = 0$, we obtain

(2.5) $v_j(t,x) \in B(I,W_r^s)$ for all $j = 1,\cdots,n$,

(2.6) $\Psi(t,x,D)d^*dv_j(t,x) \in B(I,W_r^{\kappa-2})$ for all $j = 1,\cdots,n$,

(2.7) $\Psi(t,x,D)dd^*v_j(t,x) \in B(I,W_r^\sigma)$ for all $j = 1,\cdots,n$ and $\sigma \in \mathbb{R}$.

On the other hand, from (2.1) we obtain

$$(2.8) \quad \frac{\partial}{\partial t}v + d^*dv + \sum_{k=1}^{n}\frac{\partial}{\partial x_k}(v_k v) + dq = g,$$

where $g(t,x) = \phi(t,x)f(t,x)+f'(t,x)$ and $f'(t,x) \equiv 0$ on U.

Then it follows immediately from (2.4) that

$$\Psi(t,x,D)d^*dg_j(t,x) \in B(I,W_r^{2s-3-n/r}) \quad \text{for all } j = 1,\cdots,n.$$

Applying d^*d to both sides of (2.8), we have

$$(2.9) \quad \frac{\partial}{\partial t}d^*dv - \Delta d^*dv + \sum_{k=1}^{n} d^*d\frac{\partial}{\partial x_k}(v_k v) = d^*dg,$$

since $\Delta = -dd^*-d^*d$ and $d^*d^* = 0$.

We shall apply Lemma 1 to the third term of the left-hand side of (2.9). Take a real number r' satisfying $r' \geqq r$, $r > 2$ and $s-n/r+n/r' > 0$. This is possible since $s > 0$ and $s-n/r+n/2 > n/2-1 \geqq 0$ if $r \leqq 2$. Then we have $v_j \in B(I,W_r^{s-n/r+n/r'})$ for every j by the Sobolev imbedding theorem. Hence, Lemma 1 implies

$$v_h v_k-\pi_1(v_k,v_h)-\pi_1(v_h,v_k) \in B(I,W_{r'/2}^{2s-2n/r+2n/r'}) ,$$

where $\pi_1(v_k,\cdot)$ denotes the paradifferential operator with symbol $v_k(t,\cdot)$ on $\mathbb{R}^n$ with $t \in I$ as a parameter. But, applying the Sobolev imbedding theorem again, we obtain $W^{2n-2n/r+2n/r'}_{r'/2} \subset W^{2s-n/r}_r$. It follows that

$$(2.10)\quad d^*d\frac{\partial}{\partial x_k}(v_k v) = d^*d\frac{\partial}{\partial x_k}(\pi_1(v_k,v)+\pi_1(v,v_k))+h_1 ,$$

where $(h_1)_j \in B(I,W^{2s-n/r-3}_r)$ for every $j = 1,\cdots,n$.

Next, the Leibniz rule yields

$$d^*d\frac{\partial}{\partial x_k}\pi_1(v_k,v)$$
$$= \pi_1(v_k,\nabla^3 v)+\pi_1(\nabla v_k,\nabla^2 v)+\pi_1(\nabla^2 v_k,\nabla v)+\pi_1(\nabla^3 v_k,v),$$

where the first term denotes a summation of the form

$$\sum_{h,j,\ell=1}^{n} a_{hj\ell}\pi_1(v_k, \frac{\partial^3 v}{\partial x_h \partial x_j \partial x_\ell}),$$

and other terms denote analogous summations.

For $j = 0,\cdots,3$, the Sobolev imbedding theorem implies $\nabla^j v_k \in B(I, C^{s-n/r-j})$. Hence, if $s < n/r+j$, we obtain $\pi_1(\nabla^j v_k, \nabla^{3-j}v_\ell) \in B(I,W^{2s-n/r-3}_r)$ by virtue of Lemma 1.2. This fact and an analogous argument for $\pi_1(v,v_k)$, together with (2.10), lead to

$$(2.11)\quad d^*d\frac{\partial}{\partial x_k}(v_k v) = \sum_{j=0}^{J}\{\pi_1(P_j(t,x,D),v)+\pi_1(Q_j^{(k)}(t,x,D),v_k)\} + h_2 ,$$

where $J = \min\{3, [s-n/r]\}$, the symbols $P_j(t,x,\xi)$ and $Q_j^{(k)}(t,x,\xi)$ ($k = 1,\cdots,n$) belong to $B(I,S(C^{s-n/r-j})^{3-j})$, and $(h_2)_\ell$ belongs to $B(I,W_r^{2s-n/r-3})$ for every $\ell = 1,\cdots,n$.

Substituting (2.11) into (2.9), we obtain

$$(2.12) \qquad \frac{\partial}{\partial t}d^*dv-\Delta d^*dv+\sum_{j=0}^{J}\pi_1(P_j(t,x,D),v)+\sum_{k=1}^{n}\sum_{j=0}^{J}\pi_1(Q_j^{(k)}(t,x,D),v_k)$$

$$= h_3 ,$$

where every coefficient of h_3 belongs to $B(I,W_r^{2s-n/r-3})$.

Next, we shall represent v by d^*dv and dd^*v. Put $L = \{ x \in \Omega;\ (t,x) \in V \text{ for some } t \in I \}$, and let $\chi(x)$ be a function such that $\chi(x) \equiv 1$ on L. Then $\Delta v(t,x) \equiv 0$ for $x \in \Omega\setminus L$. Hence v is represented as $G*(\Delta v)$ on Ω, where G is the Green function of the domain $\Omega' \Subset \Omega$ with smooth boundary containing $\operatorname{supp}\chi(x)$. Then, since $\chi \equiv 1$ on $\operatorname{supp} v$, we have $v = \chi v = \chi G*(\Delta v) = \chi G*(\chi\cdot\Delta v)$. On the other hand, the operator $G*$ is a pseudodifferential operator of order -2 on Ω', hence the operator $f \to \chi G*(\chi f)$ is a pseudodifferential operator of order -2 on $\mathbb{R}^n$. Let $K(x,\xi)$ denote its symbol. Then we have

$$(2.13) \quad v = K(x,D)\Delta v = -K(x,D)(d^*dv+dd^*v).$$

Now take a symbol $\psi(t,x,\xi) \in C^\infty(I,S^0)$ such that $\psi(t,x,\xi) \equiv 0$ if t is sufficiently small, $\psi(t,x,\xi) \equiv 1$ on some conic neighborhood of $(\overset{\circ}{t},\overset{\circ}{x},\overset{\circ}{\xi})$ and $\operatorname{supp}\psi$ is a compactly supported conic subset of $U\times\Gamma$. Then, applying $\psi(t,x,D)$ to (2.12) and making use of (2.13), we obtain

$$(2.14)\quad h_3 = \frac{\partial}{\partial t}\psi(t,x,D)d^*dv - \frac{\partial\psi}{\partial t}(t,x,D)d^*dv - \Delta\big(\psi(t,x,D)d^*dv\big) +$$

$$+ \sum_{k=1}^{n} \frac{\partial\psi}{\partial x_k}(t,x,D)\frac{\partial}{\partial x_k}d^*dv + (\Delta\psi)(t,x,D)d^*dv +$$

$$+ \sum_{j=0}^{J} \pi_1\big(\psi(t,x,D),\pi_1(P_j(t,x,D),-K(x,D)(d^*dv+dd^*v))\big) +$$

$$+ \sum_{k=1}^{n}\sum_{j=0}^{J} \pi_1\big(\psi(t,x,D),\pi_1(Q_j^{(k)}(t,x,D),$$

$$-K(x,D)(d^*dv_k+dd^*v_k))\big).$$

Here we remark that $\psi(t,x,\xi)$, $\frac{\partial\psi}{\partial t}(t,x,\xi)$, $\frac{\partial\psi}{\partial x_k}(t,x,\xi)$ and $\Delta\psi(t,x,\xi)$ all belong to $B(I,S^0)$, and that $K(x,\xi) \in S^{-2}$. For every $\sigma > 0$, these classes are contained in $B(I,S(C^\sigma)^0)$ and $S(C^\sigma)^{-2}$ respectively. Hence, by virtue of Lemma 3, we can replace these pseudodifferential operators by paradifferential operators if we are working modulo $B(I,W_r^\infty)$. From this we obtain

$$(2.15)\quad h_4 = \frac{\partial}{\partial t}\psi(t,x,D)d^*dv - \pi_1\Big(\frac{\partial\psi}{\partial t}(t,x,D),d^*dv\Big) - \Delta\big(\psi(t,x,D),d^*dv\big) +$$

$$+ \sum_{k=1}^{n} \pi_1\left(\frac{\partial\psi}{\partial x_k}(t,x,D),\frac{\partial}{\partial x_k}d^*dv\right) + \pi_1\big((\Delta\psi)(t,x,D),d^*dv\big) -$$

$$- \sum_{j=0}^{J} \pi_1\big(\psi(t,x,D),\pi_1\big(P_j(t,x,D),\pi_1(K(x,D),d^*dv+dd^*v)\big)\big) -$$

$$- \sum_{k=1}^{n}\sum_{j=0}^{J} \pi_1\big(\psi(t,x,D),\pi_1\big(Q_j^{(k)}(t,x,D),$$

$$\pi_1(K(x,D),d^*dv+dd^*v)\big)\big),$$

where every coefficient of h_4 belongs to $B(I,W_r^{2s-n/r-3})$.

We consider the term

$\pi_1(\psi(t,x,D),\pi_1(P_j(t,x,D),\pi_1(K(x,D),d^*dv)))$. Put $\tau = s-n/r-j \geqq 0$. Then, as before, we can replace d^*dv by $\pi_1(1,d^*dv)$ modulo $B(I,W_r^\infty)$. This implies

$$\begin{aligned}&\pi_1(\psi(t,x,D),\pi_1(P_j(t,x,D),\pi_1(K(x,D),d^*dv)))\\ &\equiv \pi_1(\psi(t,x,D),\pi_1(P_j(t,x,D),\pi_1(K(x,D),\pi_1(1-\Psi(t,x,D),d^*dv)))) +\\ &\quad + \pi_1(\psi(t,x,D),\pi_1(P_j(t,x,D),\pi_1(K(x,D),\pi_1(\Psi(t,x,D),d^*dv))))\end{aligned}$$

mod $B(I,W_r^\infty)$.

By virtue of Lemma 4, the first term of the right-hand side is equal to $\sum_\lambda \pi_1(R_\lambda(t,x,D),d^*dv)(t,x)$ mod $B(I,W_r^{s-2+2-(3-j)+\tau})$, where each $R_\lambda(t,x,\xi)$ is a product of derivatives of $\psi(t,x,\xi)$, $P_j(t,x,\xi)$, $K(x,\xi)$ and $1-\Psi(t,x,\xi)$. But, by the choice of ψ, we see immediately that $R_\lambda \equiv 0$ for every λ.

On the other hand, the condition (2.6) and Lemma 2 imply that every coefficient of

$$\pi_1(\psi(t,x,D),\pi_1(P_j(t,x,D),\pi_1(K(x,D),\pi_1(\Psi(t,x,D),d^*dv))))$$

belongs to $B(I,W_r^{\kappa-2+2-(3-j)}) \subset B(I,W_r^{\kappa-3})$.

In the same way we can calculate all terms, and we conclude that

$$\left(\frac{\partial}{\partial t} - \Delta\right)\psi(t,x,D)d^*dv = h,$$

where $\mu = \min\{2s-n/r-3,\kappa-3\}$ and each coefficient of h belongs to $B(I,W_r^\mu)$.

Hence we have, by the well-known property of the heat equation, $\psi(t,x,D)(d^{*}dv)_{j} \in B(I',W^{\mu}_{r})$ for every $j = 1,\cdots,n$ and any number μ less than $\min\{2s-n/r-1,\kappa-1\}$, where I' is any subinterval of I away from 0.

Since $\psi(t,\cdot,\cdot) \equiv 0$ for small t, we obtain $\psi(t,x,D)(d^{*}dv)_{j} \in B(I,W^{\mu}_{r})$ for all j.

On the other hand, we have $\psi(t,x,D)(dd^{*}v)_{j} \in B(I,W^{\sigma}_{r})$ for all $j = 1,\cdots,n$ and $\sigma \in \mathbb{R}$, since $d^{*}v \equiv 0$ on U. Hence $\psi(t,x,D)\Delta v_{j} \in B(I,W^{\mu}_{r})$ holds for all j.

Now we introduce other cut-off functions $\tilde{\phi}(t,x)$, $\tilde{\psi}(t,x,\xi)$ such that $\tilde{\phi} \equiv 1$ near (t,x), $\tilde{\psi} \equiv 1$ near (t,x,ξ), $\phi \equiv 1$ near $\operatorname{supp} \tilde{\phi}$ and $\psi \equiv 1$ near $\operatorname{supp} \tilde{\psi}$. Then there exists a symbol $Q(t,x,\xi) \in B(I,S^{-2})$ such that

$$Q(t,x,D)\circ\psi(t,x,D)\circ\Delta \equiv \tilde{\psi}(t,x,D)\circ\tilde{\phi}(t,x)$$

holds modulo a smoothing term. This implies $\tilde{\psi}(t,x,D)(\tilde{\phi}v_{j}) \in B(I,W^{\mu+2}_{r})$ for all j. But we have $\tilde{\phi}v_{j} = \tilde{\phi}\cdot\phi u_{j} = \tilde{\phi}u_{j}$. This completes the proof.

§3. Proof of Theorems 4 and 5.

First we remark that Theorem 4 is a special case of Theorem 5, since $\rho = \max\{\tau, s\}$. Hence we shall show the latter.

Next, applying the following proposition for every compact subset K of Γ, we can derive the assertion on p in Theorem 5 in the same way as we derived Theorem 1 from Proposition 1.

Proposition 2.

Let (u,p) be a solution of the equation (NS) or (E), and K be a compact subset of $I \times (T^*\Omega \setminus 0)$. Suppose that s is a real number and that there exist a function $\phi'(t,x) \in C_0^\infty(I\times\Omega)$ and a symbol $\psi'(t,x,\xi) \in C^\infty(I,S^0)$ such that

$$\begin{cases} \phi'(t,x) \equiv 1 \quad \text{on some neighborhood of } \pi(K), \\ \psi'(t,x,\xi) \equiv 1 \quad \text{on some conic neighborhood of } K, \\ v_j(t,x) = \phi'(t,x)u_j(t,x) \quad \text{belongs to } B(I,W_r^\tau) \quad \text{for every} \\ \qquad j = 1,\cdots,n, \text{ where } \tau \text{ is defined in } (0.4), \\ \psi'(t,x,D)v_j(t,x) \in B(I,W_r^s) \quad \text{for every } j, \\ \psi'(t,x,D)(\phi' f_j)(t,x) \in B(I,W_r^s) \quad \text{for every } j. \end{cases}$$

Then there exist a function $\tilde\phi(t,x) \in C_0^\infty(I\times\Omega)$ and a symbol $\tilde\psi(t,x,\xi) \in C^\infty(I,S^0)$ such that

$$\begin{cases} \tilde\phi(t,x) \equiv 1 \quad \text{on some neighborhood of } \pi(K), \\ \tilde\psi(t,x,\xi) \equiv 1 \quad \text{on some conic neighborhood of } K, \\ \tilde\psi(t,x,D)(\tilde\phi p)(t,x) \quad B(I,W_r^{s+1}). \end{cases}$$

Moreover, if $\psi'(t,x,\xi)$ is independent of ξ, we can choose as $\tilde\psi(t,x,\xi)$ a function independent of ξ.

Proof. First we rewrite the equation (E) in the form

$$(3.1)\quad \frac{\partial v}{\partial t} + \sum_{k=1}^{n} \frac{\partial}{\partial x_k}(v_k v) + dq = g,$$

$$\psi(t,x,D)d^* g(t,x) \in B(I,W_r^s)$$

in the same way as we obtained (2.8) from the equation (NS). Here $\psi(t,x,\xi) \in C^\infty(I,S^0)$ satisfies $\psi(t,x,\xi) \equiv 1$ near K and $\psi'(t,x,\xi) \equiv 1$ on $\operatorname{supp}\psi$.

Applying d^* to (2.8) or (3.1) and observing the equality $d^* d^* = 0$, we obtain

$$(3.2)\quad \frac{\partial}{\partial t} d^* v + \sum_{k=1}^{n} d^* \frac{\partial}{\partial x_k}(v_k v) - \Delta q = d^* g.$$

Then, as in section 2, we have

$$(3.3)\quad \frac{\partial}{\partial t} d^* v - \sum_{j,k=1}^{n} \frac{\partial^2}{\partial x_j \partial x_k}\{\pi_1(v_j,v_k)+\pi_1(v_k,v_j)\} - \Delta q = d^* g + h_1 ,$$

where $h_1 \in B(I,W_r^{2\tau-n/r-2}) \subset B(I,W_r^{s-1})$.

Next, the Leibniz rule implies

$$(3.4)\quad \sum_{j,k=1}^{n} \frac{\partial^2}{\partial x_k \partial x_j} \pi_1(v_j,v_k)$$

$$= \sum_{j,k}\Big[\pi_1\Big(\frac{\partial^2 v_j}{\partial x_k \partial x_j}, v_k\Big) + \pi_1\Big(\frac{\partial v_j}{\partial x_j}, \frac{\partial v_k}{\partial x_k}\Big) + \pi_1\Big(\frac{\partial v_j}{\partial x_k}, \frac{\partial v_k}{\partial x_j}\Big) +$$

$$+ \pi_1\Big(v_j, \frac{\partial^2 v_k}{\partial x_k \partial x_j}\Big)\Big].$$

Since $\sum_j \frac{\partial v_j}{\partial x_j} \in B(I,C^{\tau-1-n/r})$ and $\sum_j \frac{\partial v_j}{\partial x_j} \equiv 0$ near $\pi(\operatorname{supp}\psi(t,x,\xi))$, we have

$$(3.5)\qquad \pi_1\left(\psi(t,x,D),\pi_1\left(\sum_j \frac{\partial v_j}{\partial x_j},\ \sum_k \frac{\partial v_k}{\partial x_k}\right)\right)$$

$$\in B(I,W_r^{\tau-1+\tau-1+n/r}) \subset B(I,W_r^{s-1})$$

by Lemma 4.

In the same way, we have

$$(3.6)\qquad \sum_k \pi_1\left(\psi(t,x,D),\pi_1\left(\sum_j \frac{\partial^2 v_j}{\partial x_j \partial x_k},\ v_k\right)\right) \in B(I,W_r^{s-1}).$$

On the other hand, there exists a function $\chi(t,x) \in C_0^\infty(I\times\Omega)$ such that $\chi(t,x) \equiv 1$ near $\mathrm{supp}\, d^* v$ and $\mathrm{supp}\,\chi \cap \pi(\mathrm{supp}\,\psi) = \emptyset$. It follows that

$$(3.7)\qquad \pi_1\left(\psi(t,x,D),\ \sum_j \pi_1\left(v_j,\ \sum_k \frac{\partial^2 v_k}{\partial x_j \partial x_k}\right)\right)$$

$$\equiv \pi_1\left(\psi(t,x,D),\ \sum_j \pi_1\left(v_j,\pi_1\left(\chi,\ \sum_k \frac{\partial^2 v_k}{\partial x_j \partial x_k}\right)\right)\right)$$

mod $B(I,W_r^\infty)$, and this formula belongs to $B(I,W_r^{s-1})$ by Lemma 4.

Since $\psi' \equiv 1$ near $\mathrm{supp}\,\psi$, we have

$$(3.8)\qquad \pi_1\left(\psi(t,x,D),\ \sum_{j,k=1}^{n} \frac{\partial^2}{\partial x_j \partial x_k}\pi_1(v_j,v_k)\right)$$

$$\equiv \sum_{j,k} \pi_1\left(\psi(t,x,D),\pi_1\left(i\frac{\partial v_j}{\partial x_k}D_j,\ v_k\right)\right)$$

$$\equiv \sum_{j,k} \pi_1\left(\psi(t,x,D),\pi_1\left(i\frac{\partial v_j}{\partial x_k}D_j,\ \pi_1(\psi'(t,x,D),v_k)\right)\right)$$

mod $B(I,W_r^{s-1})$ by virtue of (3.4), (3.5), (3.6) and (3.7).

Next, since $\frac{\partial u_j}{\partial t} \in B(I,\mathcal{D}'(\Omega))$, we have

$\frac{\partial v_j}{\partial t} \in B(I,E'(\mathbb{R}^n)) \subset B(I,W_r^\sigma)$ for some $\sigma \in \mathbb{R}$. It follows that

$$\pi_1(\psi(t,x,D),d^*\frac{\partial v}{\partial t}) \equiv \pi_1(\psi(t,x,D), \sum_j \pi_1(\chi,d^*\frac{\partial v}{\partial t}))$$

mod $B(I,W_r^\infty)$ by Lemma 2. But Lemma 4 implies that the right-hand side belongs to $B(I,W_r^\infty)$. Hence we have

$$(3.9)\quad \pi_1(\psi(t,x,D),\frac{\partial}{\partial t}d^*v) \in B(I,W_r^\infty).$$

Applying $\pi_1(\psi(t,x,D),\cdot)$ to both sides of (3.3) and making use of (3.8) and (3.9), we obtain

$$2\sum_{j=1}^{n} \pi_1(\psi(t,x,D),\pi_1(i\frac{\partial v_j}{\partial x_k}D_j , \pi_1(\psi'(t,x,D),v_k))) + $$
$$+ \pi_1(\psi(t,x,D),\Delta q)$$
$$\in B(I,W_r^{s-1}).$$

But, since $\psi'(t,x,D)v_k \in B(I,W_r^s)$, the first term of the left-hand side belongs to $B(I,W_r^s)$. This implies $\pi_1(\psi(t,x,D),\Delta q) \in B(I,W_r^{s-1})$.

Now, making use of the Green function of Ω' as in Section 2, we obtain $\tilde{\psi}(t,x,D)(\tilde{\phi}p)(t,x) \in B(I,W_r^{s+1})$ for suitable $\tilde{\phi}$ and $\tilde{\psi}$. This completes the proof of Proposition 2.

Finally we consider $\frac{\partial u_j}{\partial t}$. As before, we can deduce that $\sum_{k=1}^{n} \frac{\partial}{\partial x_k}(u_k u_j)$ is microlocally in $B(I,W_r^{s-1})$ on K for every $j = 1,\cdots,n$. Now the conclusion is an immediate consequence of this fact and the assertion on p.

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