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Asymptotic Behaviors of Solutions of Equation for Viscous Gas Motion

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§ 1 Introduction

Burgers equation

$$(1.1)\qquad \begin{aligned} &u_t + uu_x = \mu\, u_{xx}, \quad t \ge 0,\ x \in R, \\ &u(0,x) = u_0(x), \quad x \in R. \end{aligned}$$

is a simplest model for system of equations of viscous fluids motion. It was solved explicitly by Cole and Hopf. In particular by using the explicit solution Hopf investigated the asymptotic behaviors of solutions of Burgers equation as time tends to infinity when $M = \int_{-\infty}^{\infty} u_0(x)\,dx$ is finite. By introducing a change of variables $\bar{x} = x/\sqrt{2\mu t}$, $\bar{t} = \log t$, $\bar{u} = \sqrt{(t/\mu)}\, u$, he obtained the asymptotic convergence

$$(1.2)\qquad \lim_{t\to\infty} \bar{u}(\bar{t},\bar{x}) = -G'(\bar{x}) / G(\bar{x}),$$

where

$$G(x) = \exp(-M/4\mu) \int_{-\infty}^{x} \exp(-y^2/2)\,dy + \exp(M/4\mu) \int_{x}^{\infty} \exp(-y^2/2)\,dy.$$

In the rescaled variables the Burgers equation can be written as follows:

$$(1.3)\qquad \frac{\partial \bar{u}}{\partial \bar{t}} + \bar{u}\frac{\partial \bar{u}}{\partial \bar{x}} = \frac{1}{2}\frac{\partial^2 \bar{u}}{\partial \bar{x}^2} + \frac{1}{2}\frac{\partial(\overline{xu})}{\partial \bar{x}}.$$

Thus the limit function (1.2) is a stationary solution of (1.3) with the moment $\int_{-\infty}^{\infty} \bar{u}(\bar{t},\bar{x})\, d\bar{x} = M/2\mu$, which is a conserved quantity with respect to time.

We do not know in general the asymptotic behavior of solutions for the system of equations of viscous compressible fluids motion as time tends to infinity. In [4] and [5] we treated asymptotic behaviors and equivalences for small solutions between Boltzmann equation and compressible Navier-Stokes equation as time tends to infinity. However in this case of more than two space-dimension the asymptotic behavior is described by linear partial differential equations. It is because the decay rate of solutions is so fast that the nonlinear part decays faster than the linear part as time tends to infinity . But in the one space-dimension it is not true and we have to consider the nonlinear part as well as the linear part for the asymptotic behaviors. In fact a reductive perturbation method [8] predicts that the Burgers equation describes the far field i.e., the asymptotic behavior for the general system of viscous fluid dynamical equations.

In this note we consider a system of equations of viscous barotropic gas motion and show that the asymptotic behaviors are described by two Burgers equations with different propagation speeds. A detailed proof will be published elsewhere. Similar asymptotic behaviors are investigated for the inviscid case ,i.e., for the hyperbolic conservation laws in [1] [7] and in the references in them. Thus the asymptotic forms of the solutions as time tends to infinity are different each other between the inviscid and viscous motions as noticed in [2] for the Burgers equation. The author thanks Mr. T. Ikeda for using his type writing softwares.

§ 2 Viscous Gas Motion

We consider the viscous barotropic gas motion which is governed by the following nonlinear system of two equations:

$$(2.1)\qquad \begin{aligned} &\rho_t + (\rho u)_x = 0 , \\ &(\rho u)_t + (\rho u^2+p)_x = \mu\, u_{xx} , \quad t > 0 , \quad x \in R , \end{aligned}$$

where ρ is the density, u is the velocity, $p = (a^2/\gamma)\rho^\gamma$ is the pressure for the barotropic gas, and a , γ(ratio of specific heats) and μ(viscosity coefficient) are assumed constants. The initial data

$$(2.2)\qquad \rho(0,x) = \rho_0(x) , \; u(0,x) = u_0(x) , \quad x \in R$$

are given and we want to investigate the asymptotic behavior of solutions for thr Cauchy problem (2.1)(2.2) as time tends to infinity. System (2.1) has a hyperbolic-parabolic type i.e., the first order part is hyperbolic but the velocity satisfies a parabolic equation. This is a main feature of viscous fluid dynamical equations. The Cauchy problem (2.1) (2.2) is solved globally in time by Kanel' for rather general L^2 initial data by using the Lagrangian mass coordinate. Hereafter the equilibrium state is assumed $\rho = 1$, $u = 0$.

lemma 2.1

If $\rho_0(x)-1$, $u_0(x) \in H^\ell$, $\ell \geqq 2$, $\rho_0(x) > 0$, *then there exists a unique global solution* $\rho(t,x)$, $u(t,x)$ *such that* $\rho(t,x) > 0$, $\rho(t,x)-1$, $u(t,x) \in B(0,\infty; H^\ell)$,

$$\rho_x(t,x)\ ,\ u_x(t,x)\ \varepsilon\ L^2(0,\infty;\ H^\ell)\ .$$

Here H^ℓ denotes the Sobolev space of L^2 function together with their x derivatives up to and including ℓ-th order. $B(0,T;\ H^\ell)$ denotes the bounded continuous functions of $t\ \varepsilon\ [0,T]$ with vector values in H^ℓ. $L^2(0,T;\ H^\ell)$ denotes the square summable functions of $t\ \varepsilon\ [0,T]$ with vector values in H^ℓ . We also use the space L^1 of summable functions.

Concerning the decay rate of solutions for Cauchy problem (2.1)(2.2) we have the following

lemma 2.2

If $\rho_0(x)-1\ ,\ u_0(x)\ \varepsilon\ H^\ell \cap L^1\ ,\ \ \ell \geqq 12\ ,$

and small in the norm, then

$$(2.3)\qquad |\ \partial^\alpha(\rho(t,\cdot)-1\ ,\ u(t,\cdot))\ |_{L^2}\ <\ C\ /\ (1+t)^{(1/4+\alpha/2)}\ ,\qquad \alpha = 0,1,2,3.$$

$$(2.4)\qquad |\ \partial^{4+\alpha}(\rho(t,\cdot)-1\ ,\ u(t,\cdot))\ |_{L^2}\ <\ C\ /\ (1+t)^{(2-\alpha/4)}\ ,\qquad \alpha = 0,1,\cdots,8.$$

The decay rate (2.3) *is best possible if*

$$\int\ \rho_0(x)\ -\ 1\ dx\ \neq\ 0 \quad \text{or} \quad \int\ u_0\ dx\ \neq\ 0\ .$$

This is a slight improvement of [4] [5] which is proved by the linear decay rate obtained by Fourier transform and by the energy method using the convexity of Sobolev norm.

As in [4] [5] using this decay rate we make a comparison of solutions represented in the variation of constants formula in the Fourier transform between system (2.1) and the following uniformly parabolic system

$$(2.5)\qquad \begin{aligned} \rho_{1,t} + (\rho_1 u_1)_x &= (\mu/2)\, \rho_{1,xx} \\ (\rho_1 u_1)_t + (\rho_1 u_1^2 + p(\rho_1))_x &= (\mu/2)(\rho_1 u_1)_{xx} , \end{aligned}$$

with the same initial data

$$\rho_1(0,x) = \rho_0(x) \ , \quad u_1(0,x) = u_0(x) \quad .$$

lemma 2.3

The asymptotic equivalence between ρ , u *and* ρ_1 , u_1 *can be summarized as follows:*

$$(2.6)\qquad | \ \partial^{\alpha}(\rho-\rho_1, \ u-u_1)(t,x) \ | \ < \ C \ / \ (1+t)^{(3/4-\delta+\alpha/2)} \quad ,$$

$$\alpha = 0,1,2, \quad \textit{for any } \delta > 0 \textit{ and for any } t \geqq 0 \ .$$

Since the decay estimate (2.3) is optimal in general this estimate for the difference of solutions for systems (2.1) and (2.5) is meaningful and essential for the further discussion on system (2.5).

§ 3 Asymptotic Behaviors

In order to make a further reduction of our system of equations we will use the Riemann invariants of hyperbolic part of systems (2.1) and (2.5), namely the eigenvalues and the corresponding Riemann invariants given by the following:

$$(3.1)\qquad \begin{aligned} \lambda_1 &= u - a\rho^{(\gamma-1)/2} \ , \quad r = (2a/(\gamma-1))(\rho^{(\gamma-1)/2} - 1) - u \\ \lambda_2 &= u + a\rho^{(\gamma-1)/2} \ , \quad s = (2a/(\gamma-1))(\rho^{(\gamma-1)/2} - 1) + u \ . \end{aligned}$$

By using these Riemann invariants $r = r(\rho_1, u_1)$, $s = s(\rho_1, u_1)$ the parabolic system (2.5) can be written as follows:

$$
(3.2)\quad \begin{aligned} r_t - (a + (\gamma+1)r/4 + (\gamma-3)s/4)r_x &= \mu\, r_{xx}/2 + f \;, \\ s_t - (a + (\gamma+1)s/4 + (\gamma-3)r/4)s_x &= \mu\, s_{xx}/2 + g \;, \end{aligned}
$$

where $f = (\mu/16a\rho^{(\gamma-1)})((7-\gamma)r_x^{\,2} + 2(3-\gamma)r_x s_x - (\gamma+1)s_x^{\,2})$
and $g = (\mu/16a\rho^{(\gamma-1)})((7-\gamma)s_x^{\,2} + 2(3-\gamma)r_x s_x - (\gamma+1)r_x^{\,2})$.

Here we have the nonlinear terms rr_x , ss_x , sr_x , rs_x , f and g which have the decay rate by lemma 2.2 and the definition (3.1):

$$
(3.3)\quad \begin{aligned} &|\, rr_x \,,\, ss_x \,|_{L^1} < C \,/\, (1+t) \;, \\ &|\, sr_x \,,\, rs_x \,|_{L^1} < C \,/\, (1+t) \;, \\ &|\, f \,,\, g \,|_{L^1} < C \,/\, (1+t)^{3/2} \;. \end{aligned}
$$

Since there is a difference on the decay rate between these terms, we want to compare the solution of system (3.2) with that of system (3.2) without the terms f and g ,i.e., system (3.5). But in so doing the cross terms sr_x and rs_x have not a divergent form and prevent us to obtain the estimate for the difference directly. Thus to get round this difficulty we use the hyperbolicity of the first order part of system (3.2) following an idea of Lax [6] after rewriting $sr_x = (sr)_x - rs_x$. Let us introduce the unknown functions

$$
R = rA \;, \quad S = sB \;,
$$

where

$$
\begin{aligned} A &= (2a + (\gamma-1)r/2 + (\gamma+1)s/4)^{(3-\gamma)/(\gamma+1)} \;, \\ B &= (2a + (\gamma-1)s/2 + (\gamma+1)r/4)^{(3-\gamma)/(\gamma+1)} \;. \end{aligned}
$$

Then instead of (3.2) the functions R and S satisfy

$$(3.4)\quad \begin{aligned} R_t - (aR + (\gamma+1)R^2/8A)_x - (\gamma-3)(SR/B)_x/4 &= \mu\, R_{xx}/2 + F , \\ S_t - (aS + (\gamma+1)S^2/8B)_x - (\gamma-3)(SR/A)_x/4 &= \mu\, S_{xx}/2 + G , \end{aligned}$$

where $F = F(r,s,f,g)$ and $G = G(r,s,f,g)$ consist of those terms which decay as fast as f and g ,i.e.,

$$| F , G |_{L^1} < C / (1+t)^{3/2} .$$

Using this decay rate and the fact that the quadratic terms R^2 , S^2 and RS have the divergent form in system (3.4) we can compare the solutions of (3.2) and of the following system:

$$(3.5)\quad \begin{aligned} r_{2,t} - (a + (\gamma+1)r_2/4 + (\gamma-3)s_2/4)r_{2,x} &= \mu\, r_{2,xx}/2 , \\ s_{2,t} - (a + (\gamma+1)s_2/4 + (\gamma-3)r_2/4)s_{2,x} &= \mu\, s_{2,xx}/2 , \end{aligned}$$

for $t \geqq T$, with the initial data

$$(3.6)\qquad (r_2 , s_2)(T,x) = (r , s)(T,x) .$$

The corresponding system for $R_2 = R(r_2,s_2)$ and $S_2 = S(r_2,s_2)$ is given by

$$(3.7)\quad \begin{aligned} R_{2,t} - (aR_2+(\gamma+1)R_2{}^2/8A_2)_x - (\gamma-3)(S_2R_2/B_2)_x/4 &= \mu\, R_{2,xx}/2 + F_2 , \\ S_{2,t} - (aS_2+(\gamma+1)S_2{}^2/8B_2)_x - (\gamma-3)(R_2S_2/A_2)_x/4 &= \mu\, S_{2,xx}/2 + G_2 , \end{aligned}$$

where

$$F_2 = F(r_2,s_2,0,0) , \; G_2 = G(r_2,s_2,0,0) ,$$
$$A_2 = A(r_2,s_2) , \; B_2 = B(r_2,s_2) .$$

lemma 3.1

We have the estimate for the difference of solutions

$(R-R_2, S-S_2)$ *or equivalently for* $(r-r_2, s-s_2)$

(3.8) $$| \partial^\alpha (r-r_2 , s-s_2)(t,x) |_{L^2} < C / (1+t-T)^{(1/4+\alpha/2)} (1+T)^{1/2}$$

$\alpha = 0,1,2,$ for any $t \geqq T$: fixed.

The estimate (3.8) is not so good as the estimate (2.6) which is valid for all $t \geqq 0$. But this estimate in this form is expected best, in fact if we note the special case $\gamma = 3$, it has the nonlinear terms in the nondivergent form. Here we arrived at almost diagonal system (3.5) except for cross terms $s_2 r_{2,x}$ and $r_2 s_{2,x}$. These cross terms have had the same decay rate estimate as (3.3). In order to distinguish these from the main quadratic terms $r_2 r_{2,x}$ and $s_2 s_{2,x}$ we use a property of the finite propagation speed of exponential decay with respect to x of parabolic system (3.5) which corresponds to the finite propagation speed of the hyperbolic system of the first order parts of (3.5).

lemma 3.2

If the initial data satisfy an additional exponential decay as $x \to \pm\infty$, *i.e.*,

$$| \partial^\alpha (r(0,x), s(0,x)) | < K / \cosh x \quad , \alpha = 0,1,2,3,$$

then

(3.9) $$| \partial^\alpha (r(t,x), s(t,x)) | < \min \{ C/(1+t)^{(1/2+\alpha)/2} , K e^{\beta t}/\cosh x \}$$

$\alpha = 0,1,2,3$, *where* β *is a constant.*

This is true for all systems (2.1), (2.5), (3.2), (3.4) and (3.5) because of maximum principle. Thus the solution decays exponentially with respect to x for each t. In particular the initial data (3.6) have the estimate:

$$(3.10) \quad | \, \partial^{\alpha}(r_2,\ s_2)(T,x) \, | < \min \{ C/(1+T)^{(1/2+\alpha)/2},\ K\, e^{\beta T}/\cosh x \} ,$$

$$\alpha = 0,1,2,3.$$

It follows from the maximum principle for system (3.5) along each characteristic direction we can obtain the exponential decay estimate:

lemma 3.3

Under the condition (3.10) *we have*

$$(3.11) \quad \begin{aligned} | \, \partial^{\alpha} r_2(t,x) \, | &< \min \{ C/(1+t)^{(1/2+\alpha)/2},\ K_1 e^{a(t-T)/2}/\cosh(x-a(t-T)) \}, \\ | \, \partial^{\alpha} s_2(t,x) \, | &< \min \{ C/(1+t)^{(1/2+\alpha)/2},\ K_1 e^{a(t-T)/2}/\cosh(x+a(t-T)) \}, \end{aligned}$$

$$\alpha = 0,1,2,3, \ \ \textit{for any} \ \ t \geqq T \ , \ \textit{where} \ \ K_1 = K\, e^{\beta T} .$$

Now we can distinguish the cross terms by the faster decay than (3.3) as

$$(3.12) \quad \begin{aligned} &| \, \partial^{\alpha}(r_2 s_2)(t,\ ) \, |_{L^1} < \\ &< \min \{ C/(1+t)^{(1+\alpha)/2} ,\ 2\, K_1\, e^{-at/2} (|r,s|_3) \} < \\ &< \min \{ C/(1+t)^{(1+\alpha)/2} ,\ C\, K\, e^{-a(t-(2\beta/a+1)T)/2} \} . \end{aligned}$$

$$\alpha = 1,2,3.$$

Using this decay rate we compare system (3.5) and the diagonal system

$$(3.13) \quad \begin{aligned} r_{3,t} - (a + (\gamma+1)r_3/4)r_{3,x} &= \mu\, r_{3,xx}/2 , \\ s_{3,t} + (a + (\gamma+1)s_3/4)s_{3,x} &= \mu\, s_{3,xx}/2 , \end{aligned}$$

with the initial data

$$(r_3, s_3)(T_1,x) = (r_2, s_2)(T_1,x) ,$$

where $T_1 = (3+2\beta/a)T$.

lemma 3.4

We have the estimate for the difference of solutions between systems (3.5) and (3.13) for $t \geqq T_1$

$$(3.14) \quad | \partial^{\alpha}(r_2-r_3, s_2-s_3)(t,\cdot) |_{L^2} < C / \{(1+t)^{(1/4+\alpha/2)} e^{aT} \} ,$$

$$\alpha = 0,1,2.$$

This is proved by the representation of solutions in the variation of constant formula in the Fourier transform for systems (3.5) and (3.13) and by the decay estimate (3.12).

Theorem

If the initial data are close to a constant state $(\bar{\rho}, 0)$ *in the norm in lemma 2.1 and decay exponentially as* $x \to \mp\infty$, *then the solution of system (2.1) behaves asymptotically like that of two Burgers equations (3.13) as time tends to infinity.*

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