数理解析研究所講究録
第 545 巻 1985 年 101-111

Singular Solution of Nonlinear Partial Differential Equations

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§0. Introduction

We are interested in the singularities of solutlions of nonlinear partial differential equations defined in a complex domain. Simple examples of ordinary differential equations show that there are solutions which are singular along non-characteristic surfaces. For example, $(z_2^k - z_1)^{-1}$, $k=1,2,3,\ldots$, are solutions of $\partial u/\partial z_1 = u^2$, and are singular along non-characteristic surfaces $z_1 - z_2^k = 0$. They are so-called "movable singularities" in the theory of ordinary differential equations.

In this note we first construct solutions with singulrities along non-characteristic surfaces for semilinear differential equations, which we call movable singularities for simplicity. For linear differential equations M. Zerner [5] showed a prolongation theorem across non-characteristic surfaces, so there exist no solutions with movable singularities.

Y. Tsuno [4] proved a prolongation theorem for nonlinear differential equations under some boundedness conditions. Since there exist solutions with movable singularities, some boundedness conditions of course are necessary. His results are, however, not strict. So our second aim is to weaken the boundedness conditions. In view of the first results they are best possible in general.

§1. Solutions with movable singularities

We consider the following semilinear differential equations defined in a neighborhood Ω of the origin of $\mathbb{C}^n$:

(1) $$P_m(z, D)\, u + b(z, D^B u) = 0.$$

Here we explain the notations used above:

$P_m(z, D)$ $(z=(z_1,..,z_n),$ $D=(\partial/\partial z_1,...,\partial/\partial z_n))$ is a homogeneous linear differential operator of order m whose coefficients are holomorphic in Ω.

$B:=\{\beta\in\mathbb{N}^n;\ |\beta|\leq m-1\}$ $(\mathbb{N}=\{0,1,2,...,\})$ is the set of multi-indices whose lengths are less than $m-1$.

$D^B u:=(D^\beta u;\ \beta\in B)$ is a vector whose β-th component is $D^\beta u$.

We denote by $p:=(p_\beta;\ \beta\in B)$ the variables in $\mathbb{C}^{\#B}$ where p_β corresponds to $D^\beta u$, and by $\mu:=(\mu_\beta;\ \beta\in B)\in\mathbb{N}^{\#B}$ multi-indices with $\#B$ components.

$b(z, p) := \sum_{|\mu|\leq d} b_\mu(z)\, p^\mu$ $\quad (|\mu|:=\Sigma_{\beta\in B}\ \mu_\beta,\ \ p^\mu:=\Pi_{\beta\in B}\ p_\beta^{\ \mu_\beta})$

is a polynomial in p of degree d where the coefficients b_μ are holomorphic in Ω.

We assume that Eq(1) is nonlinear, i.e., for some μ with $|\mu|\geq 2$ $b_\mu(z)$ does not vanish identically, and that the 0-th order term $b_0(z)$ vanishes identically.

Remark 1. By the Cauchy-Kovalevskaja theorem we can reduce Eq(1) to one with $b_0(z)=0$: Take a holomorphic solution $w(z)$ of Eq(1) and change the unkown function u to $v:=u-w$. Then the equation for v has no 0-th order term.

Let us put

(2) $$M:=\{\mu\in\mathbb{N}^{\#B};\ \mu \text{ s.t } b_\mu(z)\neq 0\},$$

and for each $\mu\in M$ define a linear function of $\sigma\in\mathbb{R}$ by

(3) $$y_\mu(\sigma):=|\mu|\sigma-\gamma(\mu)$$

where

(4) $$\gamma(\mu):=\textstyle\sum_{\beta\in B}\mu_\beta|\beta|.$$

DEFINITION 1. For Eq(1) we define an exponent σ_0 by

(5) $$\sigma_0:=\max_{\mu\in M,\,|\mu|\geq 2}\frac{\gamma(\mu)-m}{|\mu|-1}.$$

Remark 2. σ_0 is the greatest σ of the intersections of lines $y=y_\mu(\sigma)$ and $y=\sigma-m$. We have $-m\leq\sigma_0<m-1$.

Remark 3. σ_0 is independent of a choice of independent variables. It may, however, vary if we change unknown functions as in Remark 1.

For all $\mu\in M$, we have $y_\mu(\sigma)\geq\sigma-m$ if $\sigma\geq\sigma_0$. So we arrange the non-negative rational numbers $\{y_\mu(\sigma_0)-\sigma_0+m;\ \mu\in M\}$ and write them as

$$d_0 := 0 < d_1 < d_2 < \dots\dots < d_Q \quad (Q \geq 0).$$

Then we put

$$(6)\qquad \begin{cases} M(j) := \{\mu \in M;\ y_\mu(\sigma_0) = \sigma_0 - m + d_j\}, & j=0,1,2,\dots,Q, \\ N := \max\ \{|\mu|;\ \mu \in M(0)\}, & \\ \mathcal{N}(i) := \{\mu \in M(0);\ |\mu| = i\}, & i=2,3,\dots,N. \end{cases}$$

Note that $N \geq 2$ because $b(z, p)$ is nonlinear in p. We put

$$(7)\qquad r := \begin{cases} \text{the positive least common denominator of } d_j\text{'s} & \text{if } Q \geq 1 \\ 1, \quad \text{if } Q = 0. \end{cases}$$

We put for $\sigma \in \mathbb{R}$ and $\ell \in \mathbb{N}$

$$[\sigma;\ \ell] := \begin{cases} \sigma(\sigma-1)\dots\dots(\sigma-\ell+1), & \ell \geq 1, \\ 1, & \ell = 0. \end{cases}$$

We denote by $\langle\!\langle g_\beta \rangle\!\rangle$ the vector $(g_\beta;\ \beta \in B)$ whose β-th component is g_β.

DEFINITION 2. Let σ_0, r, N and $\mathcal{N}(i)$ be as above. We define a sequence of polynomials in $X \in \mathbb{C}$ and $\zeta := (\zeta_1, \dots, \zeta_n) \in \mathbb{C}^n$, as follows:

$$(8)\qquad S_k(z,\ \zeta;\ X) := [\sigma_0 + \tfrac{k}{r};\ m]\ P_m(z,\ \zeta) + \sum_{i=1}^{N-1} S_{k,i}(z,\ \zeta) X^i, \quad k \geq 0,$$

$$(9)\qquad S_{k,i}(z, \zeta) := \\ = \begin{cases} \sum_{\mu \in \mathcal{N}(i+1)} b_\mu(z)\ \langle\!\langle [\sigma_0;\ |\beta|]\ \zeta^\beta \rangle\!\rangle^\mu, & k=0, \\ \end{cases}$$

$$\left\{ \sum_{\mu\in\mathcal{N}(i+1)} b_\mu(z)\Big(\sum_{\beta\in B} \mu_\beta \frac{[\sigma_0+\frac{k}{r};\ |\beta|]}{[\sigma_0;\ |\beta|]} \Big) \langle\!\langle [\sigma_0;\ |\beta|]\ \zeta^\beta \rangle\!\rangle^\mu, \quad k\geq 1, \right.$$

where $\langle\!\langle [\sigma_0;|\beta|]\zeta^\beta \rangle\!\rangle^\mu := \Pi_{\beta\in B}\ ([\sigma_0;|\beta|]\zeta^\beta)^{\mu_\beta}$.

We assume that

(A.0) $\qquad S_k(z,\ \zeta;\ X) \not\equiv 0, \qquad$ for $k \geq 0$.

Let S be a germ of non-singular complex hypersurface at the origin. We assume that S is non-characteristic with respect to P_m, that is, for an irreducible defining function ϕ of S we have

(A.1) (non-characteristic condition)

$$P_m(0,\ D\phi(0)) \neq 0.$$

We want to construct a solution of Eq(1) which is singular along S, that is, a solution with movable singularities. We assume that

(A.2) There is a solution $X=u_0(z)$ of $S_0(z,\ D\phi(z);\ X)=0$ such that

(i) $u_0(z)$ is holomorphic in a neighborhood of the origin,

(ii) $S_k(0,\ D\phi(0);\ u_0(0)) \neq 0$ for $k=1,2,3,\ldots\ldots\ldots$.

Remark 4. By (A.1), $S_k(0,\ D\phi(0);\ u_0(0))$ is a polynomial in k of degree m. Hence there are at most m solutions of the equation for k: $\qquad S_k(0,\ D\phi(0);\ u_o(0)) = 0$.

THEOREM 1. Suppose (A.0), (A.1) and (A.2) are satisfied. Let $u_0(z)$ and $\phi(z)$ be as above. Then there is a solution $u(z)$ of Eq(1) represented by a convergent Puiseux series in some neighborhood Ω' of the origin:

$$u(z) = \phi(z)^{\sigma_0} \sum_{k=0}^{\infty} u_k(z)\, \phi(z)^{k/r}, \tag{10}$$

where $u_k(z)$ are holomorphic in Ω'.

We only give an outline of the proof. We assume that $u(z)$ is a priori of the form (10) and substitute it in Eq(1). Then equating the coefficients of $\phi^{\sigma_0 - m + k/r}$ to zero, we obtain transport equations:

$$\begin{cases} S_0(z,\ D\phi(z);\ u_0(z))u_0(z) = 0, \\ S_k(z,\ D\phi(z);\ u_0(z))u_k(z) + R_k = 0, \quad k \geq 0, \end{cases} \tag{11}$$

where R_k are calculated depending only on $u_0, u_1, \ldots, u_{k-1}$. By (A.2) we can determine u_k inductively and obtain a formal solution. Convergence of the formal solution is proved by the method of majorant functions. See [1] for the detail.

Remark 5. By changes of unknown functions as in Remark 1 we get solutions of Eq(1) of the form

$$\phi^{\sigma_0} \sum_{k=0}^{\infty} u_k\, \phi^{k/r} - w(z), \tag{12}$$

where w is a holomorphic solution of Eq(1). We remark that σ_0 and r may vary for each w.

§2. Prolongation across non-characteristic surfaces

Let S and ϕ be as in Section 1. Since S is non-characteristic at the origin, by a change of coordinates with $z_1=\phi$ we can reduce Eq(1) to the form

$$(13) \qquad D_1^m u = a(z, D^A u).$$

Here $A:=\{\alpha\in\mathbb{N}^n;\ |\alpha|\leq m,\ \alpha_1\leq m-1\}$ and $a(z, p)$ $(p:=(p_\alpha;\ \alpha\in A))$ is a polynomial in p with holomorphic coefficients. Solutions of the form (12) are holomorphic in $\Omega_-:=\{z\in\Omega;\ \mathrm{Re}\ z_1<0\}$ (Ω is a neighborhood of the origin and may shrink from time to time), but not in a neighborhood of the origin in general.

Our problem in this section is: If u is holomorphic in Ω_- and satisfies Eq(13), then under what conditions is it holomorphic in a neighborhood of the origin?

We put

$$(14) \qquad \tilde{M}:=\{\mu\in\mathbb{N}^{\#A};\ D_p^\mu\, a(z, p)\neq 0\} \qquad (\mu:=(\mu_\alpha;\ \alpha\in A)\in\mathbb{N}^{\#A})$$

and

$$(15) \qquad \tau_1(\mu):=\sum_{\alpha\in A} \alpha_1\cdot\mu_\alpha.$$

DEFINITION 3. We define an exponent σ_a for Eq(13) by

$$(16) \qquad \sigma_a:=\max_{\mu\in\tilde{M},\ |\mu|\geq 2} \frac{\tau_1(\mu) - m}{|\mu| - 1}.$$

Remark 6. We have $-m \leq \sigma_a < m-1$. If (A.1) and (A.2) are satisfied, then it holds that

(17) $\sigma_a := \max\{\sigma_0$; changes of unknown functions$\}$.

DEFINITION 4. Let $\sigma \leq m-1$ and $u(z)$ be holomorphic in Ω_-. We say that u is bounded of order σ in Ω_- if for $\sigma \geq 0$ we have

$$(18)\quad \begin{cases} \sup_{\Omega_-} |D_1^k u(z)| \leq \text{Const.} & \text{for } k=0,1,..,[\sigma], \\ \sup_{\Omega_-} |z_1^{k-\sigma} D_1^k u(z)| \leq \text{Const.} & \text{for } k=[\sigma]+1,..,m-1, \end{cases}$$

and for $\sigma < 0$ we have

$$(19)\quad \sup_{\Omega_-} |z_1^{-\sigma} u(z)| \leq \text{Const.}$$

For example the functions of the form (12) are bounded of order σ_0.

THEOREM 2. Let σ_a be the exponent of Eq(13) and $u(z)$ be holomorphic in Ω_- and satisfy Eq(13). If u is bounded of order σ in Ω_- with some $\sigma > \sigma_a$, then u is holomorphic in a neighborhood of the origin.

Remark 7. We can give a prolongation theorem without assuming that Eq(13) is semilinear and that $a(z, p)$ is a polynomial in p. See [1], [3].

The idea of the proof of Theorem 2 is essentially the same as that of Zerner [5]. We solve the Cauchy problem with initial data on $z_1=-\varepsilon$ and estimate radii of convergence precisely. Then letting ε tend to 0, we can show the domain of convergence contains the origin.

By Theorem 2 there exist no solutions of the form (12) with exponent $\sigma_0>\sigma_a$. By Theorem 1 and Remark 6 there exists a solution of the form (12) whose exponent is just σ_a. So Theorem 2 is best possible in general.

We finally give an example. Consider the equation

(20) $$D_1^2 u + u(D_2u)^3 = 0,$$

with

(21) $$S:\ \phi:= z_1+ cz_2 = 0, \quad (c \text{ a constant}).$$

Then $M=\{\mu=(1,0,3)\}$, $\sigma_0=(3-2)/(4-1)= \frac{1}{3}$, $r=1$, $N=4$, and

$$S_0 = \frac{1}{27}(-6\zeta_1^2 + \zeta_2^3\, X^3),$$

$$S_k = (k+\tfrac{1}{3})(k-\tfrac{2}{3})\zeta_1^2 + \frac{1}{27}(9k+4)\zeta_2^3\, X^3.$$

If $c\neq 0$ then all (A.0)~(A.2) are satisfied. Hence there is a solution with movable singularities on $z_1+ cz_2=0$. However, if $c\neq 0$, then (A.3) does not hold. We next change the unknown functions u to $v:=u-w$ where w is a holomorphic solution of Eq(20). Then v satisfies

(22) $$D_1^2 v + \{v(D_2v)^3+ w(D_2v)^3+ 3(D_2w)v(D_2v)^2+ 3wD_2w(D_2v)^2+ + 3(D_2w)^2vD_2v + 3w(D_2w)^2D_2v + (D_2w)^3\, v\} = 0.$$

For this equation we have $\sigma_0=1/2$, $r=1/2$, $N=3$, $M(0)=\{\mu=(0,0,3)\}$ and

$$S_0 = -\frac{1}{4}\zeta_1^2 + \frac{w}{8}\zeta_2^3 X^2,$$

$$S_k = \frac{1}{4}(k^2-1)\zeta_1^2 + \frac{3}{8}(1+k)w\,\zeta_2^3 X^2.$$

It is easy to see that if $c\neq 0$, then (A.0)~(A.2) are satisfied but not if $c=0$. Therefore we obtain a solution $u = \phi^{1/2} \times \ldots + w$ of Eq(20) if $c\neq 0$. The exponent σ_a of Eq(20) and Eq(22) is 1/2 if $c\neq 0$ and is $-2/3$ if $c=0$. Therefore there exist no solutions with movable singularities along $z_1+cz_2=0$ which are bounded of order $\sigma>1/2$ and $\sigma>-2/3$ for $c\neq 0$ and $c=0$, respectively.

REFERENCES

[1] T. Ishii and T. Kobayashi, in preparation.

[2] T. Kobayashi, Propagation of singularities for a first order semilinear system in C^{n+1}, to appear in Ann. Scuola Norm. Sup. Pisa.

[3] ------------, On the prolongation of solutions for quasi-linear differential equations, Proc. Japan Acad. Vol. 50 Ser. A No.2 60-62 (1984).

[4] Y. Tsuno, On the prolongation of local holomorphic solutions of nonlinear partial differential equations, J. Math. Soc. Japan, 27 454-466 (1975).

[5] M. Zerner, Domaines d´holomorphie des fonctions verifiant une equation aux derivees partielles, C. R. Acad Sc Paris, 272, 1646-1648 (1971).