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On Riemann-Hilbert transformations

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The method of Riemann-Hilbert (RH) transformations has become a powerful tool in the theory of completely integrable systems. The purpose of this note is to reveal some new aspects that lie in this method. Although our discussion is limited to the case of the self-dual Yang-Mills (SDYM) equations to make the presentation not too abstract, the ideas developed here are basically of universal nature and can be applied to various completely integrable systems.

1. Review of RH transformations

We here consider the $GL(r,\mathbb{C})$ SDYM equations ($r \geqq 2$) in the complexified Euclidean space $\mathbb{C}^4$

$$[\nabla_y, \nabla_z] = 0, \ [\nabla_{\bar{y}}, \nabla_{\bar{z}}] = 0, \ [\nabla_y, \nabla_{\bar{y}}] + [\nabla_z, \nabla_{\bar{z}}] = 0, \qquad (1)$$

where $\nabla_u = \nabla_{\partial/\partial u}$ $(u = y, z, \bar{y}, \bar{z}$, the coordinates of $\mathbb{C}^4)$ denote covariant derivations of the trivial vector bundle of rank r over

$\mathbb{C}^4$, whose connection coefficents (gauge potentials) are the unknown functions of this nonlinear system. In this expression gauge transformations can be expressed as $\nabla_u \to h^{-1} \circ \nabla_u \circ h$, $h = h(y,z,\bar{y},\bar{z})$. In what follows let us limit the consideration to holomorphic solutions defined in some domains of $\mathbb{C}^4$.

The theory of RH transformations of the SDYM equations is, just as in the cases of other completely integrable systems (cf. [2, 7, 8, 9]), based upon the existence of the associated linear system

$$(-\lambda\nabla_y + \nabla_{\bar{z}})V = 0, \quad (\lambda\nabla_z + \nabla_{\bar{y}})V = 0, \tag{2}$$

where λ is a parameter valued in the Riemann sphere $\mathbb{P}^1$, and V is an $r \times r$ matrix-valued unknown function depending on both $(y,z,\bar{y},\bar{z})$ and λ. Eqs. (1), which can be rewritten as

$$[-\lambda\nabla_y + \nabla_{\bar{z}}, \lambda\nabla_z + \nabla_{\bar{y}}] = 0,$$

are the integrability conditions of linear system (2) in the sense of Frobenius. Therefore one can always find a holomorphic and invertible matrix-valued solution V at least locally with respect to λ (although not uniquely), and such a solution conversely reconstructs the gauge potentials in a unique manner. Following the terminology of inverse scattering method, let us call such a solution V a wave function.

A RH transformation changes a wave function V into another one V' that is associated with another solution of the SDYM equations. For the moment let us suppose that V is holomorphic and invertible in a neighborhood of $\lambda = 0$. The transformation form V to V'

consists of the following steps.

i) Choose in the λ-sphere $\mathbb{P}^1$ a small circle C with center at $\lambda = 0$ such that V is holomorphic and invertible in a neighborhood of $C \cup C_+$, where C_+ and C_- denote the inside and the outside of C respectively.

ii) Give an $r \times r$ matrix-valued function $P = P(y,z,\bar{y},\bar{z},\lambda)$ that is holomorphic and invertible in a neighborhood of C and satisfies

$$(-\lambda\partial_y + \partial_{\bar{z}})P = 0, \quad (\lambda\partial_z + \partial_{\bar{y}})P = 0, \tag{3}$$

where ∂_u denote the derivations $\partial/\partial u$, $u = y,\ z,\ \bar{y},\ \bar{z}$.

iii) Form the matrix-valued function $H = VPV^{-1}$ and find a pair of matrix-valued functions $X_\pm = X_\pm(y,z,y,z,\lambda)$ that are holomorphic and invertible in some neighborhoods of $C \cup C_\pm$ respectively and satisfy the following condition

$$X_- = X_+H \quad \text{on } C. \tag{4}$$

iv) Form the matrix-valued function $V' = X_+V$.

Note that if one starts from the trivial wave function $V = 1_r$ that corresponds to the case $\nabla_u = \partial_u$ ($u = y,\ z,\ \bar{y},\ \bar{z}$), where 1_r denotes the $r \times r$ unit matrix, the above procedure coincides with Ward's construction [6] of SDYM fields.

Also note that step (iii) in the above procedure includes the uncertainty of the pair $X_\pm$ as $X_\pm \to h^{-1}X_\pm$, $h = h(y,z,\bar{y},\bar{z})$, which changes V' into $h^{-1}V'$. This however turns out to cause only a gauge transformation, because a gauge transformation $\nabla_u \to h^{-1} \circ \nabla_u \circ h$ of the covariant derivations precisely corresponds to the change $V \to h^{-1}V$ of the wave function. Ueno and Nakamura [3] formulated their transformation theory within the following gauge

$$\nabla_y = \partial_y,\ \nabla_z = \partial_z,\ \nabla_{\bar{y}} = \partial_{\bar{y}} + B_{\bar{y}},\ \nabla_{\bar{z}} = \partial_{\bar{z}} + B_{\bar{z}}, \qquad (5)$$

imposing to step (iii) the normalization condition

$$X_-(\lambda = \infty) = 1_r. \qquad (6)$$

This condition uniquely determines the pair $X_\pm$, and the resulting transformation preserves the gauge-fixing conditions $\nabla_y = \partial_y$ and $\nabla_z = \partial_z$.

2. Two different classes of RH transformations

The formulation of RH transformation reviewed in Sect. 1 involves a problem. Fix for simplicity the gauge as (5) and (6), and now consider the case where P is holomorphic and invertible not only in a neighborhood of C but also in C_+. Then so is H, therefore

$$X_- = 1_r,\ X_+ = H^{-1},\ V' = VP^{-1}.$$

This transformation $V \to VP^{-1}$, however, is in a sense a trivial one that does not change the gauge potentials. Because of this, as pointed out by Wu [7], the (infinitesimal) transformations given by Ueno and Nakamura [5] include many trivial ones that merely trivially act on the gauge potentials.

In order to fill up such a gap, one has to introduce another class of RH transformations presented by Chau [1] and Wu[7], which are defined to be transformations acting on another type of wave functions W, namely the wave functions chosen so as to be holomorphic and invertible in some neighborhoods of $\lambda = \infty$. The procedure defining the transformations is almost the same as that in Sect. 1: First choose a circle C with center at $\lambda = \infty$ such that W is holomorphic and invertible in a neighborhood of $C \cup C_-$. Then for a given matrix-valued function $Q = Q(y,z,\bar{y},\bar{z},\lambda)$ that satisfies the same conditions as P does, form $K = WQW^{-1}$ and find a pair of matrix-valued functions $Y_\pm = Y_\pm(y,z,y,z,\lambda)$ that are holomorphic and invertible in some neighborhoods of $C \cup C_\pm$ respectively and satisfy the following condition

$$Y_+ = Y_-K \quad \text{on} \quad C.$$

Finally defining W' as $W' = Y_-W$, one obtains a desired transformation $W \rightarrow W'$. (As a matter of fact, instead of working with wave functions around $\lambda = \infty$, Chau and Wu converted the problem to the preveous one through the map $\lambda \rightarrow \lambda^{-1}$.) The action of the RH transformations thus introduced is, as discussed in detail by Wu [7], essentially different from those as mentioned in Sect. 1, and in this respect if one wishes to make the whole theory more complete and symmetric, the former transformations (RH transformations "around the infinity") should be complemented to the latter ones (RH transformations "around the origin").

As one sees from the above discussion, a unified description of these two types of transformations inevitably requires to use a pair of wave functions V and W, not a single one, like those appeared in Ward's work [7]. Let us consider below how such a unified description can be realized using the pair (V,W).

For this purpose we first fix the gauge as (5). This does not loose any generality, because the first equation of eqs. (1) allows one to solve the linear systems

$$\nabla_y h = 0,\ \nabla_z h = 0 \quad \text{for} \quad h = h(y,z,y,z),$$

whose invertible matrix-valued solution then induces a gauge transformation that changes the covariant derivations into forms as (5). One can then choose a pair of wave functions V and W which are holomorphic and invertible in some neighborhoods of $\lambda = \infty$ and $\lambda = 0$ respectively such that

$$(-\lambda\partial_y + \partial_{\bar{z}} + B_{\bar{z}})V = 0,\ (\lambda\partial_z + \partial_{\bar{y}} + B_{\bar{y}})V = 0, \tag{7}$$

$$(-\lambda\partial_y + \partial_{\bar{z}} + B_{\bar{z}})W = 0,\ (\lambda\partial_z + \partial_{\bar{y}} + B_{\bar{y}})W = 0, \tag{8}$$

$$W(\lambda = \infty) = 1_r. \tag{9}$$

Note here that $B_{\bar{y}}$ and $B_{\bar{z}}$ can be reconstructed from V and W, using the coefficients of the Laurent expansion

$$V = \sum_{n=0}^{\infty} V_n\lambda^n \quad \text{around } \lambda = 0,\ W = \sum_{n=0}^{\infty} W_n\lambda^{-n} \quad \text{around} \quad \lambda = \infty, \tag{10}$$

as

$$B_{\bar{y}} = -\partial_{\bar{y}}V_0 \cdot V_0^{-1} = -\partial_z W_1,\quad B_{\bar{z}} = -\partial_{\bar{z}}V_0 \cdot V_0^{-1} = \partial_y W_1. \tag{11}$$

We then notice that the RH transformations reviewed in Sec. 1 induce transformations $(V,W) \to (V',W')$ of the pair of wave functions. In fact, defining W' as

$$W' = X_- W, \tag{12}$$

one obtains such transformations. Besides we here have the following relation

$$V'PV^{-1} = W'W^{-1}, \tag{13}$$

which characterizes such transformations, namely RH transformations "around the origin".

In view of the above characterization of RH transformations "around the origin", we see at once how to reformulate RH transformations "around the infinity", too. In fact such a transformation $(V,W) \to (V',W')$ can be similarly characterized by the following relation

$$V'V^{-1} = W'QW^{-1}. \tag{14}$$

Summing up the above discussion, one sees further that these two types of RH transformations can be composed, without depending on the order of composition, to yield finally a more general class of transformations $(V,W) \to (V',W')$ characterized by the relation

$$V'PV^{-1} = W'QW^{-1}. \tag{15}$$

This provides a unified framework for describing the various RH transformations discussed thus far. Perhaps a variety of generators of infinitesimal transformations as presented by Chau [1] and Wu [7] would be more rigorously justified in this framework, although we shall not go further into this problem here. Instead, in the next section we shall once again reformulate the above framework from a diffent point of view.

3. Infinite matrix representation of transformations

In a previous paper [4] I proposed an approach to the SDYM equations based upon an infinite matrix representation of eqs. (8), and tried to form the theories of solution and transformation. The framework developed there therefore only involved the information of W, none of V, and because of this the transformations to be described in that framework are limited to only those corresponding to RH transformations "around the infinity". If one wishes to establish a more satisfactory theory that can describe more general transformations as discussed in the previous section, the infinite matrix should be enlarged so as to include the information of both V and W. Let us below turn to this problem.

In order to define such an enlarged infinite matrix, we here introduce the following wave functions

$$\hat{V} = V_0^{-1} V = \sum_{n=0}^{\infty} \hat{V}_n \lambda^n, \quad \hat{W} = V_0^{-1} W = \sum_{n=0}^{\infty} \hat{W}_n \lambda^{-n}, \tag{16}$$

which are gauge-equivalent to V and W and subject to the following normalization

$$\hat{V}(\lambda = 0) = 1_r.$$

The corresponding covariant derivations take the following forms

$$\nabla_y = \partial_y + \hat{B}_y, \quad \nabla_z = \partial_z + \hat{B}_z, \quad \nabla_{\bar{y}} = \partial_{\bar{y}}, \quad \nabla_{\bar{z}} = \partial_{\bar{z}}, \tag{18}$$

and $\hat{V}$ and $\hat{W}$ satisfy the following equations

$$(-\lambda\partial_y + \partial_{\bar{z}} - \lambda\hat{B}_y)\hat{V} = 0, \quad (\lambda\partial_z + \partial_{\bar{y}} + \lambda\hat{B}_z)\hat{V} = 0, \tag{19}$$

$$(-\lambda\partial_y + \partial_{\bar z} - \lambda\hat{B}_y)\hat{W} = 0, \quad (\lambda\partial_z + \partial_{\bar y} + \lambda\hat{B}_z)\hat{W} = 0. \tag{20}$$

$\hat{B}_y$ and $\hat{B}_z$ can be explicitly reconstructed from $\hat{V}$ and $\hat{W}$ as

$$\hat{B}_y = -\partial_y\hat{W}_0\cdot\hat{W}_0 = \partial_{\bar z}\hat{V}_1, \quad \hat{B}_z = -\partial_z\hat{W}_0\cdot\hat{W}_0 = -\partial_{\bar y}\hat{V}_1. \tag{21}$$

We then form an infinite matrix ξ consisting of four blocks $\xi^{(m)} = (\xi^{(m)}_{ij})_{i\in\mathbb{Z},j<0}$, $m = 1, 2, 3, 4$, where $\xi^{(m)}_{ij}$ are $r \times r$ matrices, as follows:

$$\xi = \left(\begin{array}{c|c} \xi^{(1)} & \xi^{(3)} \\ \hline \xi^{(2)} & \xi^{(4)} \end{array}\right) = \left(\begin{array}{c|c} (W^*_{i-j})_{i\in\mathbb{Z},j<0} & (-\hat{W}^*_{i+j+1})_{i\in\mathbb{Z},j<0} \\ \hline (-V^*_{i+j+1})_{i\in\mathbb{Z},j<0} & (\hat{V}^*_{i-j})_{i\in\mathbb{Z},j<0} \end{array}\right)$$

$$\cdot\left(\begin{array}{c|c} (W_{i-j})_{i,j<0} & 0 \\ \hline 0 & (\hat{V}_{i-j})_{i,j<0} \end{array}\right), \tag{22}$$

where V^*_n, W^*_n, $\hat{V}^*_n$, $\hat{W}^*_n$ denote, when $n \geqq 0$, the coefficients of the Laurent expansion of V^{-1}, W^{-1}, $\hat{V}^{-1}$, $\hat{W}^{-1}$ as

$$V^{-1} = \sum_{n=0}^{\infty} V^*_n\lambda^n, \quad W^{-1} = \sum_{n=0}^{\infty} W^*_n\lambda^{-n},$$

$$\hat{V}^{-1} = \sum_{n=0}^{\infty} \hat{V}^*_n\lambda^n, \quad \hat{W}^{-1} = \sum_{n=0}^{\infty} \hat{W}^*_n\lambda^{-n},$$

but when $n < 0$, they are set together with V_n and $\hat{W}_n$ to be 0. The general notation used here for infinite matrices are the same as that of [4]. The upper left block $\xi^{(1)}$ coincides with the infinite matrix that appeared in [4], therefore this matrix ξ is actually an enlargement of the former, doubled in size. The coefficients of the Laurent expansion of the quartet $(V,W,\hat{V},\hat{W})$ appear as a part of components of ξ as indicated below.

$$\xi^{(1)} = \left(\begin{array}{cccc} & & \ddots & 0 \\ & 0 & & 1_r \\ \hline \cdots & -W_3 & -W_2 & -W_1 \\ \cdots & \cdots & \cdots & W_2^* \\ \cdots & \cdots & \cdots & W_3^* \\ \cdots & \cdots & \cdots & \cdots \end{array}\right), \quad \xi^{(3)} = \left(\begin{array}{cccc} & & 0 & \\ \hline \cdots & -V_2 & -V_1 & -V_0 \\ \cdots & \cdots & \cdots & -\hat{W}_1^* \\ \cdots & \cdots & \cdots & -\hat{W}_2^* \\ \cdots & \cdots & \cdots & \cdots \end{array}\right),$$

$$\xi^{(2)} = \left(\begin{array}{cccc} & & 0 & \\ \hline \cdots & -\hat{W}_2 & -\hat{W}_1 & -\hat{W}_0 \\ \cdots & \cdots & \cdots & -V_1^* \\ \cdots & \cdots & \cdots & -V_2^* \\ \cdots & \cdots & \cdots & \cdots \end{array}\right), \quad \xi^{(4)} = \left(\begin{array}{cccc} & & \ddots & 0 \\ & 0 & & 1_r \\ \hline \cdots & -\hat{V}_3 & -\hat{V}_2 & -\hat{V}_1 \\ \cdots & \cdots & \cdots & \hat{V}_2^* \\ \cdots & \cdots & \cdots & \hat{V}_3^* \\ \cdots & \cdots & \cdots & \cdots \end{array}\right).$$

The infinite matrix ξ has various interesting properties, among which the most important one is that eqs. (7), (8), (19) and (20) can be rewritten as

$$[-\begin{pmatrix} \Lambda & 0 \\ 0 & {}^t\Lambda \end{pmatrix}\partial_y + \partial_{\bar{z}}]\xi = \xi A, \quad [\begin{pmatrix} \Lambda & 0 \\ 0 & {}^t\Lambda \end{pmatrix}\partial_z + \partial_{\bar{y}}]\xi = \xi B, \tag{23}$$

where

$$\Lambda = (\delta_{i+1,j}1_r)_{i,j\in\mathbb{Z}} = \left(\begin{array}{cc|cc} \ddots & \ddots & 0 & \\ 0 & 0_r & 1_r & \\ \hline & & 0_r & 1_r \\ & & & \ddots \quad \ddots \end{array}\right) \quad (0_r = 0\cdot 1_r),$$

$$A = \left(\begin{array}{c|c} (-\partial_y \xi^{(1)}_{i+1,j})_{i,j<0} & (-\partial_y \xi^{(3)}_{i+1,j})_{i,j<0} \\ \hline 0 & 0 \end{array}\right),$$

$$B = \left(\begin{array}{c|c} (\partial_z \xi^{(1)}_{i+1,j})_{i,j<0} & (\partial_z \xi^{(3)}_{i+1,j})_{i,j<0} \\ \hline 0 & 0 \end{array}\right).$$

Extracting and comparing the upper left blocks that appear in eqs. (23), one obtains

$$(-\Lambda\partial_y + \partial_{\bar z})\xi^{(1)} = \xi^{(1)}A^{(1)}, \quad A^{(1)} = (-\partial_y \xi^{(1)}_{i+1,j})_{i,j<0},$$

$$(\ \Lambda\partial_z + \partial_{\bar y})\xi^{(1)} = \xi^{(1)}B^{(1)}, \quad B^{(1)} = (\ \partial_z \xi^{(1)}_{i+1,j})_{i,j<0},$$

which precisely gives the infinite matrix representation of eqs. (8) used in [4].

Let us now consider how the transformations characterized by eq. (15) can be represented in terms of ξ. The answer is as follows: First form the infinite matrix

$$\mathbb{P} = \left(\begin{array}{c|c} \mathbb{P}^{(1)} & 0 \\ \hline 0 & \mathbb{P}^{(4)} \end{array}\right) = \left(\begin{array}{c|c} (Q_{j-i})_{i,j\in\mathbb{Z}} & 0 \\ \hline 0 & (P_{i-j})_{i,j\in\mathbb{Z}} \end{array}\right), \tag{24}$$

where P_n and Q_n denote the coefficients of the Laurent expansion of P and Q as

$$Q = \sum_{n=-\infty}^{\infty} Q_n\lambda^n, \quad P = \sum_{n=-\infty}^{\infty} P_n\lambda^n.$$

Second, expressing the components of the matrix $\tilde\xi := \mathbb{P}\xi$ as

$$\tilde\xi = \left(\begin{array}{c|c} (\tilde\xi^{(1)}_{ij})_{i\in\mathbb{Z},j<0} & (\tilde\xi^{(3)}_{ij})_{i\in\mathbb{Z},j<0} \\ \hline (\tilde\xi^{(2)}_{ij})_{i\in\mathbb{Z},j<0} & (\tilde\xi^{(4)}_{ij})_{i\in\mathbb{Z},j<0} \end{array}\right),$$

and forming

$$\tilde\xi_{(-)} = \left(\begin{array}{c|c} (\tilde\xi^{(1)}_{ij})_{i,j<0} & (\tilde\xi^{(3)}_{ij})_{i,j<0} \\ \hline (\tilde\xi^{(2)}_{ij}))_{i,j<0} & (\tilde\xi^{(4)}_{ij})_{i,j<0} \end{array}\right),$$

define an infinite matrix ξ' as

$$\xi' = \tilde{\xi}\,\tilde{\xi}_{(-)}^{-1}. \tag{25}$$

Then this matrix ξ' turns out to coincide with the one constructed fron the quartet $(V',W',\hat{V}',\hat{W}')$ as ξ was constructed from $(V,W,\hat{V},\hat{W})$, therefore the transformation from ξ to ξ' gives an infinite matrix representation of the transformation from (V,W) to (V',W'). Besides if, in particular, $\mathbb{P}^{(4)}$ is the unit matrix of infinite size (in other words the induced transformation is a RH transformation "around the infinity"), the transformation $\xi^{(1)} \rightarrow \xi'^{(1)}$ of the upper left block certainly coincides with that discussed in [4]. This means that the framework developed thus far provides a generalization of my previous work.

Of course one should be careful about the fact that the above steps involve two non-trivial operations of infinite matrices, namely forming the inverse $\tilde{\xi}_{(-)}^{-1}$ and the product $\tilde{\xi}\,\tilde{\xi}_{(-)}^{-1}$, which obviously require some justification. In various interesting cases, however, these steps can be actually performed.

It should be emphasized here that the so called J-potential has already been built into our framework. In fact the leading term V_0 ($= \hat{W}_0^{-1}$) of V satisfies Yang's equation

$$\partial_y(\partial_{\bar{y}}J\cdot J^{-1}) + \partial_z(\partial_{\bar{z}}J\cdot J^{-1}) = 0, \tag{26}$$

therefore gives a J-potential. In this respect an interesting problem arises, which questions how a variety of Bäcklund transformations of Yang's equation (known or unknown) can be specified in our framework. This topic will be discussed elsewhere.

4. Concluding remarks

i) Although our discussion has been only concerned with the SDYM equations, the ideas presented here can be applied to other various completely integrable systems (if necessary, with slight modification that includes some generalization of the concept of RH transformations as remarked below (ii)). Of course, as I stressed in [4], the idea of using infinite matrices in the study of completely integrable systems has its origin in Sato's work [3], and this note still follows the lines of Sato. Nevertheless most results obtained in this note, especially the contents of Sect. 3 that make clear the meaning of RH transformations in the context of infinite matrix representation, are essentially new.

ii) The concept of RH can be further generalized by considering, instead of only considering ∞ and 0, any finite number of points $a_1, \ldots, a_{m+1}$ of the Riemann sphere and associated matrix-valued functions $U_1, \ldots, U_{m+1}$ locally defined near these points respectively. Such a transformation acts on these functions $(U_1,\ldots,U_{m+1})$ and changes them into some other ones $(U_1',\ldots,U_{m+1}')$ which are characterized by the condition that

$$U_1'P_1U_1^{-1} = U_2'P_2U_2^{-1} = \ldots = U_{m+1}'P_{m+1}U_{m+1}^{-1}, \qquad (27)$$

where $P_1, \ldots, P_{m+1}$ are holomorphic and invertible matrix-valued functions given in some neighborhoods of sufficiently small circles with centers at $a_1, \ldots, a_{m+1}$ respectively, and besides the existence of analytic continuations of $U_1'P_1U_1^{-1}$, etc. to the outside of these circles is assumed here in (27). One can also give an infinite matrix representation similar to that of Sect.3.

iii) RH transformations constructed with some algebraic procedures are of particular interest, because they often have explicit expressions and may be used as generators in constructing more complicated transformations. Here "generators" mean those of finite transformations, not of infinitesimal ones as thus far discussed in various papers. Some examples of such transformations are given by Ueno and Nakamura [5]. As a matter of fact their calculation can be reinterpreted from our point of view, and it seems likely in view of this that far more transformations of algebraic type in the above sense will be found similarly. If this is the case, one may further expect that general transformations canbe obtained by composing these spectial ones sufficiently many (if necessary, infinitely many) times [10], just as in the case of static axially symmetric gravitational fields.

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10. Y. Nakamura proposed such an idea a few years ago (unpublished), but the relation to RH transformations remained to be clarified.

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