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TOPOLOGY OF FOLDS , CUSPS AND MORIN SINGULARITIES

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Introduction.

Let $f:N \to P$ be a generic C^∞ mapping and let $\Sigma^I(f)$ be the set of singular points of f with Thom-Boardman symbol I. According to R.Thom ([13],[6]) $\Sigma^I(f)$ is a Z_2-cycle of N and its Poincaré dual cohomology class $[\Sigma^I(f)]^* \in H^*(N;Z_2)$ can be expressed as a polynomial of the Stiefel-Whitney characteristic classes of the tangent bundle TN and the induced bundle f*TP. This polynomial, called the Thom polynomial, tell us the homological location of $\Sigma^I(f)$ in N. However if $H_i(N;Z_2) = 0$ for $i = \dim.\Sigma^I(f)$, then it gives no informations about $\Sigma^I(f)$, though $\Sigma^I(f)$ may be non-trivial. Here we are interested in the topology of $\Sigma^I(f)$ and will investigate the simplest case where $P = R^p$ and f has only Morin singularities.

Results.

Let $f:N \to R^p$ be a C^∞ mapping with $\dim.N = n \geqq p$. A point $q \in N$ is a Morin singular point of type A_k of f if there exist local coordinates $(x_1,...,x_n)$ centered at q and $(y_1,...,y_p)$ centered at f(p) such that f has the form:

$$(1) \quad y_i \circ f = x_i \qquad i \leq p-1$$

$$y_p \circ f = x_p^{k+1} + \sum_{i=1}^{k-1} x_i x_p^{k-i} \pm x_{p+1}^2 \pm \dots \pm x_n^2$$

Let $A_k(f)$ denote the set of A_k-type singular points of f and $A_k(f)$ its closure.

Theorem 1. Let N be a compact manifold. If a C^∞ mapping $f:N \to R^p$ only Morin singular points, then we have

$$\chi(N) + \sum_{k=1}^{p} \chi(A_k(f)) = 0 \quad \text{mod } 2,$$

where $\chi(A)$ is the Euler number of a space A.

A_1 type singular points are called fold singular points. Let q be a fold singular point of f. Then $y_p \circ f$ in (1) becomes of the form

$$(2) \qquad y_p \circ f = x_p^{\,2} + (x_{p+1})^2 + \dots + (x_{p+\lambda-1})^2 - (x_{p+\lambda})^2 - \dots - x_n^{\,2}.$$

Now suppose that $n - p + 1$ is even, then $n - p - \lambda + 1 = \lambda$ mod 2, hence the parity of the index of the Morse function $y_p \circ f$ is independent of the choice of the coordinates. Hence the sets

$A_1^+(f) = \{ q \in N \mid q$ is a fold singular point of f with $\lambda = 0$ mod 2 . $\}$

$A_1^-(f) = \{ q \in N \mid q$ is a fold singular point of f with $\lambda = 1$ mod 2 . $\}$

are well-defined.

Let $W_i(M)$ denote the i-th Stiefel-Whitney characteristic class of a manifold M and $W_{*i}(M)$ the i-th Whitney homology class, which is Poincaré dual to $W_{m-i}(M)$, where $m = \dim. M$.

Theorem 2. Let N be a compact manifold with dim. $N = n \geqq p$. Let $f:N \to R^p$ be a C^∞ mapping which has only fold singular points. Then the following properties hold:

(a) $\chi(N) = \chi(S(f)) \mod 2$.

(b) for any k with $p-1 \geqq k \geqq 1$, we have

$$W_{*k-1}(N) = j_*(W_{*k-1}(S(f)),$$

in particular if $W_{n-k+1}(N) \neq 0$ then $W_{p-k}(S(f)) \neq 0$.

(c) if $n - p + 1$ is even, then

$$\chi(N) = \chi(S_+(f)) - \chi(S_-(f)),$$

where $S(f)$ is the singular point set of f and $S_+(f) = A_1^+(f)$, $S_-(f) = A_1^-(f)$.

A_2 type singular points are called cusp singular points. We let $K(f)$ denote the set of cusp singular points of a smooth mapping $f:N \to R^p$ and $S(f)$ the set of singular points of f. Let $i:K(f) \to S(f)$ and $j:S(f) \to N$ denote the inclusion maps respectively.

Theorem 3. Let N be a compact manifold with dim. $N = n \geqq p$. let $f:N \to R^p$ be a C^∞ mapping which has only fold and cusp singular points. Then we have

(a) $\chi(N) + \chi(S(f)) + \chi(K(f)) = 0 \mod 2,$

(b) $W_{*k}(N) + j_*(W_{*k}(S(f))) + j_*\, i_*(W_{*k}(K(f)) = 0$ for any k with $2k + 1 < p$.

Corollary of Theorem 3. If $2k + 1 < p$ and $W_{*k}(N) \neq 0$ then $W_{*k}(S(f)) + i_*(W_{*k}(K(f)) \neq 0$.

Remark. In this paper we are concerned with the topology of the singular point set of a C^∞ mapping having only some good kind of singularities and we do not touch on the existence of such good mappings. The existence problem of such good mappings have been investigated by several authors, for example [1],[2],[3], [4], [5], [7], [11], [12]. Compare our theorems in particular with Eliasberg's works [4] and [5].

The proof will be given elsewhere.

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