

FINITE DETERMINACY OF EQUIVARIANT MAP GERMS

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Equivariant Contact Equivalence

Let k denote either \mathbb{R} or \mathbb{C} and "smooth" mean C^∞ or analytic according to context. Let V and W be finite dimensional k -representations of a group G which is a compact Lie group if $k = \mathbb{R}$ and a reductive complex Lie group if $k = \mathbb{C}$. The space of (resp. equivariant) smooth map germs from $(V,0)$ to $(W,0)$ is denoted by $\mathcal{E}(V,W)$ (resp. $\mathcal{E}_G(V,W)$).

Definitions

(1) Let \mathcal{K}_G denote the group of germs, H , of equivariant diffeomorphisms of $(V \times W, 0)$ for which there exist germs of equivariant diffeomorphisms, h , of $(V,0)$ such that the following diagram commutes:-

$$\begin{array}{ccccc} (V,0) & \longrightarrow & (V \times W, 0) & \longrightarrow & (W,0) \\ \downarrow h & & \downarrow H & & \downarrow h \\ (V,0) & \longrightarrow & (V \times W, 0) & \longrightarrow & (W,0) \end{array}$$

(2) Two germs f, g in $\mathcal{E}_G(V,W)$ are \mathcal{K}_G -equivalent if there exists (H,h) in \mathcal{K}_G such that $H(h(x), f(h(x))) = (x, g(x))$, where x is a co-ordinate system for V .

(3) A germ, f , in $\mathcal{E}_G(V,W)$ is \mathcal{K}_G -finitely determined if there exists a positive integer, s , such that any g in $\mathcal{E}_G(V,W)$ with $j^s g(0) = j^s f(0)$ is \mathcal{K}_G -equivalent to f .

If the actions of G on V and W are trivial then we have the usual definitions of k -equivalence and finite determinacy

due to Mather. The above definition of \mathcal{K}_G seems to be the most natural generalisation to equivariant map germs; there are analogous definitions of \mathcal{R}_G , \mathcal{L}_G , \mathcal{A}_G and \mathcal{C}_G , but, for simplicity, only \mathcal{K}_G finite determinacy will be considered in the present paper.

Criteria for Finite Determinacy

For f in $\mathcal{E}(V,W)$ denote by tf the map $\mathcal{E}(V,V) \rightarrow \mathcal{E}(V,W)$;
 $\alpha \mapsto Df \cdot \alpha$

if f belongs to $\mathcal{E}_G(V,W)$ then tf restricts to a map

$\mathcal{E}_G(V,V) \rightarrow \mathcal{E}_G(V,W)$. Mather defined the (extended) \mathcal{K} -

tangent space of f to be the subspace of $\mathcal{E}_G(V,W)$:-

$$TK(f) := tf \mathcal{E}(V,V) + f \cdot m(W) \cdot \mathcal{E}(V,W)$$

where $m(W)$ is the maximal ideal of $\mathcal{E}(W)$, the ring of germs of smooth functions on $(W,0)$. If f belongs to $\mathcal{E}_G(V,W)$ the \mathcal{K}_G -tangent space is defined analogously by:-

$$TK_G(f) := tf \mathcal{E}_G(V,V) + (f \cdot m(W) \cdot \mathcal{E}(V,W))^G$$

where the notation $(\cdot)^G$ is used to denote the fixed point set of the natural action of G . Clearly we have

$$TK_G(f) = (TK(f))^G.$$

Theorem 1 [2], [7]

A germ f in $\mathcal{E}_G(V,W)$ is \mathcal{K}_G -finitely determined if and only if $\dim_{\mathcal{K}} \mathcal{E}_G(V,W) / TK_G(f) < \infty$. □

An immediate corollary of this is that if f in $\mathcal{E}_G(V,W)$ is \mathcal{K} -finitely determined then it is also \mathcal{K}_G -finitely determined. In fact Wall [10] proved that if $j^s f(0)$ is \mathcal{K} -sufficient, then it is also \mathcal{K}_G -sufficient.

A second characterisation of finitely determined germs states that a germ is finitely determined if and only if it has a representative which is "stable" in a punctured neighbourhood of 0 (see Wall [12] for the non-equivariant case).

Definition

A germ f in $\mathcal{E}_G(V, W)$ is (infinitesimally) K_G -stable at 0 if $T_{K_G}(f) = \mathcal{E}_G(V, W)$. Stability of germs of equivariant maps $f : V \rightarrow W$ at points $x \neq 0$ can be defined similarly after restricting f to a slice transversal to the orbit $G \cdot x$ at x , provided the orbit is closed in V .

By a closed orbit of the action of G on V we mean an orbit which is closed as a subset of V with its usual topology. If $|G|$ is finite or $k = \mathbb{R}$ and G is compact then every orbit of G on V is closed. However this is not true in the general complex case (consider, for example, the action of $G = \mathbb{C}^*$ on \mathbb{C}^2 given by $t \cdot (x, y) = (t^{-1}x, ty)$ where t is an element of \mathbb{C}^* and (x, y) of \mathbb{C}^2). In fact if $k = \mathbb{C}$ and G is a reductive complex Lie group with $\dim G \geq 1$ then there always exist non-closed orbits. The necessity of the restriction to closed orbits in the definition is due to the fact that non-closed orbits do not, in general, possess suitable slices. Fortunately we only need a definition of stability at points on closed orbits.

Theorem 2 [7]

Let $k = \mathbb{C}$. Then a germ f in $\mathcal{E}_G(V, W)$ is K_G -finitely determined if and only if there exists a G -invariant neighbourhood U of 0 in V and a G -equivariant representative

of f on U which is K_G -stable at every point in $U \setminus \{0\}$ for which the orbit $G.x$ is closed. \square

If the actions of G on V and W are trivial then the theorem reduces to the usual statement that f is K -finitely determined if and only if a representative of f is K -stable on $U \setminus \{0\}$ or, equivalently, if and only if it is transversal to 0 on $U \setminus \{0\}$.

Since a real analytic germ f in $\mathcal{E}_G(V, W)$ is K_G -finitely determined if and only if its complexification $f_{\mathbb{C}}$ is $K_{G_{\mathbb{C}}}$ -finitely determined (where $G_{\mathbb{C}}$ is the reductive complex Lie group obtained by complexifying the compact group G), this theorem also characterises K_G -finitely determined analytic germs when $k = \mathbb{R}$. Results analogous to Theorems 1 and 2 also hold for $\mathcal{R}_G, \mathcal{L}_G, \mathcal{A}_G$ and \mathcal{C}_G finite determinacy.

Invariant Maps

Suppose the action of G on W is trivial; then the following lemma is not difficult to prove.

Lemma 3. An invariant map $f : V \rightarrow W$ is K_G -stable at a point x such that $G.x$ is closed if and only if it is K -stable, and hence transversal to 0 , at x . \square

Corollary 4. If G is finite, then $f \in \mathcal{E}_G(V, W)$ is K_G -finitely determined if and only if it is K -finitely determined.

The proof of the corollary is by the lemma, Theorem 2 and the fact that all the orbits of G in V are closed. \square

If $\dim.G \geq 1$, and if $\dim.V > \dim.W > 1$, then Wall has shown [11] that no invariant map germ $f : V \rightarrow W$ with $Df(0) = 0$ can be K -finitely determined (Slodowy [10] proved a similar result for the case $\dim.W = 1$, but the situation when $\dim.V \leq \dim.W$ is unclear). However, although most invariant map germs are not K -finitely determined if $\dim.G \geq 1$, they are K_G -finitely determined. More precisely we have the following result.

Theorem 5 [8]

For any G , if G acts trivially on W , then the germs in $\mathcal{E}_G(V, W)$ which are not K_G -finitely determined lie in a subset of infinite codimension. \square

In a rather different direction we have the following proposition about real analytic germs.

Proposition 6

If $k = \mathbb{R}$ and f is a K_G -finitely determined real analytic invariant map germ then it is also C^k - K -finitely determined for all k such that $0 \leq k < \infty$.

Proof. If f is K_G -finitely determined there is a suitable representative \tilde{f} of f such that $\tilde{f}_\mathbb{C}$ is transversal to 0 at all points with closed $G_\mathbb{C}$ -orbits in $V_\mathbb{C} := V \otimes_{\mathbb{R}} \mathbb{C}$. If x is any point in V then $G_\mathbb{C} \cdot x$ is closed (Schwarz [9] p.59) and so f is transversal to 0 at all points in a punctured neighbourhood of 0 in V . The result now follows from the geometric criterion for C^k -finite determinacy (see Wall [12] Theorem 6.1). \square

Z_p - Equivariant Functions

If the action of G on W is non-trivial then we may lose the genericity, and even the existence, of K_G - finitely determined germs.

Let $G = Z_p$, identified with the p -th roots of unity; let t denote a generator of Z_p . Define an action of G on $V = \mathbb{C}^{m+n}$ by

$$t \cdot (x_1, \dots, x_m, y_1, \dots, y_n) = (tx_1, \dots, tx_m, y_1, \dots, y_n)$$

and on $W = \mathbb{C}$ by

$$t \cdot z = t^q z$$

for some q satisfying $0 \leq q < p$.

Theorem 7 [8]

(1) There exist K_G - finitely determined germs in $\mathcal{E}_G(V, W)$ if and only if $n = 0$ or $n \geq \frac{(q+m-1)!}{q!(m-1)!} - m^2$

(2) Finite determinacy with respect to K_G holds outside a subset of infinite codimension if and only if one (or more) of the following conditions holds:-

- | | |
|----------------|---|
| (a) $n = 0$ | (b) $q \leq 2$ |
| (c) $m \leq 1$ | (c) $(m, q) = (2, 3), (2, 4)$ or $(3, 3)$. |

We give a sketch of a proof of the theorem.

Any f in $\mathcal{E}_G(V, W)$ can be written

$$f(x, y) = \sum_j \bar{f}_j(x, y) \tau_j(x)$$

where $\{\tau_j\}$ is the set of monomials of degree q in x_1, \dots, x_m and the \bar{f}_j are invariant functions on V . So f is determined by a (non-unique) invariant map $\bar{f} : V \rightarrow M$, where M is the space of homogeneous forms of degree q in m variables.

Using arguments similar to some which are standard in singularity theory it can be shown that

f is K_G - stable at x in $V \setminus V^G$ if and only if it is transversal to 0 at x ,

and

f is K_G - stable at x in V^G if and only if $\bar{f}|_{V^G} : V^G \rightarrow M$ is transversal to the orbits of the natural action of $GL(m)$ on M .

Thus, by Theorem 2, K_G - finite determinacy is reduced to a set of transversality conditions on a punctured neighbourhood of 0 .

The number of these transversality conditions is finite if and only if $n = 0$ or the number of $GL(m)$ orbits in M is finite; this latter condition is satisfied if and only if $m \leq 1$ or $q \leq 2$ or $(m, q) = (2, 3)$. In these cases K_G finite determinacy is certainly generic.

More generally we can stratify M so that each stratum is foliated by $GL(m)$ orbits. If the strata of codimension $< n$ are foliated by orbits of codimension ≤ 1 in the stratum then the points where \bar{f} is not transversal to the $GL(m)$ orbits generically form a set of dimension 0 and so they can be excluded from a sufficiently small punctured neighbourhood of 0 . Conversely, if there is a stratum of codimension $< n$ which is foliated by orbits of codimension > 1 , then we can construct \bar{f} (and hence f) with points of non-transversality in any neighbourhood of 0 which can not be removed by perturbing \bar{f} slightly. This means that no extension of a suitable jet

of f can be K_G -finitely determined, and so K_G -finite determinacy can not be generic. There exists a stratification of M such that the strata of codimension $< n$ are foliated by orbits of codimension ≤ 1 if and only if we are in one of the cases of the previous paragraph or $(m,q) = (3,3)$ or $(2,4)$. This completes the proof of (2).

The necessity of the condition $n = 0$ or $n \geq \dim.M - m^2 = \frac{(q+m-1)!}{q!(m-1)!} - m^2$ in (1) is clear from the above since $\dim.M - m^2$ is the codimension of the orbits of maximal dimension in M . The sufficiency of this condition follows by more or less explicit construction of K_G -finitely determined germs. \square

Theorems 5 and 7 follow from a more general result [8] which gives necessary and sufficient conditions for R_G, C_G and K_G finite determinacy to be generic in $\mathcal{E}_G(V,W)$ for any reductive complex Lie group G and complex representations V and W . There appears to be a serious problem with proving the necessity of the conditions when $\mathbb{k} = \mathbb{R}$.

Stable Unfoldings

In complete analogy with the non-equivariant case Damon [2] proved that an equivariant map germ has a K_G -versal unfolding if and only if it is K_G -finitely determined (corresponding results hold for $R_G, \mathcal{L}_G, \mathcal{A}_G$ and C_G). However there is one aspect of unfolding theory which doesn't generalise in such a straightforward manner. Recall that for

ordinary map germs an unfolding is \mathcal{A} stable if and only if it is \mathcal{K} - versal and so the property of possessing an \mathcal{A} stable unfolding is generic; this latter fact is important in the proof of the topological stability theorem [4] and also in that of the genericity of topological \mathcal{A} - finite determinacy [6].

However for equivariant map germs this is no longer true, instead we need to replace \mathcal{K}_G by a subgroup, denoted \mathcal{K}_G^* and defined in [13]. Then it is true that an equivariant unfolding is \mathcal{A}_G - stable if and only if it is \mathcal{K}_G^* - versal and so an equivariant germ has an \mathcal{A}_G - stable unfolding if and only if it is \mathcal{K}_G^* - finitely determined. If the action of G on W is trivial then $\mathcal{K}_G^* = \mathcal{K}_G$, but in general \mathcal{K}_G^* finite determinacy is a stronger property than \mathcal{K}_G - finite determinacy and \mathcal{K}_G^* finite determinacy may not be generic even when \mathcal{K}_G - finite determinacy is.

Example. Let $G = Z_2$ with generator t and V be the $m+n$ dimensional representation defined by:-

$$t.(x_1, \dots, x_m, y_1, \dots, y_n) = (tx_1, \dots, tx_m, y_1, \dots, y_n)$$

and W the similarly defined $p+q$ dimensional representation.

Theorem 8 [8]

(1) Finite determinacy with respect to \mathcal{K}_G is generic in $\mathcal{E}_G(V, W)$ for all m, n, p and q .

(2) Finite determinacy with respect to \mathcal{K}_G^* is generic in $\mathcal{E}_G(V, W)$ if and only if one of the following holds:-

$$(a) \quad m = 0 \qquad (b) \quad p \leq 2$$

$$(c) \quad n \leq \begin{cases} m-p+q+1 & m \geq p \\ q & m < p \end{cases}$$

The non-genericity of K_G^* finite determinacy casts doubts on the possibility of an equivariant version of the topological stability theorem and Nakai [5] has found examples of pairs (M,N) of Z_2 - manifolds for which equivariant topologically stable maps are not dense and, locally, topological A_G - finite determinacy is not generic. It is perhaps interesting to note that in these examples the pairs of slice representations (V,W) of pairs of points (x,y) in $M^G \times N^G$ lie in the range $n = q+1$, $m = p-1$ and so K_G^* finite determinacy is not generic in $\mathcal{E}_G(V,W)$. It may be possible to prove an equivariant version of the topological stability theorem if K_G^* finite determinacy is generic for all possible pairs of slice representations; we might also conjecture that if K_G^* - finite determinacy is generic in $\mathcal{E}_G(V,W)$, then so is topological A_G - finite determinacy.

A Conjecture

So far we have no equivalence relation for which finite determinacy is generic for all G, V and W . However the following looks promising.

Definition

Two germs f, g in $\mathcal{E}_G(V,W)$ are $C^0 - V_G$ - equivalent if there exists a germ of an equivariant homeomorphism of $(V,0)$ taking $f^{-1}(0)$ to $g^{-1}(0)$.

Conjecture

The set of germs in $\mathcal{E}_G(V,W)$ which are not $C^0 - V_G$ - finitely determined has infinite codimension.

A proof of this conjecture could perhaps be constructed using Thom's "blowing-up" idea (e.g. [6]) together with the equivariant transversality theory and isotopy theorem of Bierstone [1] and Field [3].

Finally we note that some related work, on the infinite determinacy of equivariant germs, has been done by Wall [14].

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