

Brownian Motions on Riemann Surfaces of Inverse Functions

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§1. Introduction.

Let $B = (B_t, t \geq 0)$ be a complex Brownian motion starting at 0 defined on a probability space (Ω, \mathcal{F}, P) and f be a non-constant analytic function in the unit disc Δ . Define φ_t and W by

$$\varphi_t = \int_0^t |f'(B_s)|^2 ds,$$

up to the first exit time σ of B from Δ and

$$W = (W_t) = (f(B_{\varphi_t}^{-1})).$$

Then the process W is also a Brownian motion up to the time φ_σ . It is known that $E[\varphi(\sigma)^{p/2}] \approx \|f\|_p$ for $0 < p < \infty$ (Burkholder, Gundy and Silverstein [2]). In 1979 Davis [3] noted that φ_σ is the first exit time of the Euclidean Brownian motion W from $f(\Delta)$. Precisely let S be the Riemann surface of f^{-1} such that S is a covering surface of $f(\Delta)$ with the natural projection p and that there exists a one-to-one onto mapping f^{-1} with $f^{-1} \cdot f = p$. Such a surface is called the Riemann surface of inverse function. The

Brownian motion $W = (W_t)$ can be lifted continuously on S . Let $W^* = (W_t^*)$ be the lifted Brownian motion on S . Since the generator of W^* is $1/2$ times the Laplace-Beltrami operator corresponding to the pull-backed metric on S from the Euclidean metric on $f(\Delta)$, W^* is a Brownian motion corresponding to this metric and φ_σ is the first exit time of W^* from S .

In the present paper we shall study analogously spherical Brownian motions on Riemann surfaces of inverse functions.

§1. Result.

Let $w = f(z)$ be a non-constant meromorphic function in the z -plane to the w -sphere. We may regard f and its restriction $f|_{\{|z|<r\}}$ as one-to-one onto mappings from the complex plane C and $\{|z|<r\}$ onto Riemann surfaces of inverse functions S and S_r respectively. We may assume $S_r \subset S$. Now we can define a spherical metric on S by

$$\rho(w^*)dw^*d\bar{w}^* = \frac{dw d\bar{w}}{(1+|w|^2)^2},$$

for each local coordinate w^* with $w = p(w^*)$. Let A denote the spherical area on S , then

$$A(r, f) \equiv A(S_r) = \int_{|z| \leq r} \frac{|f'(z)|^2}{(1+|f(z)|^2)^2} dx dy.$$

Define the Ahlfors-Shimizu characteristic $T(r, f)$ by

$$T(r, f) = \int_0^r \frac{A(x, f)}{x} dx.$$

Then it is well-known that

$$T(r, f) = \frac{1}{\pi} \int_{|z| < r} \frac{|f'(z)|^2}{(1+|f(z)|^2)^2} g(z) dx dy,$$

where g is the Green's function of $\{|z| < r\}$ with a pole at 0 and $z = x+iy$.

Let $w_0^* = f(0) \in S_r$. The spherical metric ρ does not only define $A(r, f)$ and $T(r, f)$ but also generates a Brownian motion $W^* = (W_t^*)$ starting at w_0^* on S defined on some probability space $(\Omega^*, \mathcal{F}^*, P^*)$ such that

$$\lim_{t \downarrow 0} \frac{1}{t} E^* [u(W_t^*) - u(w_0^*)] = \frac{1}{2} (L_\rho u)(w_0^*), \quad (2.1)$$

for each C^2 -bounded function u on S where E^* denotes the mathematical expectation with respect to P^* and L_ρ is the Laplace-Beltrami operator corresponding to ρ . Let σ_r^* be the first exit time of W^* from S_r . Then we have,

Theorem. For each $r, r > 0$, it holds

$$E^* [\sigma_r^*] = T(r, f).$$

§3. Proof. We can construct W^* by the standard time change-argument (Blumenthal and Gettoor [1] p.212). Define φ_t by

$$\varphi_t = \int_0^t \frac{|f'(B_s)|^2}{(1+|f(B_s)|^2)^2} ds,$$

and put $\psi_t = \varphi_t^{-1}$. Then $W = (W_t) \equiv (f(B_{\psi_t}))$ is a spherical Brownian motion on the w -sphere. Let $W^* = (W_t^*)$ be a lifted process of W such that W^* has continuous paths a.s. with $p(W_t^*) = W_t$ and $W_0^* = w_0^*$. Without loss of generality we assume $f'(0) \neq 0$. Then a simple application of Itô's formula (Ikeda and Watanabe [4] p.66) shows (2.1). Since σ_r^* is the first exit time of W^* from S_r , we have

$$\sigma_r^* = \inf \{ t ; W_t^* \in S_r \}$$

$$\begin{aligned}
&= \inf \{ t ; f^{-1}(W_t^*) \in f^{-1}(S_r) \} \\
&= \inf \{ t ; |B_{\psi_t}| \geq r \} \\
&= \inf \{ \varphi_t ; |B_t| \geq r \} \\
&= \varphi_{\sigma_r}
\end{aligned}$$

where σ_r is the first exit time of B from $\{|z| < r\}$. Hence we have

$$\begin{aligned}
E[\sigma_r^*] &= E[\varphi_{\sigma_r}] \\
&= E\left[\int_0^{\sigma_r} \frac{|f'(B_s)|^2}{(1+|f(B_s)|^2)^2} ds \right].
\end{aligned}$$

Let $p(s, z) = P(s < \sigma_r, B_s \in dx dy)$ is the density function of the random variable $B_{s \wedge \sigma_r}$ with respect to the Euclidean area element. Then it is well-known (Itô-McKean [5] p.237) that

$$\int_0^{\infty} p(s, z) ds = \frac{1}{\pi} g(z).$$

This shows

$$\begin{aligned}
E[\sigma_r^*] &= \frac{1}{\pi} \int_{\{|z| < r\}} \frac{|f'(z)|^2}{(1+|f(z)|^2)^2} g(z) dx dy \\
&= T(r, f).
\end{aligned}$$

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