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ON THE SIZE OF SOME TRIGONOMETRIC POLYNOMIALS

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Let $E_n(f)$ denote the best approximation of $f \in C[0,2\pi]$, i.e.,

$$E_n(f) = \inf \|f - T_n\| ,$$

where $\| \cdot \|$ is the supremum norm and $T_n(x)$ are trigonometric polynomials of degree $\leq n$. It was Bernstein [3] who first showed the close connection between absolute convergence of the Fourier series of $f(x) \in \mathrm{Lip}\,\alpha$ and $E_n(f)$. In particular, he obtained the following theorem which shows his result being best possible.

Theorem A. For any given sequence $\varepsilon_n \downarrow 0$ such that

$$\sum_{n=1}^{\infty} \varepsilon_n/\sqrt{n} = \infty ,$$

we can find an $f \in C[0,2\pi]$ whose Fourier series is not absolutely convergent at any point at all and yet satisfies the inequality $E_n(f) \leq \varepsilon_n$.

To prove this theorem, he invented the following lemma which may well deserve an independent interest.

Lemma A. For any given natural number N, we can find a trigonometric polynomial of the form

$$T_N(x) = \sum_{N/2 \leq n \leq N} \cos(nx + \rho_n)$$

such that uniformly in x,

$$T_N(x) \ll \sqrt{N} .$$

Bernstein's original proof of this lemma is due to the theory of characters, while Bari [1] applied Kuzmin's lemma instead and Kahane[4]used Rudin-Shapiro theorem. Actually Bernstein considered the sum

$$(1)\qquad S = \sum_{A\leq n\leq B} e(a(\frac{n^2}{N} + xn)), \quad (e(u) = \exp(2\pi iu))$$

where $a > 0$, $0\leq x\leq 1$ and $A,B,N\in \mathbb{N}$ are such that $1\leq A<B\leq N$.
First we remark that if we apply Salem's lemma(Lemma 2 below)to S, then we obtain

$$(2)\qquad S \ll (\sqrt{a} + 1/\sqrt{a})\sqrt{N},$$

which holds uniformly in x,A,B.
On the one hand, from a different stand point, there is a problem of finding the polynomials

$$(3)\qquad P_N(z) = \sum_{n\leq N} c_{n,N}\, z^n ,$$

with $|c_{n,N}| = 1$ and $|z| = 1$ such that

$$(4)\qquad \sqrt{N} \ll P_N(z) \ll \sqrt{N} ,$$

for all z. Parseval's formula shows

$$(5)\qquad \max_{|z|=1} |P_N(z)| \geq \sqrt{N} .$$

See e.g. Kahane[5] for recent results. The next example of (3) in the literature seems to be the following one due to Hardy and Littlewood[cf. 7, p.199]:

$$(6)\qquad P_N(z) = \sum_{n\leq N} e(cn\log n + xn), \quad z = e(x)$$

which satisfies $P_N(z)\ll\sqrt{N}$ uniformly in x. However, as far as I know, it seems open whether it satisfies $P_N(z)\gg\sqrt{N}$ for all x. We notice that in their example the coefficients $c_n = e(cn\log n)$ are independent of N.

The main purpose of this note is to show that the size of (3) may be sometimes smaller than $\sqrt{N}$. We shall show it effectively by constructing examples.

First we prove

Theorem 1. For any given $N > 1$, we can find a sequence $c_{n,N} \in \mathbb{C}$ with $|c_{n,N}| = 1$ $(1 \leq n \leq N)$ such that

(7) $$\sum_{n \leq N} c_{n,N} z^n \ll N^{1/4},$$

for all $z \in \mathbb{C}$ with $|z| = 1$, where $\ll$ depends on z.

Proof. Consider the sum

(8) $$S_N = \sum_{n \leq N} e(xn - 2s\sqrt{n}),$$

where $0 < x \leq 1$ and $s \geq 1$ will be suitably chosen(as a function of N)later. If we put $f(t) = xt - 2s\sqrt{t}$ $(1 \leq t \leq N)$, then

$$-s/\sqrt{t} < f'(t) \leq 1 - s/\sqrt{t} \leq 1 - s/\sqrt{N}.$$

Therefore, if $4s^2 \leq t \leq N$, then

$$|f'(t)| \leq 1 - s/\sqrt{N},$$

because then $-(1 - s/\sqrt{N}) \leq -s/\sqrt{t}$.
Now we shall apply the following known lemma due to van der Corput[cf.2].

Lemma 1. If $f'(t)$ is monotone and satisfies

$$|f'(t)| \leq 1 - \varepsilon \quad , \quad (0 < \varepsilon < 1)$$

throughout (a, b), then

$$\sum_{a \leq n \leq b} e(f(n)) = \int_a^b e(f(t))dt + O(1/\varepsilon),$$

where the constant implied by O is absolute.

If we insert $\varepsilon = s/\sqrt{N}$ in the above lemma, then we obtain

$$\sum_{4s^2 \leq n \leq N} e(xn - 2s\sqrt{n}) = \int_{4s^2}^{N} e(xt - 2s\sqrt{t})dt + O(\sqrt{N}/s).$$

Thus we have

$$(9) \qquad S_N = \sum_{1 \leq n \leq 4s^2} e(xn - 2s\sqrt{n}) + \int_{4s^2}^{N} e(xt-2s\sqrt{t})dt + O(\sqrt{N}/s).$$

We appeal to the known lemma below to estimate the first sum in (9).

Lemma 2 (Salem[cf. 7, p.226]). If $f''(t) > 0$ is monotone, then

$$\sum_{a \leq n \leq b} e(f(n)) = O(\max_{a \leq t \leq b} \frac{1}{\sqrt{f''(t)}}) + O(\int_a^b (\sqrt{f''(t)} + f''(t))dt),$$

where the implied constants by O's are absolute.

Now for $f(t) = xt - 2s\sqrt{t}$ we have $f''(t) = \frac{s}{2} t^{-3/2}$. Hence by Lemma 2 we have

$$(10) \qquad \sum_{1 \leq n \leq 4s^2} e(f(n)) = O(\max_{1 \leq t \leq 4s^2} \frac{1}{\sqrt{s}} t^{3/4}) + O(\int_1^{4s^2} \sqrt{s}t^{-3/4}dt + \int_1^{4s^2} st^{-3/2}dt) = O(s) + O(s) + O(s) = O(s).$$

Next we shall estimate the integral

$$I_N = \int_{4s^2}^{N} e(xt - 2s\sqrt{t})dt.$$

If we put $t = u^2$, then

$$I_N = 2\int_{2s}^{\sqrt{N}} u \cdot e(xu^2 - 2su)du$$

$$= \frac{1}{2\pi ix} \int_{2s}^{\sqrt{N}} (e(xu^2-2su))'du + \frac{2s}{x} \int_{2s}^{\sqrt{N}} e(xu^2-2su)du$$

$$= \frac{2s}{x} \int_{2s}^{\sqrt{N}} e(xu^2 - 2su)du + O(1/x).$$

Lemma 3 [cf.6 & 7]. If $f''(t) \geq r > 0$ throughout (a, b), then

$$\int_a^b e(f(t))dt \ll 1/\sqrt{r},$$

where $\ll$ is absolute.

From this lemma we have

(11) $$\int_{2s}^{\sqrt{N}} e(xu^2 - 2su)du = O(1/\sqrt{x}).$$

Thus we obtain from (9)-(11)

$$S_N = O(s) + O(sx^{-3/2}) + O(\sqrt{N}/s) + O(1/x).$$

Finally, by choosing $s = \frac{1}{2} N^{1/4}$, we get

$$S_N = O(N^{1/4}),$$

where the implied constant by O depends on x. □

If the coefficients $c_{n,N}$ are independent of N, then the situation in general becomes more difficult and we then have the following result.

Theorem 2. For any given $\varepsilon > 0$, there exist a natural number $N_0 = N_0(\varepsilon)$ and a sequence $c_n = c_n(\varepsilon) \in \mathbb{C}$ with $|c_n| = 1$ $(1 \leq n \leq N)$ such that for all $N \geq N_0$ and z with $|z| = 1$,

$$\sum_{n \leq N} c_n z^n \ll_{\varepsilon, z} N^{2/5 + \varepsilon}$$

Proof. We only indicate the outline of the proof since it is similar to that of Theorem 1. In this case we consider the sum

$$S_N = \sum_{n \leq N} e(xn - n^c/c),$$

where $0 < x \leq 1$ and $0 < c < 1$. If we put $f(t)=xt - t^c/c$ $(2 \leq t \leq N)$, then we have by Lemma 1

$$\sum_{2 \leq n \leq N} e(f(n)) = \int_2^N e(f(t))dt + O(N^{1-c}).$$

Next we apply a known lemma [7, p.62] in order to estimate the above integral, say I(N). Then after simple calculation, we have for

$$N \geq 2(2/x)^{1/(1-c)}$$

$$I(N) - I(N/2) = O(N^{1-3c/5}),$$

where O depends on c and x. Hence substituting in N successively

$$N/2,\ N/2^2, \ldots\ldots$$

and adding them all, we get $I(N) = O(N^{1-3c/5})$.
Therefore we finally obtain

$$S_N = O(N^{1-3c/5}) + O(N^{1-c}) = O(N^{1-3c/5})$$

$$= O(N^{2/5+\varepsilon}), \quad (c = 1 - 5\varepsilon/3).$$

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