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ON THE CRITICAL PHENOMENA FOR PIECEWISE LINEAR TRANSFORMATIONS

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ABSTRACT

We will determine the decay rate of correlation for piecewise linear transformations and as an application, we will consider two critical phenomena of dynamical systems.

1. INTRODUCTION

We will determine the decay rate of correlation for a certain class of piecewise linear transformations explicitly in terms of Fredholm determinant (cf. [10] , for more general cases, cf. [11] ,[12]), and apply it to the critical phenomena in dynamical systems.

We will consider a power series Φ , called the Fredholm determinant associated with a piecewise linear transformation F, whose definition will be given in § 3, and we call solutions of $\Phi(1/z)= 0$ Fredholm eigenvalues. By γ_1, γ_2, we denote the greatest and the second greatest in modulus (γ_1= the slope λ of F). Our main theorem is stated as follows:

Theorem 1-1. Suppose that $\eta/\lambda < 1$. Then there exists an invariant measure μ which is absolutely continuous with the Lebesgue measure and the dynamical system $([0,1], \mu, F)$ is mixing. Moreover, for any pair of a function f with bounded variation and an integrable function g, we get for any $\varepsilon > 0$

$$\lim_{n\to\infty} ((\eta+\varepsilon)/\lambda)^{-n} \left\{ \int f(x)\, g(F^n x)\, d\mu - \int f\, d\mu \int g\, d\mu \right\} = 0, \tag{1.1}$$

where

$$\eta = \begin{cases} \lambda & \text{if } \quad \Phi'(1/\lambda) = 0, \\ \max\{|\tau_2|, 1\} & \text{otherwise,} \end{cases} \tag{1.2}$$

and F^n is the n-fold iterate of F:

$$F^n x = \begin{cases} x & n=0, \\ F(F^{n-1}x) & n \geq 1. \end{cases} \tag{1.3}$$

(on the Fredholm determinant, cf. [13] , [16] . Some related topics on the decay rate of correlation can be found in [2] , [6] .)

As an application of Theorem 1-1, we show two critical phenomena in dynamical systems. One is the case of β-transformations when $\lambda \downarrow 1$. This is the phase transition from disordered motion to ordered motion, The other is the unimodal linear transformations when $\lambda \downarrow \sqrt{2}$. This is the phase transition from mixing state to ergodic but not mixing state. Our second theorem is stated as follows:

Theorem 1-2. i) (β-transformations) Let N be the first return time of point 1 to the interval $(1/\lambda, 1]$:

$$F^i(1) \notin (1/\lambda, 1] \quad \text{for } 1 \leqq i \leqq N-1, \qquad (1.4)$$

$$F^N(1) \in (1/\lambda, 1].$$

Then the Fredholm eigenvalues around 1 have the following asymptotic form:

$$e^{\alpha + i\beta},$$

where

$$\alpha = \frac{1}{N}\left\{\log N + \text{small order}\right\}, \qquad (1.5)$$

$$\beta = \frac{1}{N}\left\{n\pi + \text{small order}\right\} \quad n = 0, \pm 1 \pm 2, \ldots \qquad (1.6)$$

as $\lambda \downarrow 1$. The second greatest Fredholm eigenvalue is the case $n = \pm 1$, and so we may say that the decay rate of the correlation

$$\eta/\lambda = \exp\left\{- \frac{1}{2N}\left(\frac{\pi}{\log N}\right)^2 + \text{small order}\right\}, \qquad (1.7)$$

as $\lambda \downarrow 1$.

ii) (unimodal linear transformations) The argument of the second greatest Fredholm eigenvalue equals π and the decay rate of the correlation

$$\eta/\lambda = 2\lambda^{-2} + \text{small order}, \qquad (1.8)$$

as $\lambda \downarrow \sqrt{2}$.

This theorem shows:

i) for β-transformation whose slope is sufficiently close to 1, the series of the correlation

$$\int f(x)\, g(F^n x)\, d\mu - \int f\, d\mu \int g\, d\mu$$

for a pair of a function f with bounded variation and an integrable function g decreases approximately in the order $\exp\left\{-\frac{1}{2N}\left(\frac{\pi}{\log N}\right)^2\right\}$ with the frequency of modulation approximately $1/N$,

ii) for unimodal linear transformation whose slope is sufficiently close to $\sqrt{2}$ ($\lambda > \sqrt{2}$), the series of the correlation decreases in the order $2\lambda^{-2}$ with the frequency of the modulation $\frac{1}{2}$.

Those results are the generalization of [9] and [15] .

Let us state the conditions imposed on the mapping F. Let λ be a constant ($\lambda > 1$), which we call the slope of the mapping F, and we denote subintervals $(i) = [i/\lambda, (i+1)/\lambda)$ ($0 \leqq i \leqq k-1$) and $(k) = [k/\lambda, 1]$, where k is the maximum integer which does not exceed λ. On each subinterval (i) ($0 \leqq i \leqq k-1$)

$$F'(x) = \lambda \quad \text{(we denote } \operatorname{sgn} i = +1\text{)}, \tag{1.9}$$

or

$$F'(x) = -\lambda \quad \text{(we denote } \operatorname{sgn} i = -1\text{)}, \tag{1.10}$$

and

$$\overline{F((i))} = [0,1], \tag{1.11}$$

and on the subinterval (k)

$$F'(x) = \lambda \quad \text{(that is, } \operatorname{sgn} k = +1\text{)}, \tag{1.12}$$

and

$$\min_{x\in(k)} F(x)=0. \tag{1.13}$$

2. ALPHABETS AND WORDS

Put $A=\{0,1,\ldots,k\}$, and we call each element of the set A an alphabet. We call a finite sequence of alphabets $w=a_1\ldots a_n$ a word and we define

$$(w)=\left\{x\in[0,1] : F^{i-1}(x)\in(a_i),\ 1\leq i\leq n\right\}, \tag{2.1}$$

$$|w|=n, \tag{2.2}$$

and

$$\operatorname{sgn} w=\prod_{i=1}^{n} \operatorname{sgn} a_i. \tag{2.3}$$

We consider a formal symbol ϕ which we call an empty word and we define

$$(\phi)=[0,1], \tag{2.4}$$

$$|\phi|=0, \tag{2.5}$$

and

$$\operatorname{sgn}\phi=+1. \tag{2.6}$$

For any word w, we define

$$w\phi=\phi w=w. \tag{2.7}$$

For $x\in[0,1]$, let (a_i^x) be the subinterval which contains $F^{i-1}(x)$ and we call the infinite sequence of alphabets $a_1^x a_2^x\ldots$ the expansion of x. The expansion of 1 plays an essential role throughout this paper. We call a word w admissible if $(w)\neq\phi$ and an infinite sequence of alphabets $a_1 a_2\ldots$ ($a_i\in A$) admissible

$(a_1 \ldots a_n) \neq \phi$ for any n.

3. FREDHOLM DETERMINANT

Definition 3-1. Let

$$b_n = \begin{cases} a_n^1 & \text{if } \operatorname{sgn} a_n^1 = \operatorname{sgn} a_1^1 \ldots a_{n-1}^1 = +1, \\ -a_n^1 & \text{if } \operatorname{sgn} a_n^1 = +1 \text{ and } \operatorname{sgn} a_1^1 \ldots a_{n-1}^1 = -1, \\ 1+a_n^1 & \text{if } \operatorname{sgn} a_n^1 = -1 \text{ and } \operatorname{sgn} a_1^1 \ldots a_{n-1}^1 = +1, \\ -1-a_n^1 & \text{if } \operatorname{sgn} a_n^1 = \operatorname{sgn} a_1^1 \ldots a_{n-1}^1 = -1. \end{cases} \tag{3.1}$$

Then we define

$$\Phi(z) = 1 - \sum_{n=1}^{\infty} b_n z^n, \tag{3.2}$$

and we call $\Phi(z)$ the Fredholm determinant associated with the mapping F. We call z which satisfy $\Phi(1/z) = 0$ a Fredholm eigenvalue of the mapping F.

Definition 3-2. For a point $x \in [0,1]$, we define:

i) $w(n,x)$ is the number of words w such that $|w| = n$ and $w \cdot a_1^x a_2^x \ldots$ is admissible, and we denote its generating function

$$w(z;x) = \sum_{n=0}^{\infty} w(n,x) z^n. \tag{3.3}$$

$$\text{ii)}\quad \chi(n,x)= \begin{cases} 1 & \text{if } \operatorname{sgn} a_1^1 \ldots a_n^1 = +1 \text{ and } \\ & a_1^1 \ldots a_n^1 \, a_1^x \, a_2^x \ldots \text{ is admissible,} \\ -1 & \text{if } \operatorname{sgn} a_1^1 \ldots a_n^1 = -1 \text{ and } \\ & a_1^1 \ldots a_n^1 \, a_1^x \, a_2^x \ldots \text{ is not admissible,} \\ 0 & \text{otherwise,} \end{cases} \tag{3.4}$$

and we denote its generating function

$$\chi(z;x)= \sum_{n=0}^{\infty} \chi(n,x)\, z^n, \tag{3.5}$$

where

$$\chi(0,x)= 1. \tag{3.6}$$

Theorem 3-1.

$$w(z;x)= \chi(z;x)/\Phi(z). \tag{3.7}$$

Proof. Let $s(z;x)$ be the generating function of

$$s(n,x)= \text{the number of admissible words } a_1 \ldots a_n \text{ such that } a_1 = k. \tag{3.8}$$

Then by a renewal equation for $s(z;x)$ and by

$$w(z;x)= \sum_{n=0}^{\infty} (kz)^n \, s(z;x)= (1-kz)^{-1} \, s(z;x), \tag{3.9}$$

we get the proof.

Theorem 3-2. $\tau_1 = \lambda$. (3.10)

Proof. Since

$$F(x)= \begin{cases} \lambda(x-a_1^x \lambda^{-1}) & \text{if} \quad \text{sgn } a_1^x = +1, \\ -\lambda(x-a_1^x \lambda^{-1}) & \text{if} \quad \text{sgn } a_1^x = -1, \end{cases} \tag{3.11}$$

we get

$$x= (a_1^x + \varepsilon^x)\lambda^{-1} + \text{sgn } a_1^x \, F(x)\lambda^{-1}, \tag{3.12}$$

where

$$\varepsilon^x = \begin{cases} 0 & \text{if} \quad \text{sgn } a_1^x = +1, \\ 1 & \text{if} \quad \text{sgn } a_1^x = -1. \end{cases} \tag{3.13}$$

Repeating this and taking $x= 1$, we get $\Phi(1/\lambda)= 0$. On the other hand, the topological entropy of the mapping F equals $\log|\tau_1|$ and it is not greater than $\log\lambda$. This proves the theorem.

4. THE DECAY OF CORRELATION

Theorem 4-1. Assume that $\eta/\lambda < 1$. Let

$$\rho(x)= -\lambda(\Phi'(1/\lambda))^{-1} \chi(1/\lambda;x). \tag{4.1}$$

i) Then ρ is the density of the invariant probability measure for the mapping F.

ii) The singular part of $w(z;x)$ at $z= 1/\lambda$ equals $\rho(x)/(1-\lambda z)$.

Proof. The proof of i) is almost the same as in [4]. Thus we omit this. By Theorem 3-1, 3-2 and by the assumption that $\eta/\lambda < 1$, $z = 1/\lambda$ is the singularity of first order of $w(z;x)$ and

$$\lim_{z \uparrow 1} (1-\lambda z)\, w(z;x) = \lim_{z \uparrow 1} (1-\lambda z)\, \Phi(z)^{-1}\, \chi(z;x) = \rho(x). \tag{4.2}$$

This proves the theorem.

By μ we denote the probability measure with its density ρ. Now we will prove Theorem 1-1.

Lemma 4-2. Assume that $\eta/\lambda < 1$. Then the dynamical system $([0,1], \mu, F)$ is mixing.

Proof. By (w), we denote the indicator function of the subinterval w. Then

$$\sum_{n \geq |w|} z^n \int (w)(x)\, g(F^n x)\, d\mu = \sum_{v} (z/\lambda)^{|wv|} \int g(x)\, \rho(wvx)\, dx \tag{4.3}$$

$$= -\lambda (\Phi'(1/\lambda))^{-1} \int g(x) \sum_{v} (z/\lambda)^{|wv|}\, \chi(1/\lambda; wvx)\, dx$$

$$= -\lambda (\Phi'(1/\lambda))^{-1} \int g(x)\, G^{w}(z)\, s(z/\lambda; x)\, dx,$$

where

$$G^{w}(z) = \sum_{m=0}^{\infty} \lambda^{-m}\, \mathrm{sgn}\, a_1^1 \ldots a_m^1 \sum_{v}{}^{1} (z/\lambda)^{|wv|}, \tag{4.4}$$

and $\sum_v^1$ is the sum over all words $v = a_1 \ldots a_n$ such that

i) wv is admissible,

ii) if $a_i = k$, then

$$F^{|w|+i-1}(w\, a_1 \ldots a_{i-1}) \not\ni (a_i), \tag{4.5}$$

iii) for any $x \in (wv)$ and $y \in (a^1_{m+1} \cdots a^1_{m+|wv|})$, $y > x$, (4.6)

iv) $F^{|wv|}(wv) \supset (k)$. (4.7)

On the other hand, substituting g=1 in (4.3), we get

$$\int (w)\, d\mu = \lim_{z \uparrow 1} (1-z) \sum_{n \geq |w|} z^n \int (w)(x)\, d\mu$$

$$= -\lambda(\Phi'(1/\lambda))^{-1}\, G^w(1)\, (1-k/\lambda). \tag{4.8}$$

Hence, we get

$$\lim_{z \uparrow 1} (1-z) \sum_{n \geq |w|} z^n \int (w)(x)\, g(F^n x)\, d\mu$$

$$= \int (w)\, d\mu \int g\, d\mu. \tag{4.9}$$

Since the set of words is a generator, this proves the lemma.

Theorem 4-3. Suppose that $\eta/\lambda < 1$. Then for any pair of a function with bounded variation f and an integrable function g, we get for any $\varepsilon > 0$

$$\lim_{n \to \infty} ((\eta + \varepsilon)/\lambda)^{-n} \left\{ \int f(x)\, g(F^n x)\, d\mu - \int f\, d\mu \int g\, d\mu \right\} = 0. \tag{4.10}$$

Proof. Suppose that $z = \eta$ is the zero point of first order of $\Phi(1/z)$. Then

$$\lim_{z \to \lambda/\eta} (1 - \eta z/\lambda) \sum_{n \geq |w|} z^n \int (w)(x)\, g(F^n x)\, d\mu$$

$$= -\lambda(\Phi'(1/\lambda))^{-1}\, G^w(1/\eta) \int g(x)(1-k/\lambda)\, (-\eta(\Phi'(1/\eta))^{-1})$$

$$\chi(1/\eta\, ; x)\, dx. \tag{4.11}$$

Hence there exists a constant K_1 such that

$$\lim_{z\to\lambda/\eta} \left|(1-\eta z/\lambda) \sum_{n\geq|w|} z^n \int(w)(x)\, g(F^n x)\, d\mu\right| \leqq K_1\, \eta^{-|w|} \int|g|\, dx. \qquad (4.12)$$

On the other hand,

$$\sum_{n=0}^{|w|} \left|(\lambda/\eta)^n \int(w)(x)\, g(F^n x)\, d\mu\right|$$

$$\leqq (\lambda/\eta)^{|w|}\, |w| \int(w)(x) \left\{|w|^{-1} \sum_{n=0}^{|w|} |g(F^n x)|\right\} d\mu. \qquad (4.13)$$

Hence by Lemma 4-2, there exists a constant K_2 such that

$$\sum_{n=0}^{|w|} \left|(\lambda/\eta)^n \int(w)(x)\, g(F^n x)\, d\mu\right| \leqq K_2 \eta^{-|w|} \int|g|\, dx. \qquad (4.14)$$

For a function f with bounded variation, there exists a decomposition

$$f(x) = \sum \alpha_w\, (w)(x) \qquad (4.15)$$

such that for any $0 < \gamma < 1$

$$\sum |\alpha_w|\, \gamma^{|w|} < \infty. \qquad (4.16)$$

Combining (4.12), (4.14) and (4.16) with $\gamma = (\eta+\varepsilon)/\lambda$, we get the proof.

5. CRITICAL PHENOMENA

Among the mappings which we considered in this paper, there are two critical states. One is the case when $\lambda \downarrow 1$ for β-transformations and the other is the case when $\lambda \downarrow \sqrt{2}$ for unimodal linear transformations.

Theorem 5-1. i) For a β-transformation F, that is,

$$F(x) = \lambda x \qquad (\text{mod. } 1), \tag{5.1}$$

the greatest Fredholm eigenvalues are of the asymptotic form as $\lambda \downarrow 1$:

$$e^{\alpha + i\beta}, \tag{5.2}$$

where

$$\alpha = \frac{1}{N}\left(\log N - \log\log N + \frac{\log\log N}{\log N} + \text{small order}\right) \tag{5.3}$$

$$\beta = \frac{1}{N}\left(n\pi + (-1)^{n-1}\frac{n\pi}{\log N} + \text{small order}\right) \quad n = 0, \pm 1, \pm 2, \ldots \tag{5.4}$$

The greatest Fredholm eigenvalue(= the slope λ) is the case when $n = 0$ and the second Fredholm eigenvalue are the case $n = \pm 1$. The decay rate of correlation η/λ is asymptotically of the form:

$$\eta/\lambda = \exp\left\{-\frac{1}{2N}\left(\frac{\pi}{\log N}\right)^2 + \text{small order}\right\}. \tag{5.5}$$

Here,

$$N = \min\left\{n \geq 1: \ F^n(1) \in (1/\lambda, 1]\right\}. \tag{5.6}$$

and it is of the asymptotic form:

$$N = -\frac{\log\log\lambda}{\log}\left[1 + \frac{1}{2\log\log\lambda}\left\{\frac{\log(-\log\log\lambda)}{\log\log\lambda}\right\}^2 + \right.$$

$$\left. \text{small order}\right]. \tag{5.7}$$

ii) For a unimodal linear transformation F, that is,

$$F(x) = \begin{cases} -\lambda x + 1 & \text{if} \quad x \in [0, 1/\lambda), \\ \lambda x - 1 & \text{if} \quad x \in [1/\lambda, 1], \end{cases} \tag{5.8}$$

there exists a constant g and the greatest Fredholm eigenvalue is asymptotically of the form:

$$\lambda = 2 \exp 2^{N/2} (g + \text{small order}), \tag{5.9}$$

and the second Fredholm eigenvalue is asymptotically of the form:

$$-\eta = -2 \exp 2^{N/2} (-g + \text{small order}). \tag{5.10}$$

Hence

$$\eta/\lambda = 2 \lambda^{-2} + \text{small order}. \tag{5.11}$$

The proof follows by the inverse function theorem.

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