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3次元 ベクトル場のモデュラス

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1. INTRODUCTION

It is one of the peculiarities of 3-dimensional vectorfields that the knot invariants of periodic orbits can be non trivial. One can use knot invariants, for instance, to say given dynamical systems are topologically different. In this note, we use them to study certain homoclinic phenomena.

Let X^r be a C -vectorfield on R , $r \geq 1$. A hyperbolic stationary point p is of Silnikov type if

(1) DX(P) has eigenvalues λ , $-\mu \pm i\omega$ with positive λ and μ and nonzero ω,

(2) $\lambda > \mu$,

(3) the stable and unstable manifolds of P intersects at a curve Γ .

Without loss of generality, we can suppose p is the origine.

The homoclinic orbit Γ is itself an interesting object of study. A homoclinic orbit with trivial knot type bifurcates to one with very complicated knot type. Suppose that Γ (more precisely $\bar{\Gamma}$) is unknotted. If one perturb the system so that the upper part of the unstable manifold comes back slitely above the local stable manifold. Then it twists many times arround the local unstable manifold, goes away from the origine and comes back again near the local stable manifold. If one adjusts the perturbation, then the unstable manifold hitts the stable manifold and makes new homoclinic orbit Γ' whose knot type is of 2-n torus knot where n is the number of twists. If one perturb the original homoclinic orbit Γ so that the unstable manifold hitts the local stable manifold at the third turn, then one has a homoclinic orbit of 3-n torus knot type. Moreover if one makes another homoclinic orbit from the homoclinic orbit of 3-n torus knot type, then one has a cable knot about 3-n torus knot, from which one can generates more complicated knots. In any case, if Γ' is generated from Γ by a small perturbation, then the knot type of Γ' is companion about the knot Γ . So we can introduce hierarchy relationship among the homoclinic orbits in a Silnikov bifurcation(if a 1-parameter family of vectorfields has a stasionary point of Silnikov type for a parameter value, then it also has Silnikov stationary points for infinitely many parameter values. Parameter values with Silnikov stationary point are non isolated)

Another possible application is to define a modulus of attractors in a bifurcation from Silnikov homoclinic orbit. If the eigenvalues λ and $-\mu\pm i\omega$ satisfy the inequality $\mu < \lambda < 2\mu$, then arbitralily small perturbation can generates attracting periodic orbit of the same knot type as the original homoclinic orbit. Since there are infinitely many homoclinic orbits of different knot types in a Silnikov bifurcation, there appear also infinitely many attracting periodic orbits with different knot types.

In this note, we show how knot invariant are used to define a modulus of homoclinic orbit of Silnikov stationary point.

THEOREM. Let C^1-vectorfield X_i (i=1,2) have a C^1 -linearizable stationary points p_i of Silnikov type with eigenvalues λ_i, $-\mu_i \pm i\omega_i$. Suppose that $\lambda_1/\mu_1 \neq \lambda_2/\mu_2$. Then there is no homeomorphism $h: R^3 \longrightarrow R^3$ such that

$h(p_1)=p_2$, and

h, maps each orbit of X_1 onto an orbit of X_2.

To prove the theorem, we consider 'period 2(as the Poincare map) orbits'. As a consequence of Silnikov theorem, there are infinitely many such orbits arbitrarily close to the homoclinic orbit. We use knot invariants of periodic orbits to count the number of twists around the homoclinic orbit. The ratio of the numbers of twists in the first and the second turns determine the ratio of eiginvalues λ and μ. Therefore μ/λ is determined by topological data of phase portrait, that is, μ/λ is a modulus.

For diffeomorphisms on n-manifolds it is known that if the stable manifold of a stationary point and unstable manifolds of another stationary point intersects non transversely, then there exists a modulus . However this result can not be extended to vectorfields, and a modulus of vectorfields exists only in more restricted configurations(see [4] for unified exposition on modulus of vectorfields from the foliation technics). Main reason of this difference is that topological equivalence does not preserve 'time lenth', which corresponds to 'number of points on a orbit' in the diffeomorphism case. In our 3-dimensional case, we can use knot invariants to measure the time lenth.

In the following sections we give the sketch of the proof.

2. TOPOLOGICAL INVARIANT

In this section we define a topological invariant

Take a section at a point on the homoclinic orbit, and consider periodic orbits near Γ whose period are 2 under the Poincare map on the section. Let $\mathcal{F}$ detone the set of such periodic orbits. Let $\Delta(t)$ denote the Alexander polynomial of the knot γ , $\gamma \in \Gamma$. We set

$$\eta_n = \inf \frac{\deg(\Delta_\gamma)}{\mathrm{link}(\gamma, \Gamma)}$$

where infimum is over all $\gamma \in \mathcal{F}$ with link(γ, Γ) > n (link(γ , Γ) denotes the linking number of γ and Γ). Let

$$\eta = \lim_{n \to \infty} \eta_n$$

It may seem that depends on the choice of the section. But it is independent from the choice(notice that the linking number is large iff the periodic orbit is close to the homoclinic orbit). Therefore is a topological invariant ,and if is computed from eigenvalues, then we have a modulus.

3. SILNIKOV'S THEOREM

Without loss of generality we suppose that X is linear in the unit cylinder

$$S= \{(r,\theta,z) \mid 0 < r < 1, \theta \in R, |z| \leq 1\} .$$

The orbit which starts at the point $(1,\theta_0, z_0), z_0 > 0$, leaves the cylinder at the point $(e^{-\mu T}, \theta_0 + \omega T, 1)$, where T is given by z e =1. So we can define a map

$$L: \{(1,\theta,z) \mid \theta \in R, 0 < z \leq 1\}$$
$$\longrightarrow \{(r,\theta,1) \mid 0 < r \leq 1, \theta \in R\}$$

by

$$L: (1, \theta, z) \rightarrow (z^{\mu/\lambda}, \theta - \mu/\lambda \log z, 1).$$

Notice that the orbit which starts from a point $(1, \theta, z)$ twists around the z-axis about $-\frac{1}{2\pi}\frac{\omega}{\lambda}\log z$ times.

If d is sufficiently small positive number, then we can define a diffeomorphism

$$f: \Sigma^2 \rightarrow \{(1,\theta,z) \mid \theta \in R, |z| \leq 1\}$$

defined by the flow.

Now we have a local deffeomorphism $f \circ L$. By the Silnikov's theorem there exist infinitely many horseshoes. As a consequnce, there exist infinitely many 'period-2' periodic orbits near the homoclinic orbits. If γ is a 'period-2' periodic orbit, then it comes in the cylinder at two poins with the z-coordinate $m(\gamma)$ and $M(\gamma)$, $m(\gamma) \leq M(\gamma)$. It is not hard, but messey, to prove;

PROPOSITION. $\liminf_{\varepsilon \to 0} \dfrac{\log M(\gamma)}{\log m(\gamma)} \geq \dfrac{\mu}{\lambda}$

where minimum is taken over all γ's with $M(\gamma) < \varepsilon$.

4. CALCURATION

In this section we show that the left side of the equality in the proposition is determined by knot and link invariants.

Let $N(z) = -\frac{1}{2\pi}\frac{\omega}{\lambda}\log z$. Roughly speaking, N(z) measures how many times the orbit which start at $(1, \theta, z)$ twists around the z-axis. The linking number is determined not only by the number of linking in the cylinder but also by the behavior outside the cylinder. But the behavior of the flow outside of the cylinder is very moderate(at least near the homoclinic orbit), therfore we have a conxtant C such that

$$|N(M(\gamma))+N(m(\gamma))-\mathrm{link}(\gamma,\Gamma)| \leq C$$

for all γ in $\mathcal{P}$.

It is a little bit complicated to investigate the knot invariant of γ .

First suppose that Γ is unknotted. Notice that the twists in the turn with $m(\gamma)$ can be unknotted and the knot type of γ is determined only by the turn with $M(\gamma)$. Thus there exists a constant C, which is determined by the twists outside the cylinder, such that γ has knot type of torus knot $T_{2,m}$ with

$$|N(M(\gamma))-n| \leq C.$$

In the general case in which is knotted, we have a constant C such that is the 2-n cable about , with

$$|N(M(\gamma))-n| \leq C.$$

Thus we have

$$\Delta_\gamma(t)=\Delta_\rho(t^2)\ \Delta_{T_{2,m}}(t),$$

see [1,p.144]. The degree of the polynomial $\Delta_\rho(t^2)$ is fixed, and the degree of $\Delta_{T_{2,m}}(t)$ is n-1.

Therefore

PROPOSITION. For any $\varepsilon>0$, there exists a number N such that

$$\left| \frac{deg(\Delta_\gamma)}{N(M(\gamma))} - 1 \right| < \varepsilon$$

for all $\gamma \in \mathcal{H}$ with $N(M(\gamma)) > N$.

Now the theorem is a consequence of these inequalities.

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