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On the Vlasov-Poisson limit of the Vlasov-Maxwell equation

(Dedicated to Professor Yamaguti on his 60th birthday)

京大数養 浅野 潔 (Kiyoshi Asano)
阪市大工 鵜飼 正二 (Seiji Ukai)

1. Main results

The change of the density distribution of charged gas particles is described by two types of equations: The Vlasov-Maxwell equation and the Vlasov-Poisson equation. The latter describes the motion of plasma when the magnetic field generated by the plasma is small. The purpose of this paper is to study the relation between the two equations. Roughly speaking, the solution of the Vlasov-Maxwell equation converges to the solution of the Vlasov-Poisson equation when the light velocity tends to infinity.

Let $f_i = f_i(t,x,v)$ be the density distribution of charged gas particles of the type i $(i=1,2,..,N)$ at time $t \geq 0$ and position $x \in \mathbb{R}^3$ with velocity $v \in \mathbb{R}^3$. Let $E = E(t,x)$ and $B = B(t,x)$ denote the electric and magnetic fields generated by the charged gas particles. The Vlasov-Maxwell equation is,

$$\text{(1.1)} \quad \begin{cases} \frac{\partial}{\partial t} f_i + v\cdot\nabla_x f_i + \alpha_i (E+\frac{v}{c}\times B)\cdot\nabla_v f_i = 0, \\ \frac{\partial}{\partial t} E - c\nabla_x \times B = -4\pi Jf, \\ \frac{\partial}{\partial t} B + c\nabla_x \times E = 0, \\ f_i|_{t=0} = f_{i,0}, \qquad (E,B)|_{t=0} = (E_0,B_0), \end{cases}$$

supplemented by

$$(1.2)\qquad \nabla_x \cdot E = 4\pi K f, \qquad \nabla_x \cdot B = 0.$$

Here $\cdot$ and $\times$ are the scalar and vector products in $\mathbb{R}^3$, ∇_x is the gradient in x and ∇_v in v, while c is the light velocity and $\alpha_i = q_i/m_i$, q_i and m_i being the electric charge and mass of a single particle of i-species. Further, Jf and Kf are the current and charge densities generated by $f = (f_1, f_2, \ldots, f_N)$;

$$(1.3)\qquad \begin{aligned} Jf &= \sum_{i=1}^{N} q_i \int_{\mathbb{R}^3} v f_i(t,x,v)\,dv, \\ Kf &= \sum_{i=1}^{N} q_i \int_{\mathbb{R}^3} f_i(t,x,v)\,dv. \end{aligned}$$

Notice that (1.2) can be deduced from (1.1) for $t > 0$ if it is satisfied at $t = 0$, i.e., if the initials $f_0 = (f_{1,0}, f_{2,0}, \ldots, f_{N,0})$ and (E_0, B_0) satisfy

$$(1.4)\qquad \nabla_x \cdot E_0 = 4\pi K f_0, \quad \nabla_x \cdot B_0 = 0.$$

This is a physically reasonable compatibility condition and will be assumed throughout the paper.

On the other hand, the Vlasov-Poisson equation is,

$$(1.5)\qquad \begin{aligned} &\frac{\partial}{\partial t} f_i + v \cdot \nabla_x f_i + \alpha_i E \cdot \nabla_v f_i = 0, \\ &\nabla_x \cdot E = 4\pi K f, \qquad \nabla_x \times E = 0, \\ &f_i|_{t=0} = f_{i,0}. \end{aligned}$$

This equation is formally obtained from (1.1) and (1.2) by setting $B = 0$ or $c = \infty$.

With the notations and function spaces defined by (1.19)-(1.23) below, our main results are stated in the following three theorems.

Theorem 1.1. Let $\ell \geq 3$, $\sigma \geq 0$, $\rho \geq 0$, $\beta \in \mathbb{R}$, and suppose

$$(1.6)\qquad f_{i,0} \in H^{\ell}_{\sigma,\rho,\beta} \ (1 \leq i \leq N), \qquad (E_0, B_0) \in H^{\ell}.$$

suppose further that (1.4) is fulfilled. Then, there exist positive constants C, T, γ and the following holds.

(i)(Uniform existence) For each $c \in [1,\infty)$, (1.1) has a unique classical solution $f = (f_1, f_2, \ldots, f)$ and $u = (E, B)$ on the time interval $[0,T]$ such that

(1.7) $$f \in C^0_\gamma([0,T]; H^\ell_{\sigma,\rho,\beta}) \cap C^1_\gamma([0,T]; H^{\ell-1}_{\sigma,\rho,\beta-1}),$$

(1.8) $$u \in C^0([0,T]; H^\ell) \cap C^1([0,T]; H^{\ell-1}),$$

satisfying the estimates

(1.9) $$|f|_{\ell,\sigma,\rho,\beta,\gamma,T} \leq C|f_0|_{\ell,\sigma,\rho,\beta},$$

(1.10) $$|u|_{\ell,T} \leq |u_0|_\ell + C|f_0|_{\ell,0,\rho,\beta}.$$

(ii)(Continuity in c) As functions of c, f and u satisfy

$[A.1]^\ell_{\gamma,\beta}$ $$f \in \bigcap_{j=0}^{\ell} M^j([1,\infty); C^0_\gamma([0,T]; H^{\ell-j}_{\sigma,\rho,\beta-j})),$$

$[A.2]^\ell$ $$u \in \bigcap_{j=0} M^j([1,\infty); C^0([0,T]; H^{\ell-j}).$$

Thus, the solutions exist on the interval $[0,T]$ independent of $c \in [1,\infty)$ and are continuous in c. Further, they have limits at $c = \infty$ which we will call the Vlasov-Poisson limits:

Theorem 1.2 (Existence of limit). Let f,u be as above.

(i) They can be extended to $[1,\infty]$ as functions of c so that

$[A.3]^\ell_{\gamma,\beta}$ $$f \in B^0([1,\infty]; C^0_\gamma([0,T]; H^{\ell-\varepsilon}_{\sigma,\rho,\beta-\varepsilon})), \quad \forall \varepsilon > 0,$$

$[A.4]^\ell$ $$u \in B^0([1,\infty] \times [0,T] \setminus \{(\infty,0)\}; B^{\ell-2}(\mathbb{R}^3)),$$

holds good, and in particular, they converge as $c \to \infty$ in the topology indicated here. We write the limits as

$$f^\infty = (f_1^\infty, f_2^\infty, \ldots, f_N^\infty), \quad u^\infty = (E^\infty, B^\infty).$$

(ii) $B^\infty = 0$ while (f^∞, E^∞) is a unique solution to the Vlasov-

Poisson equation (1.5) on $[0,T]$ satisfying

(1.11) $\quad (f^\infty, E^\infty) \in C^0([0,T]; H^{\ell}_{\sigma,\rho,\beta} \times \tilde{H}^{\ell+1}),$

(1.12) $\quad |f^\infty|_{\ell,\sigma,\rho,\beta,T} \leq C|f_0|_{\ell,\sigma,\rho,\beta},$

$$\|E^\infty\|_{0,T} + |\nabla_x E^\infty|_{\ell-1,T} \leq C|f_0|_{\ell,\sigma,\rho,\beta}.$$

Observe from [A.3-4] that as $c \to \infty$, f converges to f^∞ uniformly in t on $[0,T]$ while u converges to u^∞ uniformly on $[\delta,T]$ for any $\delta > 0$ but not uniformly on $(0,T]$. Physically, this implies the development of initial layer.

Also, comparing (1.7) and (1.11), we see that the limit f^∞ which is a solution to (1.5) belongs to a better function class than f, the solution to (1.1), and similarly for E^∞ and E.

Finally, we shall discuss an asymptotic expansion which is somewhat complicated due to the presence of initial layer. To simplify the notation, we introduce the operators L, Λ, A defined by

$$L(f,E;f',E') = (\partial_t f_i + v\cdot\nabla_x f_i + \alpha_i E'\cdot\nabla_v f_i + \alpha_i E\cdot\nabla_v f_i')_{i=1}^N,$$

(1.13) $\quad \Lambda f = (-4\pi Jf, 0),$

$$Au = (\nabla_x \times B, -\nabla_x \times E),$$

The first of these is an N-dimensional vector and the remaining two are 6-dimensional vectors.

We seek the expansion of the form

(1.14)
$$f = f^\infty + \sum_{j=0}^{k} c^{-j} f^j,$$
$$u = u^\infty + \sum_{j=0}^{k} c^{-j} u^j,$$

where f^∞, u^∞ are as in Theorem 1.2. The coefficients $f^j = (f_1^j, f_2^j, \ldots, f_N^j)$ $u^j = (E^j, B^j)$ still depend on c. We wish to determine them as solutions of the following equations whose derivation will be described in §5. Note that the 0-th term of the expansion (1.14) is $(f^\infty+f^0, u^\infty+u^0)$. The equation for (f^0,u^0) is the nonlinear equation,

$$(1.15)^0 \quad \begin{aligned} &L(f^0,E^0;f^\infty,E^\infty+E^0)=0,\\ &\partial_t u^0-cAu^0=P_1\Lambda f^\infty+\Lambda f^0,\\ &f^0|_{t=0}=0, \qquad u^0|_{t=0}=P_1u_0, \end{aligned}$$

where P_1 is the projection defined in §3(see the remark below (3.3)), and u_0 is the same initial as in (1.1). The equation for (f^j,u^j), $1\leq j\leq k-1$, is the linear inhomogeneous equation,

$$(1.15)^j \quad \begin{aligned} &L(f^j,E^j;f^\infty+f^0,E^\infty+E^0)=F^j,\\ &\partial_t u^j-cAu^j=\Lambda f^j,\\ &(f^j,u^j)|_{t=0}=0, \end{aligned}$$

where

$$F^j=-(\alpha_i\sum_{r=1}^{j-1}(E^r\cdot\nabla_v f_i^{j-r}+v\times B^r\cdot\nabla_v f_i^{j-r-1})+\alpha_i v\times B^0\cdot\nabla_v f_i^{j-1})_{i=1}^N,$$

and the equation for (f^k,u^k) is the nonlinear one just obtained by substituting (1.14) into (1.1) and taking account of (1.5) and $(1.15)^j$, $0\leq j\leq k-1$:

$$(1.15)^k \quad \begin{aligned} &L(f^k,E^k+c^{-1}v\times B^k;f^\infty+\sum_{j=0}^{k-1}c^{-j}f^j,E+c^{-1}v\times B)=F^k,\\ &\partial_t u^k-cAu^k=\Lambda f^k,\\ &(f^k,u^k)|_{t=0}=0, \end{aligned}$$

where F^k is a given function of (f^j,u^j), $0\leq j\leq k-1$ and their derivatives, and v,c. The expansion (1.14) is verified by the

Theorem 1.3 (Asymptotic expansion). Under the situation of Theorems 1.1 and 1.2, let $0\leq k\leq \ell-2$.

(i) The equations (1.15) can be all solved successively, uniquely on the same time interval $[0,T]$ as in Theorem 1.1 and uniformly for each $c \in [0,\infty)$, with the solutions,

$$
\begin{aligned}
&f^j \in [A.1]^{\ell-j}_{0,\beta-j}, \quad 0 \le j \le k-1,\\
(1.16) \qquad &f^k \in [A.1]^{\ell-k}_{\gamma,\beta-k},\\
&u^j \in [A.2]^{\ell-j}, \quad 0 \le j \le k,
\end{aligned}
$$

where $[A.1]^{\ell}_{\gamma,\beta}$ etc., are the function classes in Theorem 1.1. With these (f^j,u^j), (1.14) holds. Note from (1.16) that we can take $\gamma=0$ for $0 \le j \le k-1$ and that (f^j,u^j) are all uniformly bounded in c.

(ii) As $c \to \infty$, we have,

$$(1.17) \qquad (f^j,u^j) \to 0, \quad 0 \le j \le k.$$

More precisely, if (f^j,u^j) are extended to $[1,\infty]$ as functions of c with 0 at $c=\infty$, then,

$$
\begin{aligned}
&f^j \in [A.3]^{\ell-j}_{0,\beta-j}, \quad 0 \le j \le k-1, \quad f^k \in [A.3]^{\ell-k}_{\gamma,\beta-k},\\
&u^0 \in [A.4]^{\ell},\\
&u^j \in B^0([1,\infty]\times[0,T]; B^{\ell-j-2}(\mathbb{R}^3)), \quad 1 \le j \le k.
\end{aligned}
$$

(iii) Suppose, in addition to the assumption of Theorem 1.1, that $\sigma \ge 2$ and $u_0 \in H^{\ell}_2$. Then, we can strengthen the convergence (1.17) as

$$
\begin{aligned}
&|f^j|_{\ell-j-2,\sigma,\rho,\beta-j-2,T} \le dc^{-1}\log(1+cT), \quad 0 \le j \le k-1,\\
(1.18) \qquad &\|u^0\|_{\ell-2,T} \le d((1+cT)^{-1}+c^{-1}\log(1+cT)),\\
&\|u^j\|_{\ell-j-2,T} \le dc^{-1}\log(1+cT), \quad 1 \le j \le k-1,
\end{aligned}
$$

where $d = C(|f_0|_{\ell,\sigma,\rho,\beta} + \|u_0\|_{\ell})$, C being a positive constant depending only on ℓ,σ,ρ,β. A similar estimate is also available for $j=k$.

Remark. After completion of this work, we learned that Degond [5] had proved a similar result under slightly different conditions. However, his asymptotic expansion is derived rather formally.

The rest of the paper is devoted to the proof of Theorems 1.1-3. Since most of materials and tools for the proof are provided in [1, 2,7], we will give an outline only, supplementing some technical results missing there and needed here. In the next two sections we discuss linear problems associated with (1.1) while in §4 we solve (1.5), and finally in §5 we will prove Theorems 1.1-1.3.

Now we shall state the definition of the function spaces used in Theorems 1.1-3. $H^\ell = H^\ell(\mathbb{R}^3)$ is the Sobolev space of order ℓ, with the norm $|\cdot|_\ell$. Introduce the weight function

(1.19) $$\phi_{\sigma,\rho,\beta} = (1+|x|+|v|)^{\sigma} e^{\rho|v|}(1+|v|)^{\beta-\sigma},$$

and define the space $H^\ell_{\sigma,\rho,\beta}$ by

(1.20) $$H^\ell_{\sigma,\rho,\beta} = \{f=f(x,v)\,|\,\phi_{\sigma,\rho,\beta}\partial_x^\alpha\partial_v^{\alpha'} f \in L^2(\mathbb{R}^6),\ |\alpha|+|\alpha'|\le\ell\},$$

$$|f|_{\ell,\sigma,\rho,\beta} = \sum_{|\alpha|+|\alpha'|\le\ell} |\phi_{\sigma,\rho,\beta}\partial_x^\alpha\partial_v^{\alpha'} f|_{L^2(\mathbb{R}^6)}.$$

Let Ω be a (possibly closed) domain in $\mathbb{R}^m$ and Y be a Banach space with the norm $|\cdot|_Y$. $C^\ell(\Omega;Y)$ (resp. $M^\ell(\Omega;Y)$) will denote the space of Y-valued functions $h(y)$ on Ω which are strongly continuous (resp. strongly measurable) on Ω, together with derivatives $\partial_y^\alpha h$, $|\alpha|\le\ell$. We set $B^\ell(\Omega;Y) = C^\ell(\Omega;Y)\cap M^\ell(\Omega;Y)$. When $Y=\mathbb{C}^m$, we drop Y; e.g., $B^\ell(\mathbb{R}^3) = B^\ell(\mathbb{R}^3;\mathbb{C}^m)$, etc. Note that if Ω is compact, then $B^\ell(\Omega;Y) = C^\ell(\Omega;Y)$. $B^\ell(\Omega;Y)$ and $M^\ell(\Omega;Y)$ are Banach spaces with obvious norms. We write the norm of $B^\ell(\mathbb{R}^3)$ as $\|\cdot\|_\ell$, and when $\ell=0$ and $\Omega=[0,T]$, we set,

$$\|h\|_{Y,T} = \sup_{0\le t\le T} |h(t)|_Y.$$

Further, when Y is H^ℓ, $H^\ell_{\sigma,\rho,\beta}$ or $B^\ell(\mathbb{R}^3)$, we set,

$$|u|_{\ell,T} = \|u\|_{H^\ell,T},$$

(1.21) $$\|u\|_{\ell,T} = \|u\|_{B^\ell(\mathbb{R}^3),T},$$

$$|f|_{\ell,\sigma,\rho,\beta,T} = \|f\|_{H^\ell_{\sigma,\rho,\beta},T}.$$

These norms will be also used when $\Omega=[1,\infty]\times[0,T]$ in which $y=(c,t)$ moves.

The space $C^j_\gamma([0,T];H^\ell_{\sigma,\rho,\beta})$ which plays a crucial role in this paper is defined by the set of functions $f=f(t,x,v)$ such that,

$$\phi_{\sigma,\rho-\gamma t,\beta}\partial_t^r\partial_x^\alpha\partial_v^{\alpha'} f\in C^0([0,T];L^2(\mathbb{R}^6)),\ 0\le r\le j,\ |\alpha|+|\alpha'|\le\ell,$$

(1.22)

$$|f|_{\ell,\sigma,\rho,\beta,\gamma,T} = \sup_{0\le t\le T}|f(t)|_{\ell,\sigma,\rho-\gamma t,\beta},\ (j=0).$$

We have seen in [1] that to treat the factor $v\times B$ of (1.1), we must take $\gamma>0$. Finally, we also need the spaces,

$$\tilde{H}^\ell = \{u\in B^0(\mathbb{R}^3)\,|\,\nabla_x u\in H^{\ell-1}\},$$

(1.23)

$$H^\ell_\sigma = \{u|\ (1+|x|)^\sigma\partial_x^\alpha u(x)\in L^2(\mathbb{R}^3)\ ,|\alpha|\le\ell\},$$

with due norms.

2. The linear Vlasov equation

In this section we solve,

$$(2.1)\qquad \begin{aligned} &\partial_t f + v\cdot\nabla_x f + \alpha(E+c^{-1}v\times B)\cdot\nabla_v f = 0,\\ &f|_{t=s} = f_0, \end{aligned}$$

where $u=(E,B)$, f_0 are given functions. Suppose, first, that u is in the class $[A.2]^\ell$, $\ell\geq 3$, and $f_0\in C_0^{\ell+1}(\mathbb{R}^6)$ (compact support). Then, (2.1) can be solved easily by the aid of the characteristic equation associated with it. We write the solution as

$$(2.2)\qquad f = U(t,s;u,c)f_0,$$

where $U(t,s;u,c)$ is the evolution operator to (2.1) (see [1]).

Since f_0 is now assumed to be of compact support, f is also of compact support in x,v (cf. Lemma 2.6 of [1]). Noting this, we see readily that

$$(2.3)\qquad f\in M^j([1,\infty);C^0([0,T]^2;H^{\ell-j}_{\sigma,\rho,\beta})\cap C^1([0,T]^2;H^{\ell-j-1}_{\sigma,\rho,\beta-j-1})),$$

for $0\leq j\leq \ell$.

Proceeding as in the proof of Lemma 2.5 of [1], we further get,

$$(2.4)\qquad |f(t,s)|_{k,\sigma,\rho-\gamma|t-s|,\beta}\leq e^{b|t-s|}|f_0|_{k,\sigma,\rho,\beta},\quad 0\leq k\leq \ell,$$

where $\sigma,\rho\geq 0$ and

$$(2.5)\qquad \begin{aligned} &b = b_0\{(\rho+|\beta|+1)|E|_{\ell,T}+c^{-1}|B|_{\ell-1,T}+1\},\\ &\gamma\geq c_0|B|_{\ell,T}+\sigma,\quad \rho-\gamma T\geq\rho/2, \end{aligned}$$

b_0, c_0 being positive constants depending only on ℓ. Since $C_0^{\ell+1}(\mathbb{R}^6)$ is dense in $H^k_{\sigma,\rho,\beta}$, we have thus proved the

Lemma 2.1. Let $\ell\geq 3$, $\sigma\geq 0$, $\rho>0$ and $\beta\in\mathbb{R}$. Suppose $[A.2]^\ell$ for u and $f_0\in H^k_{\sigma,\rho,\beta}$, $0\leq k\leq\ell$. Let γ, T be as in (2.5). Then f of (2.2) is a unique solution to (2.1) in $[A.1]^k_{\gamma,\beta}$ (modified by $[0,T]^2$, $\phi_{\sigma,\rho-\gamma|t-s|,\beta}$ in place of $[0,T]$, $\phi_{\sigma,\rho-\gamma t,\beta}$, see (1.19,22)). Also, (2.4) holds which is a uniform estimate in c.

Recall that $u = u(c) = u(c,t)$ is a function also of c, and write (2.2) as

$$f(c,t) = U(t,s;u(c),c)f_0, \quad \text{(with } s \text{ fixed)}.$$

Let $1 \leq c \leq c'$. It follows from (2.1) that

$$|f(c,t)-f(c',t)| = \int_s^t U(t,r;u(c),c)g(c,c',r)dr,$$

with

$$g(c,c',r) = -\alpha\{E(c,r)-E(c',r)+v\times(B(c,r)/c-B(c',r)/c')\}\cdot\nabla_v f(c',r).$$

Applying (2.4), we get,

$$|f(c,t)-f(c',t)|_{0,\sigma,\rho-\gamma|t-s|,\beta-1}$$
$$\leq C\int_s^t (\|E(c,r)-E(c',r)\|_0+\|B(c,r)\|_0/c+\|B(c',r)\|_0/c')dr.$$

Consequently, if $u\in[A.4]^2$, $f(c,t)$ is continuous in c and converges as $c\to\infty$ uniformly on $[0,T]$, both strongly in $H^0_{\sigma,\rho-\gamma|t-s|,\beta-1}$. By virtue of (2.4) and the interpolation theorem, this is also the case in $H^{k-\varepsilon}_{\sigma,\rho-\gamma|t-s|,\beta-\varepsilon}$, for any $\varepsilon>0$. Thus $f(c,t)$ is strongly continuous on $[1,\infty]\times[0,T]$ in this space and the limit $f(\infty,t)$ exists.

This proves the first part of the following lemma.

Lemma 2.2. Under the situation of Lemma 2.1, suppose , in addition, that $u\in[A.4]^2$. We have,

(i) With the modification indicated in Lemma 2.1, it holds that

$$f\in[A.3]^{\ell}_{\gamma,\beta} \ .$$

(ii) $f(\infty,t)$ is a unique solution to

$$\partial_t f + v\cdot\nabla_x f + E(\infty)\cdot\nabla_v f = 0, \tag{2.6}$$
$$f|_{t=s} = f_0.$$

(iii) Moreover, $f(\infty,t)\in C^0([0,T]^2;H^k_{\sigma,\rho,\beta})$ satisfying

$$|f(\infty,t)|_{k,\sigma,\rho,\beta} \leq e^{b|t-s|}|f_0|_{k,\sigma,\rho,\beta}\ , \tag{2.7}$$

with $b = b_0\{(\rho+|\beta|+1)|E(\infty)|_{\ell,T}+1\}+\sigma$.

For the proof of (ii) and (iii), we notice that (2.6) is a special case of (2.1) with $u = (E(\infty),0)$. Then we can prove an analogue to Lemma 2.1 for (2.6), in which we can take $\gamma=0$ because the term $v\times B$ is absent. On the other hand, it is clear that $f(\infty,t)$ solves (2.6) as seen by passing to the limit in (2.1).

This indicates that $U(t,s;u,c)$ of (2.2) has a limit which gives the evolution operator to (2.6). We write this operator as $V(t,s;E(\infty))$. Thus,

(2.8) $$f(\infty,t) = V(t,s;E(\infty))f_0 .$$

Then, (2.7) gives an estimate of its operator norm in $H^k_{\sigma,\rho,\beta}$.

3. The Maxwell equation

Assuming that f is a given function, we seek a solution $u = (E,B)$ to the inhomogeneous Maxwell equation,

$$\text{(3.1)} \qquad \begin{aligned} &\partial_t u - cAu = \Lambda f, \\ &u|_{t=0} = u_0 , \end{aligned}$$

where A,Λ are as in (1.13). First, it is well known that A generates a unitary group e^{tA} on H^{ℓ}, and that

$$\text{(3.2)} \qquad \mathfrak{F} e^{tA} u_0 = e^{itA(\xi)} \hat{u}_0(\xi), \qquad u_0 = {}^t(E_0, B_0) \text{ (column vector)},$$

where $\hat{u}_0 = \mathfrak{F} u_0$ is the Fourier transform of u_0, ξ being the dual variable to x, and $A(\xi)$ is a 6×6 matrix given by

$$A(\xi) = \begin{pmatrix} 0 & -\delta(\xi) \\ \delta(\xi) & 0 \end{pmatrix},$$

$$\delta(\xi) = \begin{pmatrix} 0 & -\xi_3 & \xi_2 \\ \xi_3 & 0 & -\xi_1 \\ -\xi_2 & \xi_1 & 0 \end{pmatrix}.$$

$A(\xi)$ is the symbol of the differential operator $-iA$.

The matrix $A(\xi)$ has eigenvalues $0, |\xi|, -|\xi|$, each of multiplicity 2. Denote the corresponding eigen projections by $P_0(\xi), P_+(\xi), P_-(\xi)$, and set $P_1(\xi) = P_+(\xi) + P_-(\xi)$. We have,

$$\text{(3.3)} \qquad \begin{aligned} &P_0(\xi)u_0 = {}^t((\hat{E}_0 \cdot \xi)\xi, (\hat{B}_0, \xi)\xi), \qquad \xi = \xi/|\xi| . \\ &P_1(\xi) = I - P_0(\xi). \end{aligned}$$

All these projections are also symbols of the singular integral operators of Calderón-Zygmunt type (see [6]). We denote these singular integral operators by P_0, P_+, P_-, P_1. They are orthogonal projections in H^{ℓ} and it holds that

$$\text{(3.4)} \qquad e^{tA} = P_0 + e^{tA}P_+ + e^{tA}P_- = P_0 + e^{tA}P_1 .$$

Moreover, we have the expressions,

(3.5) $$e^{tA}P_{\pm}u_0 = (2\pi)^{-3/2}\int_{\mathbb{R}^3} e^{i(x\cdot\xi+t|\xi|)}P_{\pm}(\tilde{\xi})\hat{u}_0(\xi)d\xi.$$

Applying Lemma 5.1 of [2] to this, we have,

Lemma 3.1. (i) If $u_0\in H^{\ell}_2$ with $\ell\geq 2$, then,

(3.6) $$\|e^{tA}P_1u_0\|_{\ell-2} \leq C(1+|t|)^{-1}|u_0|_{\ell,2}, \quad t\in\mathbb{R},$$

where $\|\cdot\|_{\ell}, |\cdot|_{\ell,\sigma}$ denote the norms of $B^{\ell}(\mathbb{R}^3), H^{\ell}_{\sigma}$ respectively.

(ii) Let $u_0\in H^{\ell}$ with $\ell\geq 2$. Then,

(3.7) $$e^{tA}u_0\in B^0((-\infty,\infty);H^{\ell})\cap B^0([-\infty,\infty];B^{\ell-2}(\mathbb{R}^3)),$$
$$\|e^{tA}P_1u_0\|_{\ell-2}\to 0 \ (t\to 0).$$

Let us return to (3.1). Its unique solution is given by

(3.8) $$u(c,t) = e^{ctA}u_0 + \int_0^t e^{c(t-s)A}g(c,s)ds,$$

where $g = g(c,t) = \Lambda f(c,t)$. Let $\ell\geq 3$, $\sigma\geq 0$, $\rho,\gamma,T>0$, $\beta\in\mathbb{R}$ with $\rho-\gamma T\geq\rho/2$. Assuming $[A.1]^{\ell}_{\gamma,\beta}$ and $[A.3]^{\ell}_{\gamma,\beta}$ for f, we can easily see that

$[A.5]^{\ell}_{\sigma}$ $$g\in\bigcap_{j=0}^{\ell} M^j([0,\infty);C^0([0,T];H^{\ell-j}_{\sigma}))\cap B^0([1,\infty]\times[0,T];H^{\ell-\varepsilon}_{\sigma}), \quad \forall\varepsilon>0.$$

Furthermore, we have,

$$|g|_{\ell,\sigma}\leq a|f|_{\ell,\sigma,\rho,\beta,\gamma,T},$$

and

$$g(\infty,t)\in M^0([0,T];H^{\ell}_{\sigma})\cap B^0([0,T];H^{\ell-\varepsilon}_{\sigma}), \ \forall\varepsilon>0.$$

Using this and noting that e^{tA} and P_j, $j=0,1$ are bounded operators on H^{ℓ} with the norm 1, we apply Lemma 3.1 to (3.8) to conclude the

Lemma 3.2. Let ℓ,σ etc., be as above and assume $f\in[A.1,3]^{\ell}_{\gamma,\beta}$. Let u(c,t) be given by (3.8). We have,

(i) If $u_0\in H^{\ell}$, then $u(c,t)\in[A.2]^{\ell}\cap[A.4]^{\ell}$, and

(3.9) $|u|_{\ell,T} \le |u_0|_\ell + aT|f|_{\ell,0,\rho,\beta,\gamma,T}$.

(ii) Decompose $u(c,t)$ as

$$u(c,t) = u^\infty(t) + u^0(c,t) + u^1(c,t),$$

where

$$u^\infty(t) = P_0 u_0 + \int_0^t P_0 g(\infty,s)ds,$$

$$u^0(c,t) = e^{ctA}P_1 u_0 + \int_0^t e^{c(t-s)A}P_1 g(\infty,s)ds,$$

$$u^1(c,t) = \int_0^t e^{c(t-s)A}(g(c,s)-g(\infty,s))ds.$$

Then it holds that

$$u^\infty \in C^0([0,T];H^\ell), \quad u^0 \in [A.2]^\ell \cap [A.4]^\ell,$$

$$u^1 \in [A.5]^\ell_0,$$

and that

$$u^\infty(t) = (E^\infty(t),0) = u(\infty,t),$$

$$\nabla_x \cdot E^\infty = \nabla_x \cdot E_0 + \int_0^t \nabla_x \cdot \{-4\pi Jf(\infty,s))ds, \quad \nabla_x \times E^\infty = 0.$$

while u^0 satisfies (3.1) with $\Lambda f, u_0$ replaced by $P_1 g(\infty,t), P_1 u_0$, and so does u^1 with $\Lambda f = g(c,t)-g(\infty,s)$, $u_0=0$.

(iii) Suppose now $u_0 \in H^\ell_\sigma$, $\sigma \ge 2$. Then,

$$\|u^0(c,t)\|_{\ell-2} \le d_0\{(1+ct)^{-1}|u_0|_{\ell,2} + ac^{-1}\log(1+ct)|f(\infty)|_{\ell,2,\rho,\sigma,\beta,T}\},$$

$$\|u^1(c,t)\|_{\ell-2} \le d_0 ac^{-1}\log(1+ct)|f(c)-f(\infty)|_{\ell,2,\rho,\beta,\gamma,T}.$$

4. The Vlasov-Poisson equation

The equation (1.5) contains the Poisson equation,

(4.1) $\nabla_x \cdot E = 4\pi g, \quad \nabla_x \times E = 0,$

where $g = Kf$. This has a solution of the form,

(4.2) $E = Gg \equiv \int_{\mathbb{R}^3} G(x-y)g(y)dy,$

$G(x) = x/|x|^3.$

We note that $\nabla_x G(x) \in C^\infty(\mathbb{R}^3 \setminus \{0\})$, is of homogeneous degree -3 and has the mean value zero on S^2. Thus $\nabla_x G$ is a singular integral operator of Cálderon-Zygmund type. By easy calculation, we get,

$$\|Gg\|_0 \le C_0 |g|_2,$$

$$|\nabla_x G|_k \le C_1 |g|_k, \qquad k \ge 0.$$

Knowing this, we readily have the

Lemma 4.1. Suppose $g \in C^0([0,T]; H^{\ell-1})$ with $\ell \ge 3$ and let E be given by (4.2). Then $E \in C^0([0,T]; \tilde{H}^\ell)$ satisfying

$$\|E\|_{\tilde{H}^\ell, T} \le C|g|_{\ell-1, T}.$$

Recall the operator L of (1.13) and consider the equation,

(4.3)
$$L(f, 0; 0, E) = 0,$$
$$f|_{t=0} = f_0.$$

This is the same equation as (2.6) with s=0, so with V of (2.8), we find its solution as

$$f = V(t, 0; E) f_0.$$

Suppose $E \in C^0([0,T]; \tilde{H}^\ell)$ and $f_0 \in H^\ell_{\sigma,\rho,\beta}$ with $\ell \ge 3$, $\sigma \ge 0$, $\rho > 0$, $\beta \in \mathbb{R}^3$. Then Lemma 2.2 says that

$$f \in C^0([0,T]; H^\ell_{\sigma,\rho,\beta}),$$

satisfying (2.7) with $k=\ell$ and $s=0$. Further, we readily see that

$$Kf \in C^0([0,T];H^\ell),$$

$$|Kf|_{\ell,T} \leq C|f|_{\ell,0,\rho,\beta,T}.$$

Now we discuss the Vlasov-Poisson equation (1.5). It is a coupled equation of (4.1) and (4.3). Therefore, if g is a fixed point of the equation,

$$g = KV(t,0;Gg)f_0, \tag{4.4}$$

then (1.5) is solved by

$$f = V(t,0;Gg)f_0, \qquad E = Gg = GKf. \tag{4.5}$$

Using the results obtained so far and by the successive approximations, we can show that for any $f_0 \in H^3_{0,\rho,\beta}$ with $\rho>0$ or with $\rho=0$ and $\beta>3/2$, there exists a positive constant T and (4.4) has a unique solution $g \in C^0([0,T];H^2)$. Now we can have the

Theorem 4.1. Let $f_0 \in H^\ell_{\rho,\sigma,\beta}$, $\ell \geq 3, \sigma \geq 0, \rho > 0, \beta \in \mathbb{R}$. Then, there is a constant $T \geq 0$ and (1.5) has a unique solution of the form (4.5), satisfying:

$$f \in C^0([0,T];H^\ell_{\sigma,\rho,\beta}), \qquad E \in C^0([0,T];\tilde{H}^{\ell+1}),$$

$$|f(t)|_{\ell,\sigma,\rho,\beta} \leq e^{bt}|f_0|_{\ell,\sigma,\rho,\beta},$$

where b is that of (2.7). Moreover, T depends only on $|f_0|_{3,0,\rho,\beta}$.

Finally we shall solve the modified Vlasov-Maxwell equation appearing in (1.15):

$$L(f,E;\tilde{f},\tilde{E}) = h \quad (\text{or } L(f,E+c^{-1}v\times B;\tilde{f},\tilde{E}+c^{-1}v\times\tilde{B}) = h),$$

$$\partial_t u - cAu = Nf, \tag{4.6}$$

$$(f,u)|_{t=0} = 0.$$

Here $\tilde{f},\tilde{u},h$ are given functions. Rewrite this in the form of the

Volterra type integral equation,

$$f(t)=\int_0^t V(t,s;E)\{-\alpha E(s)\cdot\nabla_v\hat{f}(s)+h(s)\}ds,$$

or

$$f(t)=\int_0^t U(t,s;\tilde{u},c)\{-\alpha(E(s)+c^{-1}v\times B(s)\ \cdot\nabla_v\hat{f}(s)+h(s)\}ds,$$

combined with

$$u(t)=\int_0^t e^{c(t-s)A}\Lambda f(s)ds.$$

Applying Lemmas 2.2 and 3.2 to this, we can readily prove the

Lemma 4.1. Let $\ell\geq 2, \sigma\geq 0, \beta\in\mathbb{R}$ and $\rho,\gamma,T>0$ with $\rho-\gamma T\geq\rho/2$, and let

$$\tilde{f},\ h\in[A.1]^{\ell}_{0,\beta}\cap[A.3]^{\ell}_{0,\beta}\ ,\qquad \tilde{u}\in[A.2]^{\ell}\cap[A.4]^{\ell}.$$

Then there exists a unique solution to (4.6) such that

$$f\in[A.1]^{\ell-1}_{0,\beta-1}\cap[A.3]^{\ell-1}_{0,\beta-1}\quad(\text{or } f\in[A.1]^{\ell-1}_{\gamma,\beta-1}\cap[A.3]^{\ell-1}_{\gamma,\beta-1}),$$

$$u\in[A.2]^{\ell-1}\ (\text{and } u\in[A.4]^{\ell-1} \text{ if } \ell\geq 3).$$

5. Proof of Theorems 1.1-1.3

Theorem 1.1(i) has been proved in [1] using the successive approximation (the contraction mapping principle). The proof was given for $\sigma=0$ but is valid also for $\sigma>0$. This and Lemma 2.1 then prove Theorem 1.1(ii).

In order to prove Theorem 1.2, we shall recall the successive approximation mentioned above; $(f^0,u^0)=0$ and for $n\geq 1$,

$$(5.1)\qquad \begin{aligned} f^n &= U(t,0;u^{n-1},c)f_0, \\ u^n &= e^{ctA}u_0 + \int_0^t e^{c(t-s)A}\Lambda f^{n-1}(s)ds. \end{aligned}$$

Thanks to (2.4),(3.9) applied to (5.1), we can find $\gamma,T,C>0$ such that

$$(5.2)\qquad |f^n|_{\ell,\sigma,\rho,\beta,\gamma,T} + |u^n|_{\ell,T} \leq C$$

holds for all n. Then by Lemmas 2.1,2.2,3.2, it follows that

$$(5.3)\qquad f^n \in [A.1]^{\ell}_{\gamma,\beta} \cap [A.3]^{\ell}_{\gamma,\beta}\ ,\quad u^n \in [A.2]^{\ell} \cap [A.4]^{\ell},\quad n\geq 0.$$

Using these, we repeat the argument of [1] to see that

$$f^n \to f \text{ in } B^0([1,\infty);C^0_\gamma([0,T];H^{\ell-1}_{\sigma,\rho,\beta-1})),$$
$$u^n \to u \text{ in } B^0([1,\infty);C^0([0,T];H^{\ell-1})),$$

strongly as $n\to\infty$, with some limit (f,u). By (5.2) and the interpolation theorem, this convergence is also true if $\ell-1,\beta-1$ are replaced by $\ell-\epsilon,\beta-\epsilon$, for any $\epsilon>0$. This and (5.3) then imply that for any $\delta>0$,

$$u^n \to u \text{ strongly in } B^0([1,\infty]\times[\delta,T];B^{\ell-2}(\mathbb{R}^3)).$$

Now the first half of Theorem 1.2 follows since the limit (f,u) obviously coincides with the solution of Theorem 1.1, and the latter half comes directly from Lemmas 2.2,3.2 and Theorem 4.1.

The asymptotic expansion in Theorem 1.3 is obtained as follows. First, let f^∞ be that of Theorem 1.2 and assume the expansion, (5.4) $$f = f^{\infty} + \sum_{j=0}^{k} c^{-j} f^{j}.$$

Substitute this into (3.8) which holds for our (f,u) by going to the limit in (5.1). Then we have,

(5.5) $$u = \tilde{u}^{0} + \sum_{j=1}^{k} c^{-j} u^{j},$$

where

$$\tilde{u}^{0} = e^{ctA} u_{0} + \int_{0}^{t} e^{c(t-s)A} \Lambda (f^{\infty}(s) + f^{0}(s))ds,$$

$$u^{j} = \int_{0}^{t} e^{c(t-s)A} \Lambda f^{j}(s)ds, \quad 1 \le j \le k.$$

Using (3.4), we decompose $\tilde{u}^{0}$ further as

$$\tilde{u}^{0} = u^{\infty} + u^{0},$$

$$u^{\infty}(t) = P_{0}u_{0} + \int_{0}^{t} P_{0} \Lambda f^{\infty}(s)ds,$$

$$u^{0}(t) = e^{ctA} P_{1} u_{0} + \int_{0}^{t} e^{c(t-s)A} (P_{I} \Lambda f^{\infty}(s) + \Lambda f^{0}(s))ds.$$

By Lemma 3.2 and since our f^{∞} is that of Theorem 1.2, u^{∞} defined above is just that of Theorem 1.2. Also, recalling that (3.8) is a unique solution to (3.1), we see that u^{j} solves (formally) the Maxwell equation in $(1.15)^{j}$, $0 \le j \le k$. Substitute (5.4) and (5.5) into (1.1) to deduce the equation for f^{j} in $(1.15)^{j}$. Now the proof of Theorem 1.3 can be completed by the help of Lemmas 2.2, 3.2 and 4.1, and by proceeding as in the proof of Theorems 1.1, 1.2 and 4.1. The detail is omitted.

References

[1] Asano, K., On local solutions of the initial value problem for the Vlasov Maxwell equation (1984), Preprint.

[2] Asano, K., On the incompressible limit of the compressible Euler equation (1985), Preprint.

[3] Asano, K. and Ukai, S., On the fluid dynamical limit of the Boltzmann equation, Lec. Notes in Num. Appl. Math. 6 (1983) 1-19, Kinokuniya/Noth-Hallond.

[4] Bardos, C. and Degond, P., Global existence for the Vlassov-Poisson equation in 3 sapce variables with small initial data, Internal report n^{o} 101, Centre de Math. Appl. E.P.P. (1983)

[5] Degond, P., Local existence of solutions of the Vlasov-Maxwell equations and convergence to the Vlasov-Poisson equations for infinite light velocity (1984), Preprint.

[6] Mizohata, S., The theory of partial differential equation, Cambridge Univ. Press (1973).

[7] Ukai, S., The incompressible limit and the initial layer of the compressible Euler equation (1984), Preprint.

K.A.;
Dep. of Math.
Yoshida College
Kyoto Univ.
Kyoto

S.U.;
Dep. of Appl. Phys.
Osaka city Univ.
Osaka