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CLASSIFICATION OF SEMI-REGULAR GROUP DIVISIBLE DESIGNS WITH $\lambda_2 = \lambda_1 + 1$ *

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Group divisible (GD) designs with parameters $v = mn, b, r, k, \lambda_1, \lambda_2$ satisfying $\lambda_2 = \lambda_1 + 1$ have strong statistical significance in terms of optimality. In this paper, we attempt to classify semi-regular GD designs satisfying $\lambda_2 = \lambda_1 + 1$ by expressing all the parameters in terms of at most four integral parameters. As special cases, available series of semi-regular GD designs can be derived.

1. Introduction

The largest, simplest and perhaps most important class of 2-associate partially balanced incomplete block designs is known as group divisible (GD). A GD design is an arrangement of v (= mn) treatments in b blocks such that each block contains k ($< v$) distinct treatments; each treatment is replicated r times; and the treatments can be divided into m groups of n (≥ 2) treatments each, any two treatments occurring together in λ_1 blocks if they belong to the same group, and in λ_2 blocks if they belong to different groups. For the usual incidence matrix N of the GD design, NN' has eigenvalues $r - \lambda_1$ (= θ_1, say) and $rk - \lambda_2 v$ (= θ_2, say) other than rk, with the respective multiplicities $m(n - 1)$ and $m - 1$.

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Depending on values of the eigenvalues, GD designs are classified into three subtypes: (a) singular if $\theta_1 = 0$; (b) semi-regular (SR) if $\theta_1 > 0$ and $\theta_2 = 0$; (c) regular if $\theta_1 > 0$ and $\theta_2 > 0$.

From a well-known relation $r(k - 1) = (n - 1)\lambda_1 + n(m - 1)\lambda_2$, it holds that $\theta_1 - \theta_2 = n(\lambda_2 - \lambda_1)$. Hence, if $|\theta_1 - \theta_2| = 1$, then any GD design does not exist. Furthermore, if $|\theta_1 - \theta_2|$ is a prime, p, say, then $n = p$ and $|\lambda_2 - \lambda_1| = 1$. Note that in a singular GD design $\lambda_1 > \lambda_2$; in an SRGD design $\lambda_2 > \lambda_1$. From a point of view of statistical optimality, it is known (cf. Takeuchi [4]) that a GD design with $\lambda_2 = \lambda_1 + 1$ is A- and E-optimal. In the above sense, a restriction "$\lambda_2 = \lambda_1 + 1$" has a special meaning on existence and optimality. We shall here consider GD designs satisfying $|\lambda_1 - \lambda_2| = 1$ and attempt to classify them in a closed form. The case of SRGD designs, in particular, will be considered in detail.

2. Singular and regular designs

In a singular GD design, it is known (cf. Bose and Connor [1])that the existence of a balanced incomplete block (BIB) design with parameters $v^*, b^*, r^*, k^*, \lambda^*$ is equivalent to the existence of a singular GD design with parameters $v = nv^*$, $b = b^*$, $r = r^*$, $k = nk^*$, $\lambda_1 = r^*$, $\lambda_2 = \lambda^*$ for every n. Hence a singular GD design satisfying $\lambda_1 = \lambda_2 + 1$ is only of the form as $v = mn$, $b = m$, $r = m - 1$, $k = (m - 1)n$, $\lambda_1 = m - 1$, $\lambda_2 = m - 2$, which can always be constructed from a trivial BIB design with parameters $v^* = b^* = m$, $r^* = k^* = m - 1$, $\lambda^* = m - 2$.

In a regular GD design, though there are possibilities of $\lambda_1 - \lambda_2 = \pm 1$, Mukerjee, Kageyama and Bhagwandas [2] characterized a regular GD design satisfying $rk - \lambda_2 v = 1$ and $\lambda_2 = \lambda_1 + 1$ as a symmetrical design whose parameters are expressed in terms of only two integral parameters. It seems to be difficult to characterize a regular GD design satisfying

$\lambda_1 - \lambda_2 = \pm 1$ without further restrictions on parameters.

3. Characterization of SRGD designs

The following observations will be helpful in the sequel. Consider the equation

$$px - qy = w, \tag{1a}$$

where p and q are relatively prime positive integers and w is a non-negative integer. Given p,q and w, it is easily seen that (1a) has positive integral-valued solutions (x,y). Furthermore, if (x_1,y_1) and (x_2,y_2) are any two distinct positive integral-valued solutions of (1a), then either $x_1 < x_2$, $y_1 < y_2$ or $x_1 > x_2$, $y_1 > y_2$. Hence there exists a solution, say (x^*,y^*) of (1a), depending on p,q and w, such that if $(\bar{x},\bar{y})$ be any other solution then $x^* < \bar{x}$, $y^* < \bar{y}$. The solution (x^*,y^*) will be called the minimal solution of (1a). It may be seen that every positive integral-valued solution of (1a) is of the form

$$(x^* + tq,\ y^* + tp) \quad (t = 0,1,2,\dots).$$

In particular, the minimal solution of

$$px - qy = 1 \tag{1b}$$

will be denoted by (x_o,y_o), where, of course, $x_o = x_o(p,q)$ and $y_o = y_o(p,q)$ are functions of p and q. Also, with x_o defined as above, the minimal solution of

$$px - qy = x_o \tag{1c}$$

will be denoted by (g_o,h_o), where $g_o = g_o(p,q)$ and $h_o = h_o(p,q)$ are functions of p and q. Since p and q are relatively prime, one has

$$\{(qj + 1)_{\text{mod } p}: j = 1,2,\dots,p\} = \{0,1,\dots,p - 1\}$$

and hence

$$y_o \leq p. \tag{2}$$

It may further be seen that y_o and p are relatively prime.

Consider now an SRGD design with parameters $v = mn, b, r, k, \lambda_1, \lambda_2$, where

$$rk - \lambda_2 v = 0, \tag{3}$$

and

$$\lambda_2 = \lambda_1 + 1. \tag{4}$$

The relation (3), together with $r(k - 1) = (n - 1)\lambda_1 + n(m - 1)\lambda_2$, implies

$$r = n + \lambda_1. \tag{5}$$

Since for an SRGD design k must be an integral multiple of m (cf. Raghavarao [3]), let

$$k = cm, \tag{6}$$

where c is a positive integer and by (3)-(6),

$$c = n(\lambda_1 + 1)/(n + \lambda_1) = (\lambda_1 + 1) - (\lambda_1 + 1)\lambda_1/(n + \lambda_1). \tag{7}$$

Also, by (5)-(7),

$$b = vr/k = (n + \lambda_1)^2/(\lambda_1 + 1). \tag{8}$$

Clearly, n and λ_1 are such that both b and c are positive integers. Defining

$$a = n + \lambda_1, \; s = \lambda_1 + 1, \tag{9}$$

it follows from (7) and (8) that $s(s - 1)/a$ and a^2/s are both integral-valued. This holds trivially if $s = 1$ (i.e. $\lambda_1 = 0$), in which case by (4)-(8), the parameters of the design are of the form

$$v = mn, \; b = n^2, \; r = n, \; k = m, \; \lambda_1 = 0, \; \lambda_2 = 1. \tag{10}$$

Consider now the further case $s > 1$ (i.e. $\lambda_1 \geq 1$). Let d represent the integer $s(s - 1)/a$. Then

$$a = s(s - 1)/d. \tag{11}$$

Evidently, there exists a unique factorization of d such that

$$d = pq, \tag{12}$$

and s/p and $(s - 1)/q$ are integral-valued. Here p and q are relatively prime since so are s and $s - 1$. Let

$$s/p = x, \quad (s - 1)/q = y. \tag{13}$$

Note that x and y have to be positive integers, since $s > 1$. Under (13), $px - qy = 1$, and, therefore, by our earlier discussion x and y must be of the form

$$x = x_o + tq, \quad y = y_o + tp \quad (t = 0,1,2,\ldots), \tag{14}$$

where (x_o, y_o) is the minimal solution of (1b). By (11)-(14),

$$s = px = p(x_o + tq), \tag{15a}$$

$$s - 1 = qy = q(y_o + tp), \tag{15b}$$

$$a = s(s - 1)/d = (x_o + tq)(y_o + tp). \tag{16}$$

In the above $t \geq 1$, for $t = 0$ implies that $a/s = y_o/p \leq 1$ (by (2)), i.e. $a \leq s$, which is impossible from (9) and the fact $n \geq 2$.

Now by (15a), (16),

$$a^2/s = (x_o + tq)(y_o + tp)^2/p,$$

which must be integral-valued. As noted earlier, y_o and p and hence $y_o + tp$ and p are relatively prime. Therefore, $x_o + tq$ must be an integral multiple of p. Let $z = (x_o + tq)/p$. Then $pz - qt = x_o$, and comparing this with (1c), z and t are of the form

$$z = g_o + fq, \quad t = h_o + fp \quad (f = 0,1,2,\ldots), \tag{17}$$

g_o and h_o being as defined earlier. Hence

$$(x_o + tq)/p = [x_o + (h_o + fp)q]/p = (x_o + h_o q)/p + fq = g_o + fq, \tag{18}$$

since (g_o, h_o) is a solution of (1c).

By (15)-(18),

$$s = p^2(g_o + fq),$$

$$s - 1 = q[y_o + (h_o + fp)p],$$

$$a = p(g_o + fq)[y_o + (h_o + fp)p].$$

Hence by (4)-(9),

$$n = a - (s - 1) = [y_o + (h_o + fp)p][p(g_o + fq) - q], \tag{19a}$$

$$v = mn = m[y_o + (h_o + fp)p][p(g_o + fq) - q], \tag{19b}$$

$$b = a^2/s = (g_o + fq)[y_o + (h_o + fp)p]^2, \tag{19c}$$

$$r = a = p(g_o + fq)[y_o + (h_o + fp)p], \tag{19d}$$

$$c = s - s(s - 1)/a = p[p(g_o + fq) - q],$$

$$k = cm = mp[p(g_o + fq) - q], \tag{19e}$$

$$\lambda_1 = s - 1 = q[y_o + (h_o + fp)p], \tag{19f}$$

$$\lambda_2 = s = p^2(g_o + fq), \tag{19g}$$

where $m(\geq 2)$, $f(\geq 0)$, $p(\geq 1)$, $q(\geq 1)$ are integral-valued, p and q are relatively prime and y_o, g_o, h_o are functions of p and q as defined earlier.

Thus for an SRGD design with $\lambda_2 = \lambda_1 + 1$, the parameters must be of the form (10) or (19a-g). It is seen that the parameters of the design can be expressed in a closed form in terms of at most four integral parameters. It may, further, be remarked that the four parameters involved in (19a-g) are again not all independent since p and q have to be relatively prime. The series (10) occurs frequently in the available literature as one of the main series of GD designs.

The relations (10) and (19a-g) provide a natural classification of SRGD designs with $\lambda_2 = \lambda_1 + 1$. The designs with parameters as in (19a-g) may be further subclassified according to m,f,p and q. Incidentally, from (10) and (19a-g), an SRGD design with $\lambda_1 = 1$ and $\lambda_2 = 2$ is non-existent.

In a large number of SRGD designs with $\lambda_2 = \lambda_1 + 1$, v is an integral multiple of k and it may be interesting to investigate this situation as a special case of (10) and (19a-g). For the series in (10), v is trivially an integral multiple of k. Consider, therefore, the series described in (19a-g). Note that by (6),(7),(9),(14) and (16),

$$v/k = (n + \lambda_1)/(\lambda_1 + 1) = a/s = (y_o + tp)/p,$$

and hence the integrality of v/k implies that y_o/p is an integer. Now by (2), and the fact that y_o and p are relatively prime, one must have p = 1. If p = 1, then for arbitrary positive integer q, it is easy to check that $x_o = q + 1$, $y_o = 1$, $g_o = 2q + 1$, $h_o = 1$, and hence (19a-g) reduce to

$$n = (f + 2)[(f + 1)q + 1], \quad v = m(f + 2)[(f + 1)q + 1],$$
$$b = (f + 2)^2[(f + 2)q + 1], \quad r = (f + 2)[(f + 2)q + 1], \tag{20}$$
$$k = m[(f + 1)q + 1], \quad \lambda_1 = (f + 2)q, \quad \lambda_2 = (f + 2)q + 1,$$

where $m(\geq 2)$, $f(\geq 0)$, $q(\geq 1)$ are integers. Combining (10) and (20), the parameters of an SRGD design with $\lambda_2 = \lambda_1 + 1$, and, further, with v as an integral multiple of k, may be expressed in a compact form as

$$n = (\ell + 1)(\ell\alpha + 1), \quad v = m(\ell + 1)(\ell\alpha + 1), \quad b = (\ell + 1)^2(\ell\alpha + \alpha + 1),$$
$$r = (\ell + 1)(\ell\alpha + \alpha + 1), \quad k = m(\ell\alpha + 1), \quad \lambda_1 = (\ell + 1)\alpha, \quad \lambda_2 = \ell\alpha + \alpha + 1,$$

where $m(\geq 2)$, $\ell(\geq 1)$, $\alpha(\geq 0)$ are integers.

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